The Vehicle Routing Problem with Drones: Some Worst-Case Results

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Introduction: Motivation of VRPD

- Amazon (U.S.),
- Google (U.S.),
- FedEx (U.S.),
- DHL (Germany),
- SingPost (Singapore),
- Alibaba (China), and others are researching the use of commercial drones

Source: http://www.channelnewsasia.com/

Figure 1: SingPost drone testing
The use of drones can reduce the time and cost of package delivery.

Drones have limited range and payload.

We want a model in which the trucks serve as both delivery vehicles and mobile depots for the drones.
Introduction: The basic VRPD model

- **Trucks**
  - A homogeneous fleet of $m$ trucks
  - The speed of a truck is 1 unit/hr
  - The capacity of a truck is $C$ packages
  - There is no limit on the duration of a truck route

- **Drones**
  - Every truck carries $k$ identical drones
  - A drone follows the same distance metric as the truck
  - The speed of a drone is $\alpha$ units/hr
  - The capacity of a drone is 1 package
  - The battery life is unlimited
  - The time required to prepare the drone for another launch is instantaneous
Introduction: The basic VRPD model

- **Customers**
  - Every customer demands 1 package
  - Every demand can be delivered either by a truck or a drone

- **Coordination**
  - Trucks can continue to serve customers after a drone is dispatched
  - Drones can be dispatched and picked up only at nodes (customer locations and the depot)
  - A drone dispatched at a node must be picked up by the same truck, at the same or at a different node
Introduction: The basic VRPD model

- The objective is to minimize the completion time, i.e., the duration from the time when the first truck is dispatched from the depot with its drones to the time when the last truck returns to the depot with all its drones.

- The optimal objective function value is denoted by $Z(\text{VRPD}^{(k)}_{m,\alpha})$
Introduction: A VRPD example

The truck route is represented by the solid black line and the drone routes are represented by the blue and red dashed lines.

Figure 2: A VRPD$_{1,1}^{(2)}$ example
Introduction: A VRPD example

\[ Z(TSP) = 8 + 6\sqrt{2} \approx 16.485 \]

\[ Z^f(VRPD^{(2)}_{1,1}) = 12 \]

\[ \frac{Z(TSP)}{Z^f(VRPD^{(2)}_{1,1})} \approx 1.374 \]

What is the theoretical maximum savings we can get from using the drones?

**Figure 3: Optimal TSP solution**
Worst-case analysis

- We want to give theoretical bounds on the benefit from using drones.

- We compare two associated problems $P_t$ and $P_{td}$ with the same set of customers:
  - In $P_t$, the fleet consists of trucks only.
  - In $P_{td}$, the fleet consists of trucks and drones.
Worst-case analysis

- We denote the optimal solutions by $Z(P_t)$ and $Z(P_{td})$ and show that
  
  • There is a number $B \geq 1$ such that
    \[
    \frac{Z(P_t)}{Z(P_{td})} \leq B, \quad \text{and}
    \]
  
  • There is either an instance $P^*$ such that
    \[
    \frac{Z(P^*_t)}{Z(P^*_td)} = B
    \]
  
  • Or a sequence of instances $\{P^*_n\}$ such that
    \[
    \lim_{n \to \infty} \frac{Z(P^*_t,n)}{Z(P^*_td,n)} = B
    \]
## Worst-case analysis

<table>
<thead>
<tr>
<th></th>
<th>$P_t$</th>
<th>$P_{td}$</th>
<th>$\sup{Z(P_t)/Z(P_{td})}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>TSP</td>
<td>$\text{VRPD}^{(k)}_{1,\alpha}$</td>
<td>$\alpha k + 1$</td>
</tr>
<tr>
<td>2</td>
<td>TSP</td>
<td>$\text{VRPD}^{(k)}_{m,\alpha}$</td>
<td>$m(\alpha k + 1)$</td>
</tr>
<tr>
<td>3</td>
<td>$\text{VRP}^*_m$</td>
<td>$\text{VRPD}^{(k)}_{m,\alpha}$</td>
<td>$\alpha k + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$\text{VRPD}^{(k)}_{m,\alpha}$</td>
<td>$\text{VRPD}^{(k)}_{m,\beta}$</td>
<td>$\beta/\alpha$</td>
</tr>
</tbody>
</table>

**Table 1:** Some of the problems studied

1 $\text{VRP}_m^*$: A min-max VRP with a fleet of $m$ trucks.
Worst-case analysis: Structure of the proofs

- To show that $B$ is an upper bound of $\frac{Z(P_t)}{Z(P_{td})}$
  - Start with the optimal $P_{td}$ solution, $Z(P_{td})$
  - Construct a feasible $P_t$ solution, $Z^f(P_t)$
  - Such that the $Z^f(P_t)$ is bounded above by $B \ast Z(P_{td})$

- To show that $B$ is the least upper bound of $\frac{Z(P_t)}{Z(P_{td})}$
  - Construct an instance $P^*$
  - Such that $\frac{Z(P^*_t)}{Z^f(P^*_{td})} = B$
  - Therefore, $\frac{Z(P^*_t)}{Z(P^*_{td})} \geq \frac{Z(P^*_t)}{Z^f(P^*_{td})} = B$
An example of the theorems: VRPD\textsubscript{m,\(\alpha\)} vs VRP\textsubscript{m}^* 

**Theorem**

*If the triangle inequality is valid, then*

\[
\frac{Z(VRP_m^*)}{Z(VRPD_{m,\alpha}^{(k)})} \leq \alpha k + 1,
\]

*and the bound is tight.*
VRPD_{m,k\alpha}^{(1)} vs VRPD_{m,\alpha}^{(k)}

Is it better to use one faster drone (with speed $k\alpha$) or to use $k$ slower drones (with speed $\alpha$)?

- The worst-case bounds are the same
- $k$ slower drones have larger combined capacity
- One faster drone is at least as efficient in terms of the combined speed
VRPD\(^{(1)}_{m,k\alpha}\) vs VRPD\(^{(k)}_{m,\alpha}\)

**Figure 4:** A faster drone is more efficient
\[ \text{VRPD}_{m,k\alpha}^{(1)} \text{ vs VRPD}_{m,\alpha}^{(k)} \]

Two drones of speed 2 vs one drone of speed 4

(a) Fleet A: Completion time = 6
(b) Fleet B: Completion time = 7.5

**Figure 5:** More slower drones are better
Comments on the worst-case analysis

- The drones have the same distance metric as the trucks
- The drone battery life is unlimited
- The objective function considers only the completion time
Extension: Different distance metrics

**Theorem**

Let $Q_t$ and $Q_d$ be the distance matrices followed by trucks and drones, respectively. Then,

$$\frac{Z(\text{VRP}^*, Q_t)}{Z(\text{VRPD}^{(k)}_{m,\alpha}, Q_t, Q_d)} \leq \frac{Z(\text{VRP}^*_m, Q_t)}{Z(\text{VRP}^*_m, Q_d)} (\alpha k + 1).$$

Let $H = \max_{i \neq j} \{ Q_t(i, j)/Q_d(i, j) \}$. Then

$$\frac{Z(\text{VRP}^*_m, Q_t)}{Z(\text{VRPD}^{(k)}_{m,\alpha}, Q_t, Q_d)} \leq H(\alpha k + 1).$$
Extension: Different distance metrics

- A fleet of one truck loaded with one drone
- Radius of inner circle = 1
  Radius of outer circle = 2
- $\alpha = \frac{4}{\pi} \sqrt{5 - 2\sqrt{2}} \approx 1.876$
- The truck is restricted to the edges shown; the drone can fly as the crow flies

Figure 6: Different truck and drone distance metrics
Extension: Different distance metrics

(a) Restricted to the edges:
\[ Z(TSP, Q_t) = 2 + 5.5\pi \]

(b) Using crow-fly distances:
\[ Z(TSP, Q_d) = 8\sqrt{5} - 2\sqrt{2} \]

Figure 7: TSP solutions with different distance metrics
Extension: Different distance metrics

\[ Z(TSP, Q_t) \approx 19.3, \quad Z(TSP, Q_d) \approx 11.8 \]

\[ \alpha k + 1 \approx 2.876 \]

\[ \frac{Z(TSP, Q_t)}{Z(TSP, Q_d)}(\alpha k + 1) \approx 4.702 \]

\[ H(\alpha k + 1) \approx 1.7445 \times 2.8765 \approx 5.018 \]
Extension: Different distance metrics

The actual speed-up ratio \((2 + 5.5\pi)/(2\pi) \approx 3.068\)

\[(a) \ Z(TSP, Q_t) = 2 + 5.5\pi\]

\[(b) \ Z(VRPD_{1,\alpha}^{(1)}, Q_t, Q_d) = 2\pi\]

**Figure 9:** TSP and VRPD solutions
Conclusions

- We introduced the VRPD along with a number of model assumptions

- We proved several worst-case results with respect to the VRPD

- We extended the VRPD model and gave additional results
Future work to be explored

- The relationship between the VRPD and other problems
- Graph specific analysis
- Computational studies
Thank you!

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