Variance bounds on the permanent and transitory components of stochastic discount factors ☆

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ABSTRACT

In this paper, we develop lower bounds on the variance of the permanent component and the transitory component, and on the variance of the ratio of the permanent to the transitory components of SDFs. Exactly solved eigenfunction problems are then used to study the empirical attributes of asset pricing models that incorporate long-run risk, external habit persistence, and rare disasters. Specific quantitative implications are developed for the variance of the permanent and the transitory components, the return behavior of the long-term bond, and the comovement between the transitory and the permanent components of SDFs.

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1. Introduction

In an important contribution, Alvarez and Jermann (2005) lay the foundations for deriving bounds when the stochastic discount factor (hereby, SDF), from an asset pricing model, can be decomposed into a permanent component and a transitory component [see also the contribution of Hansen and Scheinkman (2009) and Hansen, Heaton, and Li (2008)]. Our objective in this paper is to propose a bounds framework, in the context of permanent and transitory components, and we show that our bounds are fundamentally different from Alvarez and Jermann (2005). A rationale for developing our bounds is that the bounds postulated in Alvarez and Jermann (2005) hinge on the return properties of the long-term discount bond, the risk-free bond, and a generic equity portfolio. Our approach generalizes to an asset space with a dimension greater than three, and it departs

by relying on the variance-measure (as in Hansen and Jagannathan, 1991).

Building on our treatment, we address the key questions of (i) how useful our bounds are in assessing asset pricing models, and (ii) how our bounds compare, in the theoretical and empirical dimension, with the counterparts in Alvarez and Jermann (2005) (wherever applicable). The first question is pertinent to economic modeling, while the second question is pertinent to the incremental value-added and the tightness of our bounds in empirical applications.

In our setup, we develop a lower bound on the variance of the permanent component of the SDF and a lower bound on the variance of the transitory component of SDFs, and then a lower bound on the variance of the ratio of the permanent to the transitory components of SDFs. The lower bound on the variance of the permanent/transitory component can be viewed as a generalization of the Alvarez and Jermann (2005) bounds. Our lower bound on the variance of the ratio of the permanent to the transitory components of SDFs allows us to assess whether asset pricing models are capable of describing the additional dimension of joint pricing across markets, and has no analog in Alvarez and Jermann (2005). A salient feature of our bounds is that they incorporate information from average returns as well as the variance-covariance matrix of returns from a generic set of assets. We show that the bound implications for the permanent component of SDFs in the three-asset case is considerably weaker than those reported in our multiple-asset context.

An essential link between our variance bounds and asset pricing is the eigenfunction problem of Hansen and Scheinkman (2009), which facilitates an analytical expression for the permanent and transitory components of the SDF for an asset pricing model. The posited bounds, in conjunction with an analytical solution to the eigenfunction problem, can provide a setting for discerning whether the time-series properties underlying an asset pricing model are consistent with observed data from financial markets.

To provide wider foundations for our empirical examination, we focus on the eigenfunction problem for a broad class of asset pricing models, namely, (i) long-run risk [Bansal and Yaron (2004) as also parameterized in Kelly (2009)), (ii) external habit persistence (Campbell and Cochrane (1999) as also parameterized in Bekaert and Engstrom (2010)], and (iii) rare disasters (as parameterized in Backus, Chernov, and Martin, 2011). Then we are led to ask a question of economic interest: What can be learned about asset pricing models that consistently price the long-term discount bond, the risk-free bond, the equity market, and a multitude of other assets. The importance of studying long-run risk, external habit persistence, and rare disasters under a common platform is also recognized by Hansen (2009).

A number of new insights can be garnered about the performance of asset pricing models in the context of equity and bond data. One implication of our findings is that the variance of the permanent component of the SDF in models is of an order lower than the corresponding bound reflected in returns of bonds, equity market, and

portfolios sorted by size and book-to-market. The model with rare disasters exhibits the highest variance of the permanent component, a feature linked to occasional consumption crashes.

Next, we observe that the transitory component of the SDF in models fails to meet the lower variance bound restriction. This bound is tied to the Sharpe ratio of the long-term bond. We further characterize the expected return (variance) of a long-term bond, and find that while each model quantitatively depicts the real return of a risk-free bond, they fall short in reproducing the long-term bond properties. This metric of inconsistency matters since prospective models typically offer reconciliation with the equity premium, while often ignoring the return of long-term bonds. Our inquiry uncovers that the mis-specified transitory component is a source of the limitation in describing the return behavior of long-term bonds.

We make two key observations with respect to the joint dynamics of permanent and transitory components of SDFs. First, we show that it is possible to recover the comovement between the permanent and the transitory components from historical data without making any distributional assumptions. While the data tell us that the two components should move in the same direction, our analysis reveals that this feature is not easily imitated by the models. Second, we show that the model-based variance of the ratio of the permanent to the transitory components of the SDFs is insufficiently high, conveying the need to describe more plausibly the joint dynamics of the returns of bonds and other assets.

Taken all together, our non-parametric approach highlights the dimensions of difficulty in reconciling returns data under some parameterizations of asset pricing models. Our work belongs to a list of studies that searches for the least misspecified asset pricing model, and explores their implications, as outlined, for example, in Bansal, Kiku, and Yaron (2009), Beeler and Campbell (2009), and Yang (2010, 2011) Our variance bounds could be adopted as a complementary device to examine the validity of a model, apart from matching sample moments, slope coefficients from predictive regressions, and correlations. The eigenfunction problem offers further guidance for asset price modeling.

2. Bounds on the permanent and transitory components

This section presents theoretical bounds related to the unconditional variance of permanent and transitory components of SDFs, and the ratio of the permanent to the transitory components of SDFs.

Since asset pricing models often face a hurdle of explaining asset market data based on unconditional bounds, we develop our results in terms of unconditional bounds, instead of the sharper conditional bounds.

2.1. Motivation for developing bounds based on the properties of a generic set of asset returns

We adopt notations similar to that in Alvarez and Jermann (2005), and let $\{M_t\}$ be the process of strictly positive pricing kernels. As in Hansen and Richards

(1987), we use the absence of arbitrage opportunities to specify the current price of an asset that pays D_{t+k} at time $t+k$ as

$$V_t[D_{t+k}] = E_t\left(\frac{M_{t+k}}{M_t} D_{t+k}\right), \tag{1}$$

where $E_t(.)$ represents the conditional expectation operator. The SDF from t to $t+1$ is represented by M_{t+1}/M_t.

To differentiate returns offered by different types of assets, we first define $R_{t+1,k}$ as the gross return from holding, from time t to $t+1$, a claim to one unit of the numeraire to be delivered at time $t+k$. Then, the return from holding a discount bond with maturity k from time t to $t+1$, and the long-term discount bond are, respectively,

$$R_{t+1,k} = \frac{V_{t+1}[1_{t+k}]}{V_t[1_{t+k}]} \quad \text{and} \quad R_{t+1,\infty} \equiv \lim_{k\to\infty} R_{t+1,k}. \tag{2}$$

The case of $k=1$ in Eq. (2) corresponds to the gross return of a risk-free bond.

Next we denote by R_{t+1}, the gross return of a broad equity portfolio or the equity market. Such an asset captures the equity risk premium, and plays a key role in the formulations of Alvarez and Jermann (2005).

To build on the asset space, it is of interest to define $\mathbf{R}_{t+1} = (R_{t+1,1}, R_{t+1}, \mathbf{R}_{t+1,a})'$, which constitutes an $n+2$-dimensional column vector of gross returns. The n-dimensional vector $\mathbf{R}_{t+1,a}$ contains a finite number of risky assets that excludes the equity market and the long-term discount bond.

Consider the set of SDFs that consistently price $n+3$ assets, that is, the long-term discount bond, the risk-free bond, the equity market, and additionally n other risky assets,

$$\mathbb{S} \equiv \left\{ \frac{M_{t+1}}{M_t} : E\left(\frac{M_{t+1}}{M_t} R_{t+1,\infty}\right) = 1 \quad \text{and} \quad E\left(\frac{M_{t+1}}{M_t} \mathbf{R}_{t+1}\right) = \mathbf{1} \right\}, \tag{3}$$

where $\mathbf{1}$ is a column vector of ones conformable with $\mathbf{R}_{t+1}$, and $E(.)$ represents the *unconditional* expectation operator. The mean of the SDF is given by $\mu_m \equiv E((M_{t+1}/M_t)1_{t+1})$.

In the discussions to follow, we refer to $Var[u] = E(u^2) - (E(u))^2$ as the variance-measure for some random variable u, and use it to quantify the importance of transitory and permanent components of SDFs.

The L-measure-based bounds framework in Alvarez and Jermann (2005) is intended specifically for a long-term discount bond, a risk-free bond, and a single equity portfolio, whereas Eq. (3) allows one to expand the asset space to a dimension beyond three. Our motivation for studying variance-measure-based bounds, as opposed to L-measure-based bounds, will be articulated in Section 3.

There are reasons to expand the set of assets in our analysis. Note that the crux of the bounds approach is that the bounds are derived under the assumption that (i) the transitory component prices the long-term bond, (ii) the permanent component prices other assets, and (iii) the SDF correctly prices the entire set of assets, while accounting for the relation between the transitory and the permanent components. In this context, we show that the bounds implication for the three-asset case of Alvarez and Jermann (2005) is considerably weaker than those

reported in our empirical illustrations involving many risky assets.

Equally important, our variance bounds treatment is intended for all sorts of assets, which is in the vein of, among others, Shanken (1987); Hansen and Jagannathan (1991); Snow (1991); Cecchetti, Lam, and Mark (1994); Kan and Zhou (2006); Luttmer (1996); Balduzzi and Kallal (1997), and Bekaert and Liu (2004). In this sense, the viability of asset pricing models can now be judged by their ability to satisfactorily accommodate the risk premium on a spectrum of traded assets, and not just the equity premium.

2.2. Bounds on the permanent component of SDFs

Alvarez and Jermann (2005, Proposition 1), and Hansen and Scheinkman (2009, Corollary 6.1) show that any SDF can be decomposed into a transitory component and a permanent component. Inspired by their analyses, we presume that there exists a decomposition of the pricing kernel M_t into a transitory and a permanent component of the type:

$$M_t = M_t^T M_t^P, \quad \text{with} \quad E_t(M_{t+1}^P) = M_t^P. \tag{4}$$

The permanent component M_t^P is a martingale, while the transitory component M_t^T is a scaled long-term interest rate. In particular,

$$R_{t+1,\infty} = \left(\frac{M_{t+1}^T}{M_t^T}\right)^{-1}, \tag{5}$$

which follows from Alvarez and Jermann (2005, Assumptions 1 and 2, and their proof of Proposition 2). The transitory component prices the long-term bond with $E((M_{t+1}^T/M_t^T)R_{t+1,\infty}) = 1$. Completing the description of the decomposition (4), the transitory and permanent components of the SDF can be correlated.

We assume that the variance-covariance matrix of $\mathbf{R}_{t+1}$, $\mathbf{R}_{t+1}/R_{t+1,\infty}$, $\mathbf{R}_{t+1}/R_{t+1,\infty}^2$ are each nonsingular. Our proof follows.

Proposition 1. Suppose the relations in Eqs. (4) and (5) hold. Then the lower bound on the unconditional variance of the permanent component of SDFs $M_{t+1}/M_t \in \mathbb{S}$ is

$$Var\left[\frac{M_{t+1}^P}{M_t^P}\right] \geq \sigma_{pc}^2 \equiv \left(\mathbf{1} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)' \Omega^{-1} \left(\mathbf{1} - E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right), \tag{6}$$

where $\Omega \equiv Var[\mathbf{R}_{t+1}/R_{t+1,\infty}]$.

Proof. See Appendix A.

We note that our variance bound is general and not specific to the Alvarez and Jermann (2005) or Hansen and Scheinkman (2009) decomposition. The bound applies to any SDF that can be decomposed into a permanent and a transitory component.

The σ_{pc}^2 in inequality (6) is computable, given the return time-series of long-term bond, risk-free bond, equity market, and other assets. Our development facilitates a variance bound on the permanent component of

the SDF that can accommodate the return properties of the desired number of assets contained in $\mathbf{R}_{t+1}$.

Inequality (6) bounds the variance of the permanent component of the SDF, which can be a useful object for understanding what time-series assumptions are necessary to achieve consistent risk pricing across a multitude of asset markets. In addition to using the information content of returns across different asset classes, the bound is essentially model-free and can be employed to evaluate the empirical relevance of the permanent component of any SDF, regardless of its distribution. The permanent component of the SDF from any asset pricing model should respect the bound in (6).

The quadratic form for the lower bound in (6) departs fundamentally from the corresponding L-measure-based bound for the three-asset case in Alvarez and Jermann (2005, Eq. (4)):

Lower bound on the permanent component in

$$\text{Alvarez and Jermann is } E(\log(R_{t+1}/R_{t+1,\infty})). \qquad (7)$$

Under their approach, it is the expected return spread that unpins the bound. In contrast, the stipulated bound in (6) combines information from the average returns and the variance-covariance matrix of returns.

Although the Hansen and Jagannathan (1991) bound was not developed to differentiate between the permanent and the transitory components of the SDF, the variance bound on the permanent component σ_{pc}^2 is receptive to an interpretation, as in the Hansen and Jagannathan (1991) bound (see also Cochrane and Hansen, 1992). To appreciate this feature, note that $\mathbf{R}_{t+1}/R_{t+1,\infty} \simeq \mathbf{1}+\log(\mathbf{R}_{t+1})-\log(R_{t+1,\infty})\mathbf{1}$, and suppose that $\mathbf{R}_{t+1}$ consists of a risk-free bond, equity market, and equity portfolios. Then following Hansen and Jagannathan (1991) and Cochrane (2005), the variance bound on the permanent component of SDFs can be interpreted as the maximum Sharpe ratio when the investment opportunity set is composed of the risk-free bond with excess return relative to the long-term bond, the equity market with excess return relative to the long-term bond, and the equity portfolios with excess return relative to the long-term bond.

In a manner akin to the lower bound on the volatility of the SDF in Hansen and Jagannathan (1991), we establish, via Eq. (18), that σ_{pc}^2 corresponds to the volatility of the permanent component of SDFs exhibiting the lowest variance. Thus, there is a key difference between the lower bound on the variance of SDFs and the lower bound on the variance of the permanent component of SDFs.

Koijen, Lustig, and Van Nieuwerburgh (2010) highlight the economic role of $Var[M_{t+1}^P/M_t^P]/Var[M_{t+1}/M_t]$ in affine models. Our formulation leads to a bound on the ratio (proof is in Bakshi and Chabi-Yo, 2011):

$$\frac{Var\left[\dfrac{M_{t+1}^P}{M_t^P}\right]}{Var\left[\dfrac{M_{t+1}}{M_t}\right]} \geq \frac{\sigma_{pc}^2}{\sigma_{pc}^2+1-\mu_m^2}. \qquad (8)$$

The upshot is that the lower bound on the *size* of the permanent component of the SDFs is also analytically

distinct from its L-measure and three-asset based counterpart in Alvarez and Jermann (2005, Eq. (5)).

The relative usefulness of our bounds in empirical applications is the focal point of the exercises in Sections 3.3, 5.1, and 5.6. Our contention in Section 3.3 is also that the bounds appear quantitatively stable to how the return of a long-term bond is proxied.

2.3. Bounds on the transitory component of SDFs

While a central constituent of any SDF is the permanent component, a second constituent is the transitory component, which equals the inverse of the gross return of an infinite-maturity discount bond and governs the behavior of interest rates. Absent a transitory component, the excess returns of bonds are zero, which contradicts empirical evidence (Cochrane and Piazzesi, 2005). To gauge the ability of SDFs to explain aspects of the bond market data, while consistently pricing the remaining set of assets $\mathbf{R}_{t+1}$, as described in Eq. (3), we provide a lower bound on the variance of the transitory component of SDFs.

Proposition 2. *Suppose the relations in Eqs.* (4) *and* (5) *hold. Then the lower bound on the unconditional variance of the transitory component of SDFs* $M_{t+1}/M_t \in \mathbb{S}$ *is*

$$Var\left[\frac{M_{t+1}^T}{M_t^T}\right] \geq \sigma_{tc}^2 \equiv \frac{\left(1-E\left(\dfrac{M_{t+1}^T}{M_t^T}\right)E(R_{t+1,\infty})\right)^2}{Var[R_{t+1,\infty}]}. \qquad (9)$$

Proof. See Appendix A.

The variance of the transitory component of any SDF that consistently prices the long-term bond should be higher than the bound depicted in Eq. (9). There is no analog in Alvarez and Jermann (2005) to our analytical bound on $Var[M_{t+1}^T/M_t^T]$.

We can also characterize the *upper bound* on the size of the transitory component, namely, the counterpart to Alvarez and Jermann (2005, Proposition 3), as:

$$\frac{Var\left[\dfrac{M_{t+1}^T}{M_t^T}\right]}{Var\left[\dfrac{M_{t+1}}{M_t}\right]} \leq \frac{Var\left[\dfrac{1}{R_{t+1,\infty}}\right]}{(\mathbf{1}-\mu_m E(\mathbf{R}_{t+1}))'(Var[\mathbf{R}_{t+1}])^{-1}(\mathbf{1}-\mu_m E(\mathbf{R}_{t+1}))}, \qquad (10)$$

where the denominator of (10) is the lower bound on the variance of the SDFs, as in Hansen and Jagannathan (1991, Eq. (12)).

The quantity on the right-hand side of Eq. (9) is tractable and computable from the returns data. A particular observation is that the bound in (9) is a parabola in $(E(M_{t+1}^T/M_t^T),\sigma_{tc}^2)$ space, and σ_{tc}^2 is positively associated with the square of the Sharpe ratio of the long-term bond. We potentially contribute by using our bound (9) to assess the bond market implications of asset pricing models.

2.4. Bounds on the ratio of the permanent to the transitory components of SDFs

A third feature of SDFs is their ability to link the behavior of the bond market to other markets. In this regard, a construct suitable for understanding cross-market relationships is the variance of $(M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T)$, which captures the importance of the permanent component relative to the transitory component.

Proposition 3. Suppose the relations in Eqs. (4) and (5) hold. Then the lower bound on the unconditional variance of the ratio of the permanent to the transitory components of SDFs is

$$Var\left[\frac{\frac{M_{t+1}^P}{M_t^P}}{\frac{M_{t+1}^T}{M_t^T}}\right] \geq \sigma_{pt}^2 \equiv \left(\mathbf{1} - \mu_{pt}E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right)\right)' \Sigma^{-1}\left(\mathbf{1} - \mu_{pt}E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right)\right),$$

(11)

where $\Sigma \equiv Var[\mathbf{R}_{t+1}/R_{t+1,\infty}^2]$ and $\mu_{pt} \equiv E((M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T))$.

Proof. See Appendix A.

For a given $E((M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T))$, Proposition 3 provides the lower bound on $Var[(M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T)]$, the purpose of which is to assess whether the SDFs can explain joint pricing restrictions across markets. The importance of linking markets has been emphasized by Campbell (1986); Campbell and Ammer (1993), Baele, Bekaert, and Inghelbrecht (2010), Colacito, Engle, and Ghysels (2011), and David and Veronesi (2008). Our bound in Proposition 3 is new, with no counterpart in Alvarez and Jermann (2005).

Further note that $Var[(M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T)]$ can be cast in terms of the mixed moments of the permanent and the transitory components of the SDF. Later we derive an explicit implication regarding a form of dependence between M_{t+1}^P/M_t^P and M_{t+1}^T/M_t^T that can be inputed from the data. In one extreme, a pricing framework devoid of a comovement between M_{t+1}^P/M_t^P and M_{t+1}^T/M_t^T amounts to a flat term structure, a feature falsified by the data.

2.5. Further discussion and relation to Hansen and Jagannathan (1991)

Recapping our results so far, the first type of bounds we propose is on the variance of the permanent component of SDFs. Such a bound is beneficial for characterizing the restrictions imposed by time-series assumptions in asset pricing models. Next, we derive a lower bound on the variance of the transitory component of SDFs, which is a potentially useful tool for disentangling which time-series assumptions on the consumption growth process can more aptly capture observed features of the bond market. Finally, we provide a lower bound on the variance of the relative contribution of the permanent to the transitory component of SDFs, which can establish the link between bond pricing and the pricing of other assets. Our bounds, thus, provide a set of dimensions along which one could appraise asset pricing models.

When a diagnostic test, for instance, the Hansen and Jagannathan (1991) variance bound, rejects an asset pricing model, it often fails to ascribe model failure specifically to the inadequacy of the permanent component of SDFs, the transitory component of SDFs, or to a combination of both. Thus, our framework offers the pertinent measures to investigate which dimension of the SDF can be modified, when the goal is to capture return variation either in a single market or across markets. We elaborate on this issue by analytically solving an eigenfunction problem for asset pricing models in the class of long-run risk, external habit persistence, and rare disasters, and then invoking our Propositions 1, 2, and 3.

Each of the bounds derived in Propositions 1, 2, and 3 are unconditional bounds. In Appendix B, we scale the returns by conditioning variables and propose bounds that incorporate conditioning information.

3. Comparison with Alvarez and Jermann (2005) bounds

Germane to our bounds are two central questions: In what way are these bounds distinct from the corresponding bounds in Propositions 2 and 3 in Alvarez and Jermann (2005)? How useful are our proposed bounds, and how do they compare, say, along the empirical dimension, with Alvarez and Jermann (2005)?

To address these questions, we first note that Alvarez and Jermann define the L-measure (entropy) of a random variable u as

$$L[u] \equiv f[E(u)] - E(f[u]), \quad \text{with } f[u] = \log(u). \quad (12)$$

Using $L[u]$ as a measure of volatility, Alvarez and Jermann develop their bounds in terms of the L-measure. Under their characterizations, a one-to-one correspondence exists between the L-measure and the variance-measure of $\log(u)$, when u is distributed lognormally, as in $L[u] = \frac{1}{2}Var[\log(u)]$. Still, discrepancies between the two measures can get magnified under departures from lognormality, for example, Kelly (2009); Bekaert and Engstrom (2010), and Backus, Chernov, and Martin (2011), as we show, where neither the SDF nor the permanent component are lognormally distributed, as captured by $|L[u] - \frac{1}{2}Var[\log(u)]| > 0$.

The distinction between our treatment and in that of Alvarez and Jermann (2005) is hereby studied from three perspectives. First, we illustrate some differences in M_{t+1}^P/M_t^P and M_{t+1}^T/M_t^T across the L-measure and the variance-measure in an example economy. Second, we highlight some conceptual differences in the characterization of bounds, focusing on the three-asset setting of Alvarez and Jermann, i.e., we rely on the return properties of the long-term bond, the risk-free bond, and the equity market. Third, we provide a comparison of bounds in the data dimension while maintaining the three-asset setting.

3.1. L-measure versus the variance-measure in an example economy

Suppose the log of the pricing kernel evolves according to an AR(1) process,

$$\log(M_{t+1}) = \log(\beta) + \varsigma\log(M_t) + \varepsilon_{t+1}, \quad \text{where} \quad \varepsilon_{t+1} \sim \mathcal{N}(0, \sigma_\varepsilon^2), \quad (13)$$

with $|\varsigma| < 1$ as in Alvarez and Jermann (2005, page 1997). In this setting, the log excess bond return at maturity k is

$$\log\left(\frac{R_{t+1,k}}{R_{t+1,1}}\right) = \frac{\sigma_\varepsilon^2}{2}(1-\varsigma^{2(k-1)}).$$

It can be shown that the permanent component of the SDF is

$$\frac{M_{t+1}^P}{M_t^P} = \exp\left(-\varsigma^2\frac{\sigma_\varepsilon^2}{2} + \varsigma\varepsilon_{t+1}\right), \quad \text{and hence,}$$

$$M_t^P \text{ is a martingale} \quad E\left(\frac{M_{t+1}^P}{M_t^P}\right) = 1. \tag{14}$$

The transitory component is

$$\frac{M_{t+1}^T}{M_t^T} = \exp\left(-\frac{\sigma_\varepsilon^2}{2}(2-\varsigma^{2(k-1)}) - \log(\beta) - (\varsigma-1)\log(M_{t+1})\right),$$
$$\text{for a large } k. \tag{15}$$

Now,

$$L\left[\frac{M_{t+1}^P}{M_t^P}\right] = \varsigma^2\frac{\sigma_\varepsilon^2}{2}, \quad \text{and} \quad L\left[\frac{M_{t+1}^T}{M_t^T}\right] = (1-\varsigma)^2\frac{\sigma_\varepsilon^2}{2}\left(\frac{1-\varsigma^{2(t+1)}}{1-\varsigma^2}\right). \tag{16}$$

Further, under our treatment, the variance of the permanent component is $Var[M_{t+1}^P/M_t^P] = \exp(\varsigma^2\sigma_\varepsilon^2) - 1$, and

$$Var\left[\frac{M_{t+1}^T}{M_t^T}\right] = \left(\exp\left((\varsigma-1)^2\frac{\sigma_\varepsilon^2}{2}\left(\frac{1-\varsigma^{2(t+1)}}{1-\varsigma^2}\right)\right) - 1\right)$$
$$\times \exp\left(-\sigma_\varepsilon^2(2-\varsigma^{2(k-1)}) - 2\varsigma^{(t+1)}\log(\beta)\right.$$
$$\left. + (1-\varsigma)^2\frac{\sigma_\varepsilon^2}{2}\left(\frac{1-\varsigma^{2(t+1)}}{1-\varsigma^2}\right)\right), \tag{17}$$

and they do not coincide with the L-theoretic counterparts in (16). The basic message, namely, that there are intrinsic differences between the two dispersion measures, also obtains under conditioning information.

3.2. Under what situations, the framework based on the variance-measure may be preferable?

As can be inferred from the properties of the L-measure (see Appendix A in Alvarez and Jermann, 2005), the bounds they derive are designed for the situation in which the SDFs correctly price *at most* three assets: the long-term discount bond, the risk-free bond, and the equity portfolio, with no obvious way to generalize to the dimension of asset space beyond three.

We offer bounds that are derived under the condition that the permanent and the transitory components of the SDFs correctly price a finite number of assets. Our evidence in Section 5.1 puts on a firmer footing the notion that the variance bound on the permanent component is also considerably sharper. The novelty of our bounds is that they exploit the information in both the average returns and the variance-covariance matrix of asset returns.

Even in the setting of the long-term discount bond, the risk-free bond, and the equity portfolio, some conceptual differences between the two treatments can be highlighted. Among a set of M_{t+1}/M_t^P that correctly price asset returns, we denote by M_{t+1}^{*P}/M_t^{*P} the permanent component of SDFs with the lowest variance. It is

$$\frac{M_{t+1}^{*P}}{M_t^{*P}} = 1 + \begin{pmatrix} 1 - E\left(\frac{R_{t+1,1}}{R_{t+1,\infty}}\right) \\ 1 - E\left(\frac{R_{t+1}}{R_{t+1,\infty}}\right) \end{pmatrix}' \left(Var\left[\begin{pmatrix} \frac{R_{t+1,1}}{R_{t+1,\infty}} \\ \frac{R_{t+1}}{R_{t+1,\infty}} \end{pmatrix}\right]\right)^{-1}$$
$$\times \begin{pmatrix} \frac{R_{t+1,1}}{R_{t+1,\infty}} - E\left(\frac{R_{t+1,1}}{R_{t+1,\infty}}\right) \\ \frac{R_{t+1}}{R_{t+1,\infty}} - E\left(\frac{R_{t+1}}{R_{t+1,\infty}}\right) \end{pmatrix}. \tag{18}$$

Hence, in the three-asset setting, Eq. (18) can be viewed as the solution to the problem:

$$\min_{M_{t+1}^P/M_t^P} Var\left[\frac{M_{t+1}^P}{M_t^P}\right] \quad \text{subject to } E\left(\frac{M_{t+1}^P}{M_t^P}\frac{R_{t+1,1}}{R_{t+1,\infty}}\right) = 1,$$
$$E\left(\frac{M_{t+1}^P}{M_t^P}\frac{R_{t+1}}{R_{t+1,\infty}}\right) = 1, \text{and } E\left(\frac{M_{t+1}^P}{M_t^P}\right) = 1. \tag{19}$$

In general, the variance of M_{t+1}^{*P}/M_t^{*P} equals the lower bound on the variance of the permanent component of SDFs. Therefore, our analysis makes it explicit that, regardless of the probability distribution of M_{t+1}^P/M_t^P, our results pertain to bounds on variance. The message worth conveying is that the variance of the permanent (and transitory) component from an asset pricing model is denominated in the same units of riskiness as the bounds recovered from the data, a trait that our framework shares also with the studies of Hansen and Jagannathan (1991); Bernardo and Ledoit (2000), and Cochrane and Saa-Requejo (2000).

Exhibiting a specific distributional property, the Alvarez and Jermann (2005) lower bound is the L-measure of the permanent component: $E(\log(R_{t+1}/R_{t+1,\infty}))$. Yet it is not possible to find an analytical expression for the permanent component of SDFs, namely, $\tilde{M}_{t+1}^P/\tilde{M}_t^P$ such as $L[\tilde{M}_{t+1}^P/\tilde{M}_t^P] = E(\log(R_{t+1}/R_{t+1,\infty}))$. To establish this argument, we use the definition of the L-measure, whereby

$$L\left[\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P}\right] = \log(1) - E\left(\log\left(\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P}\right)\right),$$
$$\left(\text{since } E\left(\frac{\tilde{M}_{t+1}^P}{\tilde{M}_t^P}\right) = 1\right). \tag{20}$$

Therefore, we can at most deduce that $E(\log(\tilde{M}_{t+1}^P/\tilde{M}_t^P)) = E(\log(R_{t+1,\infty}/R_{t+1}))$, and it may not be possible to recover $\tilde{M}_{t+1}^P/\tilde{M}_t^P$ analytically in terms of asset returns. It seems that the lower bound on the L-measure of the permanent component of SDFs does not satisfy the definition of the L-measure.

3.3. Lessons from a comparison with Alvarez and Jermann (2005) bounds in the data dimension

Still, some empirical questions remain with respect to observed data in the financial markets: What is gained by generalizing the L-measure-based setup in Alvarez and Jermann (2005)? In what sense do our proposed bounds quantitatively differ from Alvarez and Jermann (2005)? How sensitive are our variance bounds when one

surrogates $R_{t+1,\infty}$ by the return of a bond with a reasonably long maturity?

To facilitate these objectives, here we follow Alvarez and Jermann (2005) in the choice of three assets, the monthly sample period of 1946:12 to 1999:12 (637 observations), as well as expressing the point estimates from the L-measure and, hence, from the variance-measure in *annualized* terms in Table 1. Specifically, we rely on data (http://www.econometricsociety.org/suppmat.asp?id=61&vid=73&iid=6&aid=643) on the return of a long-term bond, the return of a risk-free bond, and the return of a single equity portfolio (optimal growth portfolio based on ten CRSP size-decile portfolios and the equity market).

The maturity of a long-term bond is guided by data considerations and allowed to take a value of 20, 25, or 29 years. Also reported in Table 1 are the 90% confidence intervals (in square brackets), which are based on 50,000 random samples of size 637 from the data and a block bootstrap.

There are three lessons that can be drawn. First, our exercise suggests that our bounds are broadly different from Alvarez and Jermannn. Second, when the maturity of the bond is altered from 20 to 29 years, the bound on $L[M_{t+1}^P/M_t^P]$ varies little, whereas the bound σ_{pc}^2 on $Var[M_{t+1}^P/M_t^P]$ varies somewhat from 0.799 to 0.933.

This differential sensitivity can be ascribed to the fact that σ_{pc}^2 (as in Eq. (6)) is determined by both average returns and the variance-covariance matrix of asset returns. Finally, the discrepancy between the bounds on $L[M_{t+1}^P/M_t^P]$ and $\frac{1}{2}Var[M_{t+1}^P/M_t^P]$ is large in the data, implying that the permanent component *is far from* being distributed normally in logs. This finding suggests that higher-moments of M_{t+1}^P/M_t^P may be relevant to asset pricing.

4. Eigenfunction problem and the transitory and permanent components of SDFs in asset pricing models

This section contributes by deriving an analytical solution to the eigenfunction problem of Hansen and Scheinkman (2009) to determine the transitory and permanent components of the SDF. Featured is a strand of asset pricing models that have drawn considerable support in their ability to depict stylized properties of aggregate equity market returns and risk-free bond returns.

Specifically, we focus on models in (a) the long-run risk class (Bansal and Yaron, 2004), (b) the external habit persistence class (Campbell and Cochrane, 1999), and (c) the rare consumption disasters class (Rietz, 1988; and Barro, 2006). Within each class we adopt a generalization of the SDF that invokes departure from

Table 1

Comparison of our variance bounds with the L-measure-based bounds in Alvarez and Jermann (2005), when $R_{t+1,\infty}$ is surrogated by the return of a bond with maturity of 20, 25, or 29 years.

Here we follow Alvarez and Jermann (2005), both in the choice of *three* assets and the sample period. The bounds hinge on the return properties of long-term discount bond, risk-free bond, and single equity portfolio (optimal growth portfolio based on ten CRSP size-decile portfolios and the equity market). We proxy $R_{t+1,\infty}$ by bond returns with maturity ranging from 20 years to 29 years. The monthly data used in the construction of the bounds is from 1946:12 to 1999:12, and taken from http://www.econometricsociety.org/suppmat.asp?id=61&vid=73&iid=6&aid=643. Reported are the estimates (*annualized, in bold*) of the bounds from the data, along with the 90% confidence intervals (in square brackets). To obtain the confidence intervals, we create 50,000 random samples of size 637 (number of observations) from the data, where the sampling is based on 12 blocks. The displayed lower bound on $Var[M_{t+1}^P/M_t^P]$ and $Var[M_{t+1}^P/M_t^P]/Var[M_{t+1}/M_t]$ are, respectively, based on Eqs. (6) and (8), while the upper bound on $Var[M_{t+1}^T/M_t^T]/Var[M_{t+1}/M_t]$ is based on Eq. (10). The expressions for the bounds on the L-measure are presented in Alvarez and Jermann (2005, Propositions 2 and 3).

	$R_{t+1,\infty}$ is proxied by the return of a bond with *maturity*:		
	20 years	25 years	29 years
Lower bound on $Var\left[\frac{M_{t+1}^P}{M_t^P}\right]$	0.799	0.909	0.933
	[0.646, 0.942]	[0.710, 1.089]	[0.675, 1.151]
Lower bound on $L\left[\frac{M_{t+1}^P}{M_t^P}\right]$	0.201	0.205	0.217
	[0.178, 0.226]	[0.179, 0.232]	[0.184, 0.251]
$L\left[\frac{M_{t+1}^P}{M_t^P}\right] - \frac{1}{2}Var\left[\frac{M_{t+1}^P}{M_t^P}\right]$ (Difference in bounds)	-0.198	-0.249	-0.249
	$[-0.274, -0.118]$	$[-0.343, -0.145]$	$[-0.364, -0.116]$
Lower bound on $Var\left[\frac{M_{t+1}^P}{M_t^P}\right]/Var\left[\frac{M_{t+1}}{M_t}\right]$	0.884	0.896	0.898
	[0.866, 0.904]	[0.878, 0.917]	[0.876, 0.923]
Lower bound on $L\left[\frac{M_{t+1}^P}{M_t^P}\right]/L\left[\frac{M_{t+1}}{M_t}\right]$	1.018	1.039	1.098
	[0.964, 1.075]	[0.964, 1.117]	[0.977, 1.222]
Upper bound on $Var\left[\frac{M_{t+1}^T}{M_t^T}\right]/Var\left[\frac{M_{t+1}}{M_t}\right]$	0.039	0.055	0.102
	[0.029, 0.047]	[0.041, 0.067]	[0.075, 0.125]
Upper bound on $L\left[\frac{M_{t+1}^T}{M_t^T}\right]/L\left[\frac{M_{t+1}}{M_t}\right]$	0.079	0.111	0.199
	[0.066, 0.090]	[0.092, 0.129]	[0.165, 0.232]

log-normality, the purpose of which is to enrich the setting for researching the broader relevance of bounds in Propositions 1, 2, and 3. The asset pricing models delineated next, along with those of Bansal and Yaron (2004) and Campbell and Cochrane (1999), are at the core of the empirical investigation.

4.1. Solution to the eigenfunction problem for a model incorporating long-run risk

To determine the transitory and permanent components of the SDF through the eigenfunction problem, consider the modification of the long-run risk model proposed in Kelly (2009). The distinguishing attribute is that the model incorporates heavy-tailed shocks to the evolution of (log) nondurable consumption growth g_{t+1}, which are governed by a tail risk state variable Λ_t.

$$
\begin{aligned}
g_{t+1} &= \mu + x_t + \sigma_g \sigma_t z_{g,t+1} + \sqrt{\Lambda_t} W_{g,t+1}, \\
x_{t+1} &= \rho_x x_t + \sigma_x \sigma_t z_{x,t+1},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\sigma_{t+1}^2 &= \overline{\sigma}^2 + \rho_\sigma(\sigma_t^2 - \overline{\sigma}^2) + \sigma_\sigma z_{\sigma,t+1}, \\
\Lambda_{t+1} &= \overline{\Lambda} + \rho_\Lambda(\Lambda_t - \overline{\Lambda}) + \sigma_\Lambda z_{\Lambda,t+1},
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
z_{g,t+1}, z_{x,t+1}, z_{\sigma,t+1}, z_{\Lambda,t+1} &\sim \text{i.i.d } \mathcal{N}(0,1), \\
W_{g,t+1} &\sim \text{Laplace}(0,1).
\end{aligned}
\tag{23}
$$

Following Bansal and Yaron (2004), x_t is a persistently varying component of the expected consumption growth rate, and σ_t^2 is the conditional variance of consumption growth with unconditional mean $\overline{\sigma}^2$.

The z shocks are standard normal and independent. In addition to gaussian shocks, the consumption growth depends on non-gaussian shocks W_g, where the W_g shocks are Laplace-distributed variables with mean zero and variance 2, and independent. W_g shocks are independent of z shocks. The model maintains the tradition of Epstein and Zin (1991) recursive utility.

Proposition 4. The transitory and permanent components of the SDF in the model of Kelly (2009) are

$$
\frac{M_{t+1}^T}{M_t^T} = \upsilon \exp(-c_1(x_{t+1} - x_t) - c_2(\sigma_{t+1}^2 - \sigma_t^2) - c_3(\Lambda_{t+1} - \Lambda_t)),
$$

$$
\text{and } \frac{M_{t+1}^P}{M_t^P} = \frac{M_{t+1}}{M_t} \bigg/ \frac{M_{t+1}^T}{M_t^T},
\tag{24}
$$

where υ is defined in (C.13), and the coefficients c_1, c_2, and c_3 are defined in (C.14). The expression for M_{t+1}/M_t is presented in (C.1) of Appendix C.

Proof. See Appendix C.

Eq. (24) is obtained by solving the eigenfunction problem of Hansen and Scheinkman (2009, Corollary 6.1):

$$
E_t\left(\frac{M_{t+1}}{M_t} \frac{1}{M_{t+1}^e}\right) = \frac{\upsilon}{M_t^e},
\tag{25}
$$

where the parameter υ is the dominant eigenvalue. The conjectured M_{t+1}^e that satisfies (25) determines $M_t^T = \upsilon^t M_t^e$ and $M_t^P = M_t / M_t^T$.

While the transitory component of the SDF is lognormally distributed, the permanent component of the SDF,

and the SDF itself, are not lognormally distributed. The non-gaussian shocks W_g are meant to amplify the tails of the permanent component of the SDF and the SDF. Eq. (24) makes it explicit how the sources of the variability in the transitory and permanent components of the SDF can be traced back to (i) the specification of the preferences, and (ii) the dynamics of the fundamentals.

4.2. Solution to the eigenfunction problem for a model incorporating external habit persistence

Bekaert and Engstrom (2010) propose a variant of the Campbell and Cochrane (1999) model where (i) the dynamics of consumption growth g_t consist of two fat-tailed skewed distributions, and (ii) the SDF is

$$
\frac{M_{t+1}}{M_t} = \beta \exp(-\gamma g_{t+1} + \gamma(q_{t+1} - q_t)),
\tag{26}
$$

where β is the time preference parameter, γ is the curvature parameter, and $q_t \equiv \log(C_t/(C_t - H_t)) = \log(1/S_t)$. For external habit H_t and consumption C_t, the variable S_t is the surplus consumption ratio and, hence, q_t represents the log of the *inverse* surplus consumption ratio (see also Santos and Veronesi, 2010; and Borovicka, Hansen, Hendricks, and Scheinkman, 2011). Uncertainty in this economy is described by

$$
\begin{aligned}
g_{t+1} &= \overline{g} + x_t + \sigma_{gp}\omega_{p,t+1} - \sigma_{gn}\omega_{n,t+1}, \\
x_t &= \rho_x x_{t-1} + \sigma_{xp}\omega_{p,t} + \sigma_{xn}\omega_{n,t},
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
q_{t+1} &= \mu_q + \rho_q q_t + \sigma_{qp}\omega_{p,t+1} + \sigma_{qn}\omega_{n,t+1}, \\
\omega_{p,t+1} &= \text{ge}_{t+1} - p_t \text{ and } \omega_{n,t+1} = \text{be}_{t+1} - n_t,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
p_t &= \overline{p} + \rho_p(p_{t-1} - \overline{p}) + \sigma_{pp}\omega_{p,t}, \\
n_t &= \overline{n} + \rho_n(n_{t-1} - \overline{n}) + \sigma_{nn}\omega_{n,t},
\end{aligned}
\tag{29}
$$

$$
\text{ge}_{t+1} \sim \text{Gamma}(p_t, 1), \quad \text{be}_{t+1} \sim \text{Gamma}(n_t, 1),
\tag{30}
$$

where p_t and n_t are the conditional mean of the good environment and bad environment shocks denoted by ge_{t+1} and be_{t+1}, respectively. The distinguishing attribute of the model is that it incorporates elements of external habit persistence together with long-run consumption risk.

Proposition 5. The transitory and permanent components of the SDF in Bekaert and Engstrom (2010) are

$$
\frac{M_{t+1}^T}{M_t^T} = \upsilon \exp(\gamma(q_{t+1} - q_t) - c_2(x_{t+1} - x_t) - c_3(p_{t+1} - p_t)
$$

$$
-c_4(n_{t+1} - n_t)), \quad \text{and} \quad \frac{M_{t+1}^P}{M_t^P} = \frac{M_{t+1}}{M_t} \bigg/ \frac{M_{t+1}^T}{M_t^T},
\tag{31}
$$

where υ is defined in (D.2), the coefficients c_2 through c_4 are defined in (D.3), and M_{t+1}/M_t is as in (26).

Proof. See Appendix D.

Besides generating the statistically observed risk-free return, the equity premium, and moments of consumption growth, among the noteworthy model features are its ability to generate time-variation in risk premiums and consistency with risk-neutralized equity return moments.

One implication of the solution (31) is that the model embeds a transitory component of the SDF which comoves with changes in q_t. In contrast, the permanent component is detached from variations in q_t.

4.3. Solution to the eigenfunction problem for a model incorporating rare disasters

We consider a version of the asset pricing model of Rietz (1988); Barro (2006), and Gabaix (2009). Uncertainty about consumption growth is modeled following Backus, Chernov, and Martin (2011) as

$$\log(g_{t+1}) = w_{t+1} + z_{t+1}, \quad \text{with } w_{t+1} \sim \text{i.i.d. } \mathcal{N}(\mu, \sigma^2),$$
$$z_{t+1}|J_{t+1} \sim \text{i.i.d. } \mathcal{N}(\theta J_{t+1}, \delta^2 J_{t+1}), \text{ and} \tag{32}$$

J_{t+1} is i.i.d. Poisson random variable with density $\dfrac{\omega^j}{j!}e^{-\omega}$

for $j \in \{0, 1, 2, \ldots\}$, and mean ω. $\tag{33}$

The distinguishing attribute of this model is that z_{t+1} produces sporadic crashes in consumption growth and serves as a device to produce a fat-tailed distribution of consumption growth. In the model, (w_{t+1}, z_{t+1}) are mutually independent over time, and the SDF is of the form (under time-separable power utility):

$$\frac{M_{t+1}}{M_t} = \beta g_{t+1}^{-\gamma} = \exp(\log(\beta) - \gamma w_{t+1} - \gamma z_{t+1}), \tag{34}$$

where β is time preference parameter, and γ is the coefficient of relative risk aversion.

Proposition 6. The transitory and permanent components of the SDF in the rare disasters model are

$$\frac{M_{t+1}^T}{M_t^T} = \frac{1}{R_{t+1,\infty}}, \quad \text{and}$$
$$\frac{M_{t+1}^P}{M_t^P} = R_{t+1,\infty}\exp(\log(\beta) - \gamma w_{t+1} - \gamma z_{t+1}), \tag{35}$$

where

$$R_{t+1,\infty}^{-1} = \exp\left(\log(\beta) - \gamma\mu + \frac{1}{2}\gamma^2\sigma^2\right)$$
$$\times \sum_{j=0}^{\infty} \frac{e^{-\omega}\omega^j}{j!}\exp\left(-\gamma\theta j + \frac{1}{2}\gamma^2\delta^2 j\right)$$

is a constant.

Proof. See Appendix E in Bakshi and Chabi-Yo (2011). □

Within the setting of (32) and (33), the model inherits the property that M_{t+1}/M_t and M_{t+1}^P/M_t^P are not lognormally distributed. It can be shown that $E(M_{t+1}^P/M_t^P)^\ell = (\beta R_{t+1,\infty})^\ell \exp(\omega(e^{0.5\gamma^2\delta^2\ell^2 - \gamma\theta\ell} - 1) + 0.5\gamma^2\sigma^2\ell^2 - \gamma\mu\ell)$, for $\ell = 2, 3, \ldots$, which furnishes the moments of the permanent component, while maintaining $E(M_{t+1}^P) = M_t^P$.

5. Empirical application to asset pricing models

To lay the groundwork for the empirical examination, the analysis of this section starts by highlighting the tightness of our lower bound on the variance of the permanent component in the context of a finite number

of assets. Then we elaborate on the performance of asset pricing models under our metrics of evaluation, including the lower bound restrictions on the permanent and transitory components of SDFs.

5.1. Description of the set of asset returns and the tightness of the variance bound

Stepping outside of the three-asset setting in Table 1, recall that $\mathbf{R}_{t+1}$ is the return vector that is correctly priced by the SDF along with $R_{t+1,\infty}$ (see Eq. (3)). Whereas our variance bound on the permanent component σ_{pc}^2 (see Eq. (6)) exhibits dependence on $E(\mathbf{R}_{t+1}/R_{t+1,\infty})$ and $Var[\mathbf{R}_{t+1}/R_{t+1,\infty}]$, tractable expressions are not yet available for L-measure-based bounds when there are more than three assets.

Two questions are pertinent to our development and to our empirical comparison of asset pricing models: (1) Does the variance bound σ_{pc}^2 get incrementally sharper when the SDF is required to correctly price more assets? (2) How does the tightness of the new bound fare relative to Alvarez and Jermann (2005)?

To answer these questions, we consider a set of monthly equity and bond returns over the period 1932:01 to 2010:12 (our choice of the start date circumvents missing observations). The source of risk-free bond and equity returns is the data library of Kenneth French, while the source of intermediate and long-term government bond returns is Morningstar (Ibbotson). Real returns are computed by deflating nominal returns by the Consumer Price Index inflation. For our illustration, we consider four different $\mathbf{R}_{t+1}$:

(i) SET A: Risk-free bond, equity market, intermediate government bond, and 25 Fama-French equity portfolios sorted by size and book-to-market;
(ii) SET B: Risk-free bond, equity market, intermediate government bond, ten size-sorted, and six size and book-to-market, sorted equity portfolios;
(iii) SET C: Risk-free bond, equity market, intermediate government bond, and six size and book-to-market, sorted equity portfolios; and,
(iv) SET D: Risk-free bond, equity market, and intermediate government bond.

The lower bound on the permanent component, σ_{pc}^2, reported below, imparts two conclusions:

	Set A	Set B	Set C	Set D	AJ
Lower bound σ_{pc}^2 (Eq. (6))	0.1254	0.0835	0.0674	0.0296	0.0041

First, σ_{pc}^2 declines from SET A to SET D, suggesting that expanding the number of assets in $\mathbf{R}_{t+1}$ leads to a bound that is intrinsically tighter. Second, the estimates of σ_{pc}^2 generated from SET A through D are sharper relative to Alvarez and Jermann (2005, Eq. (4)), which is represented by the entry marked AJ. The gist of this exercise is that $\sigma_{pc}^2/2$ based on the return properties of SET A (SET D) is about 15 (four) times sharper than the L-measure-based lower bound on the permanent component, thereby substantiating its incremental value in asset pricing applications.

5.2. A setting for model evaluation, granted that each model calibrates to some data attributes

Turning to the themes of our study, we exploit the exactly solved eigenfunction problems in Propositions 4–6, to provide the building blocks for our empirical study in a few ways:

- Our interest is in assessing the ability of a model to produce realistic permanent and transitory components of the SDF and whether their variance respects the proposed lower bounds. We rely on a statistic generated from a simulation procedure and the associated p-value;
- Analytical solutions to the eigenfunction problem facilitate a quantitative implication regarding the return of a long-term bond, whereby

$$E(r_{t+1,\infty}) = E(R_{t+1,\infty} - 1) = E\left(\frac{M_t^T}{M_{t+1}^T} - 1\right),$$

$$Var[r_{t+1,\infty}] = E\left(\frac{M_t^T}{M_{t+1}^T} - E\left(\frac{M_t^T}{M_{t+1}^T}\right)\right)^2. \qquad (36)$$

The return moments of a long-term bond implicit in an asset pricing model are, thus, computable, and could be benchmarked to those from a suitable proxy to ascertain their plausibility. Campbell and Viceira (2001), among others, provide an impetus to develop asset pricing models that also generate reasonable return behavior for the long-term bond.

The bounds yardstick, when combined with (36), could serve as a differentiating diagnostic when competing models (i) each calibrate closely to the mean and standard deviation of consumption growth, and (ii) offer conformity with the historical real return of risk-free bond and the real return of equity market.

Additionally, and equally relevant, we extract the implication of each model for the comovement between the permanent and transitory components of the SDF, in a manner to be described shortly via Eqs. (37) and (38), and examine its merit relative to the one imputed from the data.

Primitive parameters are chosen consistently according to the models of Kelly (2009), Bekaert and Engstrom (2010), and Backus, Chernov, and Martin (2011), as displayed in Tables Appendix-I through Appendix-III of Bakshi and Chabi-Yo (2011), respectively, together with those of Bansal and Yaron (2004) and Campbell and Cochrane (1999). Even though we refrain from presenting the full-blown solution to the eigenfunction problem for the last two models to save space, our objective in implementing all five models is to offer a unifying picture of how each asset pricing model performs under our yardsticks of evaluation.

While the conditional variances are amenable to closed-form characterization, the unconditional variances are tractable only via simulations, except for the rare disaster model given i.i.d. uncertainties. Accounting for this feature, each model is simulated using the dynamics of consumption growth and other state variables, for instance, as in (21)–(23), for the long-run risk model of Kelly (2009), over a single simulation run of 360,000 months (30,000 years). Then we build the time-series of model-specific $\{M_{t+1}^P / M_t^P\}$ and $\{M_{t+1}^T / M_t^T\}$, according to the solution of the eigenfunction problem, and we calculate the unconditional moments. Using a single simulation run to infer the population values for the entities of interest is consistent with, among others, the approach of Campbell and Cochrane (1999) and Beeler and Campbell (2009).

5.3. Models could appeal to utility specifications and dynamics of fundamentals that magnify the permanent component of SDFs

Our thrust is to examine whether models produce sensible dynamics for the permanent component of the SDF. Such an analysis can enable insights into how economic fundamentals are linked to SDFs, and how the performance of an asset pricing model could be improved by altering the properties of the permanent component of SDFs.

At the outset, we report, for each model, the variance of the permanent component of the SDF, $Var[M_{t+1}^P / M_t^P]$, in Panel A of Table 2. Reported in the final column is the lower bound σ_{pc}^2 calculated based on SET A, along with the 90% confidence intervals, shown in square brackets, from a block bootstrap, when sampling is done with 15 blocks. It is helpful to think in terms of σ_{pc}^2 from SET A, since this set corresponds to a universe of equity portfolios whose return properties are the subject of much scrutiny in the empirical asset pricing research (e.g., Malloy, Moskowitz, and Vissing-Jorgensen, 2009).

Our implementations reveal that the monthly $Var[M_{t+1}^P / M_t^P]$ implied by the models of Kelly (2009) and Bansal and Yaron (2004) is 0.0374 and 0.0342, respectively, while that of the models of Bekaert and Engstrom (2010) and Campbell and Cochrane (1999) is 0.0280 and 0.0234, respectively. The most pronounced value of 0.0580 is obtained under the model of Backus, Chernov, and Martin (2011).

Going further, we formulate the restriction: $\sigma_{pc}^2 - Var[M_{t+1}^P / M_t^P] \leq 0$ for a candidate asset pricing model, which allows one to elaborate on whether a model respects the lower bound [beyond eye-balling estimates; see also Cecchetti, Lam, and Mark, 1994]. Then inference regarding this restriction can be drawn via repeated simulations. For this purpose, we rely on a finite-sample simulation of 948 months (1932:01 to 2010:12), and we choose 200,000 replications. The proportion of the replications satisfying $\sigma_{pc}^2 - Var[M_{t+1}^P / M_t^P] \leq 0$ can be interpreted as a p-value for a one-sided test. This p-value is shown in curly brackets in Table 2, and a low p-value indicates rejection. Pertinent to this exercise, our evidence reveals that the variance of the permanent component of the SDF from each model fails to meet the lower bound restriction of 0.1254 per month. Importantly, the reported p-values provide some support for the contention that the model-based variances of the permanent component are reliably lower than σ_{pc}^2. A likewise conclusion emerges when σ_{pc}^2 is computed from SET B and SET C

Table 2
Assessing the restriction on the variance of the permanent component of SDFs from asset pricing models.

We compute the variance of the permanent component of the SDF, $Var[M_{t+1}^P/M_t^P]$, via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculations are based on model parameters tabulated in Bakshi and Chabi-Yo (2011, Tables Appendix-I through Appendix-III), and the reported values are the averages from a single simulation run of 360,000 months. The reported lower bound σ_{pc}^2 on the permanent component (see Eq. (6) of Proposition 1) is based on the return properties of $\mathbf{R}_{t+1}$, corresponding to SET A, and the long-term bond. We keep the maturity of the long-term bond to be 20 years, as also in Alvarez and Jermann (2005, p. 1993). The monthly data used in the construction of σ_{pc}^2 is from 1932:01 to 2010:12 (948 observations), with the 90% confidence intervals in square brackets. To compute the confidence intervals, we create 50,000 random samples of size 948 from the data, where the sampling in the block bootstrap is based on 15 blocks. Reported below the estimates of $Var[M_{t+1}^P/M_t^P]$ are the p-values, shown in curly brackets, which represent the proportion of replications for which model-based $Var[M_{t+1}^P/M_t^P]$ exceeds σ_{pc}^2 in 200,000 replications of a finite sample simulation over 948 months. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. Reported annualized mean, standard deviation, and first-order autocorrelation of consumption growth in Panel B are obtained by following the convention of aggregating monthly consumption series to annual. Shown in Panel C are the average real return of the risk-free bond and the average return of equity market, all based on a single simulation run. The 90% bootstrap confidence intervals on the return of risk-free bond and equity are [0.0013,0.0048] and [0.0527,0.1189], respectively.

	Long-run risk		External habit		Rare	Lower
	Kelly	Bansal-Yaron	Bekaert-Engstrom	Campbell-Cochrane	disasters	bound, σ_{pc}^2 (based on SET A)
Panel A: Permanent component of the SDF, monthly						
$Var[M_{t+1}^P/M_t^P]$	0.0374	0.0342	0.0280	0.0234	0.0580	0.1254
	{0.000}	{0.000}	{0.001}	{0.000}	{0.099}	[0.045, 0.131]
						Data
Panel B: Consumption growth, annualized						
Mean	0.0245	0.0189	0.0380	0.0189	0.0205	0.0341
Standard deviation	0.0494	0.0298	0.0175	0.0150	0.0298	0.0285
First-order autocorrelation	0.248	0.517	0.243	0.000	0.000	0.227
Panel C: Real return of a risk-free bond and equity, average (annualized)						
Real return of risk-free bond, $R_{t+1,1}-1$	0.0240	0.0252	0.0040	0.0094	0.0200	0.0017
Real return of equity, $R_{t+1}-1$	0.0603	0.0802	0.0583	0.0724	0.0593	0.0864

(not reported). For example, the highest p-value of 0.099 (0.164) corresponds to the model with rare disasters for SET A (SET C).

While the models differ markedly in their capacity to generate a volatile permanent component, it is noteworthy that each model parametrization reasonably mimics the equity premium and the real risk-free return, while simultaneously calibrating closely to the first-two moments of consumption growth (see Panels B through C of Table 2). Thus, there appears to be a tension, within a model, between matching the sample average of equity returns and risk-free returns, versus generating a minimum volatility of the permanent component stipulated by theory and as inferred from the data. We further expand on this point when discussing the implication of the models for long-term bond returns.

While the backbone of the Kelly (2009) model is to incorporate tails in nondurable consumption growth, it tends to elevate the variance of the permanent component, modestly relative to Bansal and Yaron (2004). Specifically, in the formulation of Kelly (2009), the distributions of log SDF and the log of the permanent component of the SDF are symmetric with fat tails. From the documented results, we infer that long-run risk models could accentuate the variability in the permanent component of SDFs by incorporating more flexible tail properties in the SDF and in the permanent component of the SDF. We recognize nonetheless that there is insufficient evidence favoring the presence of skewness in

nondurable consumption growth. Thus, an avenue to enrich long-run risk models is to incorporate durable consumption in the dynamics of the real economy. In the spirit of our results, Yang (2011) provides evidence that durable consumption growth is left-skewed and exhibits time-varying volatility.[2]

Judging by our results, the approach in Bekaert and Engstrom (2010) does not appear to substantially improve upon the variance of the permanent component relative to Campbell and Cochrane (1999). This finding is somewhat unexpected, given a flexible modeling of consumption growth as well as the term structure of interest rates, in conjunction with a modified process for marginal utility.

What could be a rationale for the highest estimate of the variance of the permanent component of the SDF in an asset pricing model with rare disasters? This outcome deserves two comments. First, the high variability of the permanent component arises from occasional crashes in the consumption growth process, and this success comes at the expense of somewhat unrealistic consumption growth higher-moments. For example, the skewness of

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[2] Recent studies include, among others, Bansal, Dittmar, and Kiku (2009), Constantinides and Ghosh (2008), Croce, Lettau, and Ludvigson (2008), Ferson, Nallareddy, and Xie (2010), Kaltenbrunner and Lochstoer (2010), Koijen, Lustig, Van Nieuwerburgh, and Verdelhan (2010), Lettau and Ludvigson (2004), Piazzesi and Schneider (2006), and Zhou and Zhu (2009).

the annualized log consumption growth in the model with rare disasters is −11.02, with a kurtosis of 145.06, as discussed also in Backus, Chernov, and Martin (2011, Table III). Second, the model with rare disasters admits the most right-skewed and fat-tailed distribution of the permanent component among all of our models.

In summary, although the list of models under consideration is far from exhaustive, they still embed different utility specifications and specify the long-run and short-run risk in distinct ways (see also Hansen, 2009). Yet, a common thread among the models is their inability to meet the lower bound restriction on the variance of the permanent component of SDFs. The larger lesson being that the SDF of the asset pricing models could be refined to accommodate a permanent component featuring a larger variance. Our metrics of assessment can provide a perspective on how the dynamics of consumption growth and fundamentals could be modified, or preferences could be generalized, to improve the working of asset pricing models.

5.4. Success of models is confounded by their lack of consistency with aspects of the bond market

The methods of this paper allow us to contemplate two additional questions: (i) Which asset pricing model conforms with the lower bound on the transitory component σ_{tc}^2 (Eq. (9) of Proposition 2)?, and (ii) What are the quantitative implications of each model for the behavior of the long-term bond returns?

It bears emphasizing that while the lower bound σ_{pc}^2 is independent of the mean of the permanent component by construction, the lower bound on the transitory component exhibits dependence on the estimate of the mean $E(M_{t+1}^T/M_t^T)$ across models. With this feature in mind, we report σ_{tc}^2, in relation to the estimate of $E(M_{t+1}^T/M_t^T)$, which is then compared to $Var[M_{t+1}^T/M_t^T]$ produced by a

model in Table 3. The crux of our finding is that models generate insufficient $Var[M_{t+1}^T/M_t^T]$ relative to the lower bound σ_{tc}^2. This conclusion is confirmed through the p-values that examine the restriction $\sigma_{tc}^2 - Var[M_{t+1}^T/M_t^T] \leq 0$. We again obtain this p-value for each model by appealing to a finite sample simulation with 200,000 replications. It is seldom that $Var[M_{t+1}^T/M_t^T]$ is greater than the minimum volatility restriction on the transitory component, and the reported p-values are all below 0.01. Our approach essentially identifies a source of the misalignment of asset pricing models with aspects of the bond market data.

Moving to the second question of interest, Table 4 summarizes the model implications of the transitory component for the moments of the long-term bond. The key point to note is that calibrations geared toward replicating the equity return and the risk-free return can miss basic aspects of the long-term bond market. For example, the Bansal and Yaron, 2004, and the Bekaert and Engstrom, 2010 models imply an expected annualized long-term bond return of −14.26% and −0.21%, respectively. Given the real return of a long-term bond averages 2.43%, there appears to be a gap between the prediction of the models and the data.

That the misspecified transitory component is a source of the incongruity of models with bond market data is also revealed through the standard deviation of long-term bond returns. Specifically, when the models fail to generate plausible dynamics of the transitory component, they can introduce a wedge between the volatility of long-term bond returns implied by a model versus the data counterpart. Here, it can be seen that the Bansal and Yaron (Bekaert and Engstrom) model implies an annualized standard deviation of 12.41% (2.36%), which deviates from the data value of 8.87%. While the extant literature has largely focused efforts on rationalizing equity return volatility (see, for instance, Schwert, 1989; Wachter,

Table 3

Assessing the restriction on the variance of the transitory component of SDFs from asset pricing models.

We compute $Var[M_{t+1}^T/M_t^T]$ via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculations are based on model parameters tabulated in Bakshi and Chabi-Yo (2011, Tables Appendix-I through Appendix-III), and the reported values are the averages from a single simulation run of 360,000 months. The reported lower bound σ_{tc}^2 on the transitory component is based on Eq. (9) of Proposition 2. We keep the maturity of the long-term bond to be 20 years, and the real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. The monthly data used in the construction of σ_{tc}^2 are from 1932:01 to 2010:12 (948 observations), with the 90% confidence intervals in square brackets. To compute the confidence intervals, we create 50,000 random samples of size 948 from the data, where the sampling in the block bootstrap is based on 15 blocks. Reported below the estimates of $Var[M_{t+1}^T/M_t^T]$ are the p-values, shown in curly brackets, which represent the proportion of replications for which model-based $Var[M_{t+1}^T/M_t^T]$ exceeds σ_{tc}^2 in 200,000 replications of a finite sample simulation over 948 months. Because the transitory component of the SDF in the model with rare disasters is a constant, its p-value entry is shown as "na."

	Long-run risk		External habit		Rare
	Kelly	Bansal-Yaron	Bekaert-Engstrom	Campbell-Cochrane	disasters
$Var[M_{t+1}^T/M_t^T]$	1.7×10^{-8} {0.000}	1.9×10^{-3} {0.000}	3.5×10^{-5} {0.000}	2.2×10^{-6} {0.000}	0.0000 na
Lower bound, σ_{tc}^2	9.8×10^{-7} [0.0000, 0.0000]	4.3×10^{-1} [0.3404, 0.5026]	7.7×10^{-3} [0.0041, 0.0141]	7.8×10^{-4} [0.0005, 0.0016]	2.0×10^{-4} [0.0000, 0.0004]
$E(M_{t+1}^T/M_t^T)$	0.9980	1.0147	1.0002	0.9987	0.9983

Table 4
Implications of the transitory component of the SDF for the real return of long-term bond.

Parameters tabulated in Bakshi and Chabi-Yo (2011, Tables Appendix-I through Appendix-III), in conjunction with the solution of the eigenfunction problems, are used to generate $\{M_{t+1}^T/M_t^T\}$ via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. Through a single simulation run of 360,000 months, we compute the population values corresponding to:

$$E(r_{t+1,\infty}) = E(R_{t+1,\infty}-1) = E\left(\frac{M_t^T}{M_{t+1}^T}-1\right), \quad Var[r_{t+1,\infty}] = E\left(\frac{M_t^T}{M_{t+1}^T}-E\left(\frac{M_t^T}{M_{t+1}^T}\right)\right)^2.$$

For comparison, also reported are the annualized mean and standard deviation of the real return of long-term bond for the monthly data over 1932:01 to 2010:12. The 90% confidence intervals are shown in square brackets, created from 50,000 random samples of size 948 from the data where the sampling in the block bootstrap is based on 15 blocks.

	$E(r_{t+1,\infty})$	$\sqrt{Var[r_{t+1,\infty}]}$
Real return of long-term bond from models, annualized		
Long-run risk, Kelly	0.0243	0.0002
Long-run risk, Bansal-Yaron	−0.1426	0.1241
External habit, Bekaert-Engstrom	−0.0021	0.0236
External habit, Campbell-Cochrane	0.0159	0.0069
Rare disasters	0.0201	0.0000
Real return of long-term bond in the data, annualized		
1932:01–2010:12 sample	0.0243	0.0887
	[0.0081, 0.0404]	[0.0829, 0.0946]

2011), far less effort has been devoted to rationalizing bond return volatility.

The goal to study the behavior of the return of a long-term bond has the flavor of Beeler and Campbell (2009, Table IX), whereby we explore the possible misalignment of long-term bond returns across models using the transitory component and by solving the eigenfunction problem. Our push to consider the long-term bond return as one criterion in model assessment is also guided by Alvarez and Jermann (2005). Specifically, they argue that, absent the permanent component of the SDF, the maximum risk premium in the economy is reflected in the long-term bond return. The versatility of an asset pricing model also lies in its ability to generate credible properties of the long-term bond returns, as also elaborated in a different context by Campbell and Viceira (2001).

5.5. Models face challenges capturing the relation between M_{t+1}^P/M_t^P and M_{t+1}^T/M_t^T implicit in the data

Motivating the mixed performance of the models so far, we further ask: What can be discerned about the relation between the transitory and the permanent components, given that they jointly price a given set of assets? In the analysis to follow, we address this question from two angles.

We first examine the comovement between the transitory and the permanent components embedded in an asset pricing model. More concretely, the solution to the eigenfunction problem allows us to deduce the left-hand

side below (using $E(M_{t+1}^P/M_t^P) = 1$):

$$Cov\left[\frac{M_{t+1}^P}{M_t^P}, \frac{M_{t+1}^T}{M_t^T}\right] = E\left(\frac{M_{t+1}}{M_t}\right) - E\left(\frac{1}{R_{t+1,\infty}}\right). \tag{37}$$

Equivalently, it imparts the following quantitative implication:

$$\frac{Cov\left[\frac{M_{t+1}^P}{M_t^P}, \frac{M_{t+1}^T}{M_t^T}\right]}{Var\left[\frac{M_{t+1}^T}{M_t^T}\right]} = \underbrace{\frac{E\left(\frac{1}{R_{t+1,1}}\right) - E\left(\frac{1}{R_{t+1,\infty}}\right)}{Var\left[\frac{1}{R_{t+1,\infty}}\right]}}_{\text{can be inputed from the data, and is model}-\text{free}}. \tag{38}$$

Note that the left-hand side of (38) can be recovered as the slope coefficient from the OLS regression: $M_{t+1}^P/M_t^P = a_0 + b_0(M_{t+1}^T/M_t^T) + \epsilon_{t+1}$ in the model-specific simulations. The variance of M_{t+1}^T/M_t^T is a convenient normalization that enables the quantity on the right-hand side to be inputed from the data. Asset pricing theory is silent on the joint distribution of the permanent and the transitory components of SDFs.

Table 5 summarizes (i) the slope coefficient imputed from the data, (ii) the estimate of b_0 from the regression in a single simulation run, and (iii) 95th and 5th percentiles of the b_0 estimates from a finite sample simulation with 200,000 replications.

Observe, however, that the numerator on the right-hand side of Eq. (38), is to first-order, the expected return spread of the long-term bond over the risk-free bond (since $1/(1+x) \approx 1-x$). Thus, a fundamental trait of the data is that it supports a positive covariance between the

Table 5
Implication of the models for the comovement between the permanent and the transitory components of SDFs.

Parameters tabulated in Bakshi and Chabi-Yo (2011, Tables Appendix-I through Appendix-III), in conjunction with the solution of the eigenfunction problems, are used to generate $\{M_{t+1}^P/M_t^P\}$ and $\{M_{t+1}^T/M_t^T\}$ via simulations, respectively, for the models that incorporate long-run risk and external habit persistence. Through a single simulation run of 360,000 months, we perform the OLS regression $M_{t+1}^P/M_t^P = a_0 + b_0(M_{t+1}^T/M_t^T) + \epsilon_{t+1}$, thereby inferring the slope coefficient $b_0 = Cov[M_{t+1}^P/M_t^P, M_{t+1}^T/M_t^T]/Var[M_{t+1}^T/M_t^T]$. In addition, we generate the distribution of b_0 in 200,000 replications of a finite sample simulation over 948 months, and report the values of the mean, the 95th percentile, and the 5th percentile. For comparison, reported also is the inputed value of $E(1/R_{t+1,1}) - E(1/R_{t+1,\infty})/Var[1/R_{t+1,\infty}]$ for the monthly data over 1932:01 to 2010:12. The 90% confidence intervals are shown in square brackets, created from 50,000 random samples of size 948 from the data where the sampling in the block bootstrap is based on 15 blocks. Since the transitory component in the model with rare disasters is a constant, the slope coefficient is identically zero, which renders the finite sample simulation redundant. For this reason, some entries are shown as "na."

	Estimate of b_0 in the regression: $\frac{M_{t+1}^P}{M_t^P} = a_0 + b_0 \left(\frac{M_{t+1}^T}{M_t^T} \right) + \epsilon_{t+1}$			
	Single simulation run b_0, population	Finite sample simulation (distribution of b_0)		
		Mean	95th	5th
Long-run risk, Kelly	−5.88	−5.88	57.68	−69.56
Long-run risk, Bansal-Yaron	−2.43	−2.44	−2.22	−2.65
External habit, Bekaert-Engstrom	−16.25	−15.55	−9.28	−22.73
External habit, Campbell-Cochrane	0.23	0.00	0.19	−0.28
Rare disasters	0.00	na	na	na
Imputed from the data, b_0				
1932:01–2010:12 sample	1.96 [1.07, 3.96]			

permanent and the transitory components of SDFs. The value of b_0 implied from the data is 1.96.

A comparison of the slope coefficients obtained through our simulations elicits the observation that, on average, three (one) out of five models produce negative (positive) slope coefficient whose magnitudes contrast the data counterpart. The 95th and 5th percentiles for the b_0 distribution in the models of Bansal and Yaron and Bekaert and Engstrom are negative, suggesting that $b_0 < 0$ appears to be an intrinsic feature of their models. Overall, this evidence illustrates the ambivalence of the models in replicating the association between the permanent and the transitory components implicit in the data.

In addition, recognize in our context that a statistic to gauge the performance of models in capturing the joint pricing of bonds and other assets is the lower bound on $Var[(M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T)]$. Complementing the picture from Tables 2 and 3, the results in Table 6 show that asset pricing models are unable to describe the behavior of joint pricing implicit in the bound. The p-values that examine the restriction $\sigma_{pt}^2 - Var[(M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T)] \leq 0$ are typically small, refuting another restriction suggested by the theory.

The relation between the permanent and the transitory components of the SDF has information content for modeling asset prices. A particular lesson to be gleaned is that asset pricing models could be enriched to better describe the joint dynamics of the transitory and the permanent components of SDFs as also reflected in bond

risk premium, a positive equity risk premium, and the risk premium on a broad spectrum of assets.

5.6. Models may reproduce some asset market phenomena, but find it onerous to satisfy bounds

Synthesizing elements of Tables 2, 3, and 6, we pose three additional questions in turn.

First, why is it that formulations from an asset pricing model can come close to duplicating the observed equity premium, and at the same time not satisfy the lower bound on the permanent component of the SDF for a broad set of asset returns? We expand on this seemingly contradictory observation by computing the lower bound on the permanent component when the SDF is required to satisfactorily price the long-term bond, the risk-free bond, and the market equity [returning to the set of assets in Alvarez and Jermann (2005)]. For this set of assets, $\sigma_{pc}^2 = 0.0185$, and the minimum p-value examining the restriction $\sigma_{pc}^2 - Var[M_{t+1}^P/M_t^P] \leq 0$ is 0.314 across models. The nuance is that asset pricing models can meet the restrictions for a narrower set of assets, but not the broader set of assets that includes the 25 (6) Fama-French equity portfolios (as in SET A (SET C)).

As one enlarges the set of assets in $\mathbf{R}_{t+1}$, the permanent component of the SDF must be generalized to cope with the risk premiums on a wider array of assets. This prompts the next question: What is our incremental value beyond the L-measure in comparing asset pricing models? The analysis below (with p-values computed as before)

Table 6

Assessing the restriction on the variance of the ratio of the permanent to the transitory components of SDFs from asset pricing models.

We compute $(Var[M_{t+1}^P]/M_t^P/(M_{t+1}^T/M_t^T)]$ via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculations are based on model parameters tabulated in Bakshi and Chabi-Yo (2011, Tables Appendix-I through Appendix-III), and the reported values are the averages from a single simulation run of 360,000 months. The reported lower bound σ_{pt}^2 on the ratio of the permanent to the transitory components is based on Eq. (11) of Proposition 3. The bound is based on the return properties of $\mathbf{R}_{t+1}$, corresponding to SET A, and the long-term bond. Monthly data used in the construction of σ_{pt}^2 are from 1932:01 to 2010:12 (948 observations), with the 90% confidence intervals in square brackets. To compute the confidence intervals, we create 50,000 random samples of size 948 from the data, where the sampling in the block bootstrap is based on 15 blocks. Reported below the estimates of $Var[(M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T)]$ are the p-values, shown in curly brackets, which represent the proportion of replications for which model-based $Var[(M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T)]$ exceeds σ_{pt}^2 in 200,000 replications of a finite sample simulation over 948 months.

	Long-run risk		External habit		Rare
	Kelly	Bansal-Yaron	Bekaert-Engstrom	Campbell-Cochrane	disasters
$Var\left[\frac{M_{t+1}^P}{M_t^P} \Big/ \frac{M_{t+1}^T}{M_t^T}\right]$	0.0375	0.0449	0.0298	0.0234	0.0582
	{0.000}	{0.000}	{0.001}	{0.000}	{0.094}
Lower bound, σ_{pt}^2	0.1309	0.1842	0.1237	0.1264	0.1324
	[0.0481, 0.1381]	[0.0868, 0.1982]	[0.0429, 0.1296]	[0.0448, 0.1327]	[0.0491, 0.1399]
$E\left(\frac{M_{t+1}^P}{M_t^P} \Big/ \frac{M_{t+1}^T}{M_t^T}\right)$	1.0018	0.9916	1.0008	1.0012	1.0019

speaks to the relevance of the tightness of our bounds in empirical applications:

stochastic discount factors. At the center of our approach are three theoretical results, one related to a lower bound

	Kelly	Bansal-Yaron	Bekaert-Engstrom	Campbell-Cochrane	Rare Disasters	Lower bound
$L[\frac{M_{t+1}^P}{M_t^P}]$	0.0176	0.0171	0.0100	0.0115	0.0032	**0.0041**
p-value	{0.986}	{0.990}	{0.999}	{0.994}	{0.254}	

In particular, our results point to a scenario in which models generate high (low) p-values with the L-measure (variance-measure), as seen by comparing the entries for p-values in Table 2. It is the stringency of the lower variance bound, together with a set of empirically relevant assets in $\mathbf{R}_{t+1}$, that enhances the flexibility of the bounds approach to differentiate between competing asset pricing models.

Building on our observations, we finally ask whether an asset pricing model can satisfy the Hansen and Jagannathan (1991) bound for a broad set of assets, for instance, SET A, and yet fail the lower bound on the permanent component in Proposition 1. Bakshi and Chabi-Yo (2011, Lemma 1) establish the relevance of a bound on $Cov[(M_{t+1}^P/M_t^P)^2,(M_{t+1}^T/M_t^T)^2]$, with the implication that further theoretical work is needed to get a firmer grasp of the joint distribution of M_{t+1}^p/M_t^p and M_{t+1}^T/M_t^T. More can be learned about the functioning of asset pricing models through the lens of permanent and transitory components of SDFs.

6. Conclusions and extensions

This paper presents a variance bounds framework in the context of permanent and transitory components of

on the variance of the permanent component, another on the lower bound on the variance of the transitory component, and also a lower bound on the variance of the ratio of the permanent to the transitory components of stochastic discount factors. Instrumental to the tasks at hand, we establish the tightness of our variance bounds relative to Alvarez and Jermann (2005), and we show that our bounds can be useful in asset pricing applications. A specific conclusion is that bound implications for the permanent component of the stochastic discount factors in the setting of Alvarez and Jermann (2005) are considerably weaker than those reported in our context of a generic set of assets. Our analysis furnishes bounds that incorporate information from average returns as well as the variance-covariance matrix of returns.

Combining the variance bounds with the eigenfunction problem offers guidance for asset price modeling in several ways. First, we present the solution to the eigenfunction problem for five asset pricing models in the class of long-run risk, external habit persistence, and rare disasters. This solution justifies the calculation of the moments of the transitory and the permanent components, as well as all its mixed moments. Second, we corroborate that models face a particular impediment satisfying the lower bound restrictions imposed by our

bounds, even when the models are successful in matching the equity premium and the return of the risk-free bond. Third, we find that the models are not compatible with the return properties of the long-term bond. Finally, while the data support a positive comovement between the transitory and the permanent components, our analysis reveals that this feature is not easily reconciled within our parametrization of asset pricing models.

Our work could be extended. While our focus is directed toward primary assets, one could modify the analysis to include stochastic discount factors that also satisfactorily price out-of-the-money put options on the market as well as claims on market variance. Incorporating such claims can impose further hurdles on an asset pricing model.

In addition, our framework is amenable to investigating the suitability of stochastic discount factors to address several asset pricing puzzles together, for instance, by combining elements of the value premium, the equity premium, the risk-free return, and the bond risk premium. In this regard, our work is related to, among others, Ai and Kiku (2010), Aït-Sahalia, Parker, and Yogo (2004), Cochrane and Piazzesi (2005), Koijen, Lustig, and Van Nieuwerburgh (2010), Lettau and Wachter (2007), Routledge and Zin (2010), Santos and Veronesi (2010), and Zhang (2005). Overall, our variance bounds approach, when combined with the eigenfunction problem, could refine the quest for a better understanding of asset returns.

Appendix A. Proofs of unconditional bounds

In the results that follow, we provide proofs of Propositions 1–3.

Proof of Proposition 1. The proof is by construction. Recognize that

$$E\left(\left(\frac{M_{t+1}^P}{M_t^P}-E\left(\frac{M_{t+1}^P}{M_t^P}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}-E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\right)$$
$$= \mathbf{1}-E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right), \qquad (A.1)$$

where the right-hand side of (A.1) is obtained by noting that $E(M_{t+1}^P/M_t^P)=1$ and $R_{t+1,\infty}=(M_{t+1}^T/M_t^T)^{-1}$. Denote

$$\Omega \equiv E\left(\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}-E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}-E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)'\right)$$
$$\text{and}\quad \mathbf{B}\equiv\mathbf{1}-E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \qquad (A.2)$$

Multiply the right-hand side of (A.1) by $\mathbf{B}'\Omega^{-1}$ to obtain:

$$\mathbf{B}'\Omega^{-1}\mathbf{B}=E\left(\left(\frac{M_{t+1}^P}{M_t^P}-E\left(\frac{M_{t+1}^P}{M_t^P}\right)\right)\left(\mathbf{B}'\Omega^{-1}\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}-E\left(\mathbf{B}'\Omega^{-1}\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\right),$$
$$(A.3)$$

$$\leq\left(Var\left[\frac{M_{t+1}^P}{M_t^P}\right]\right)^{1/2}\times\left(Var\left[\mathbf{B}'\Omega^{-1}\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right]\right)^{1/2}. \qquad (A.4)$$

Given that $Var[\mathbf{B}'\Omega^{-1}(\mathbf{R}_{t+1}/R_{t+1,\infty})]$ equals $\mathbf{B}'\Omega^{-1}\mathbf{B}$, our application of the Cauchy-Schwarz inequality implies that $(\mathbf{B}'\Omega^{-1}\mathbf{B})^{1/2}\leq(Var[M_{t+1}^P/M_t^P])^{1/2}$. Thus, we have established the bound in Eq. (6). □

Proof of Proposition 2. For brevity, denote $\mu_{tc}\equiv E(M_{t+1}^T/M_t^T)$, which is the mean of the transitory component of the SDF. Again the proof is by construction, whereby we recognize that

$$E\left(\left(\frac{M_{t+1}^T}{M_t^T}-E\left(\frac{M_{t+1}^T}{M_t^T}\right)\right)(R_{t+1,\infty}-E(R_{t+1,\infty}))\right)=1-\mu_{tc}E(R_{t+1,\infty}),$$
$$(A.5)$$

since $E((M_{t+1}^T/M_t^T)R_{t+1,\infty})=1$. Multiplying both sides of (A.5) by $(1-\mu_{tc}E(R_{t+1,\infty}))/Var[R_{t+1,\infty}]$, and using the Cauchy-Schwarz inequality validates the bound in Eq. (9) of Proposition 2. □

Proof of Proposition 3. Observe that

$$E\left(\left(\frac{\frac{M_{t+1}^P}{M_t^P}}{\frac{M_{t+1}^T}{M_t^T}}-E\left(\frac{\frac{M_{t+1}^P}{M_t^P}}{\frac{M_{t+1}^T}{M_t^T}}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}-E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right)\right)\right)$$
$$= \mathbf{1}-\mu_{pt}E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right), \qquad (A.6)$$

where recalling the notation $\mu_{pt}=E((M_{t+1}^P/M_t^P)/(M_{t+1}^T/M_t^T))$. We denote

$$\Sigma\equiv E\left(\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}-E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right)\right)\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}-E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right)\right)'\right)$$
$$\text{and}\quad \mathbf{D}\equiv\mathbf{1}-\mu_{pt}E\left(\frac{\mathbf{R}_{t+1}}{R_{t+1,\infty}^2}\right). \qquad (A.7)$$

Multiply (A.6) by $\mathbf{D}'\Sigma^{-1}$ and apply the Cauchy-Schwarz inequality to the left-hand side of (A.6). □

Appendix B. Proofs of unconditional bounds that incorporate conditioning information

Proof of Propositions 1–3 with conditioning variables. For tractability of exposition, we denote by z_t the set of conditioning variables. The variable z_t predicts $\mathbf{R}_{t+1}$. We note that

$$E\left(\left(\frac{M_{t+1}^P}{M_t^P}-E\left(\frac{M_{t+1}^P}{M_t^P}\right)\right)\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}-E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)\right)$$
$$= E(z_t'\mathbf{1})-E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \qquad (B.1)$$

Eq. (B.1) follows, since the permanent component of the pricing kernel is a martingale. Now, denote

$$\Theta\equiv E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}-E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right)\right)^2 \quad\text{and}\quad \mathbf{H}\equiv E(z_t'\mathbf{1})-E\left(\frac{z_t'\mathbf{R}_{t+1}}{R_{t+1,\infty}}\right). \quad (B.2)$$

Multiplying (B.1) by $\mathbf{H}'\Theta^{-1}$ and applying the Cauchy-Schwarz inequality, we obtain $\mathbf{H}'\Theta^{-1}\mathbf{H}\leq Var[M_{t+1}^P/M_t^P]$. The same approach can be used to derive Propositions 2 and 3 with conditioning variables. □

Appendix C. Solution to the eigenfunction problem in Kelly (2009)

Our end-goal is to present the permanent and the transitory components of the SDF by solving the

eigenfunction problem. To save space, the intermediate steps are provided in Bakshi and Chabi-Yo (2011).

Under the dynamics of the real economy posited in (21)–(23) and recursive utility, the SDF is,

$$\log\left(\frac{M_{t+1}}{M_t}\right) = \xi_t + \mathcal{D}_1(g_{t+1}-E_t(g_{t+1})) + \mathcal{D}_2(x_{t+1}-E_t(x_{t+1}))$$
$$+ \mathcal{D}_3(\sigma^2_{M,t+1}-E_t(\sigma^2_{M,t+1})) + \mathcal{D}_4(\Lambda_{t+1}-E_t(\Lambda_{t+1})),$$
(C.1)

where the innovation in market volatility $\sigma^2_{M,t+1}-E_t(\sigma^2_{M,t+1})$ is displayed in (C.11). We determine

$$\mathcal{D}_1 = -\lambda_g, \quad \mathcal{D}_2 = -\lambda_x, \quad \mathcal{D}_3 = -\frac{\lambda_\sigma}{(\sigma_g^2+\kappa_1^2 A_x^2\sigma_x^2)},$$
$$\mathcal{D}_4 = \frac{2\lambda_\sigma}{(\sigma_g^2+\kappa_1^2 A_x^2\sigma_x^2)} - \lambda_\Lambda,$$
(C.2)

with $\xi_t = \theta\log(\beta) - \frac{\theta}{\psi}(\mu+x_t) + (\theta-1)E_t(r_{M,t+1})$. Furthermore,

$$\lambda_g = 1-\theta+\frac{\theta}{\psi}, \quad \lambda_x = (1-\theta)\kappa_1 A_x, \quad \lambda_\sigma = (1-\theta)\kappa_1 A_\sigma,$$
$$\lambda_\Lambda = (1-\theta)\kappa_1 A_\Lambda, \quad \text{and}$$
(C.3)

$$A_x = \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho_x}, \quad A_\Lambda = \frac{\theta\left(1-\frac{1}{\psi}\right)\left(1-\frac{1}{\psi}\right)}{1-\kappa_1\rho_x},$$
$$A_\sigma = \frac{\theta}{2}\left(\frac{\left(1-\frac{1}{\psi}\right)^2\sigma_g^2+\kappa_1^2 A_x^2\sigma_x^2}{1-\kappa_1\rho_\sigma}\right),$$
(C.4)

$$A_0 = \frac{\log(\beta)+\left(1-\frac{1}{\psi}\right)\mu+\kappa_0+\kappa_1(A_\sigma\overline{\sigma}^2(1-\rho_\sigma)+A_\Lambda\overline{\Lambda}(1-\rho_\Lambda))+\frac{1}{2}\theta\kappa_1^2(A_\sigma^2\sigma_\sigma^2+A_\Lambda^2\sigma_\Lambda^2)}{1-\kappa_1}.$$
(C.5)

The innovation in market volatility is derived using the expression for excess return:

$$r_{M,t+1}-E_t(r_{M,t+1}) = \kappa_1 A_\Lambda\sigma_\Lambda z_{\Lambda,t+1} + \sigma_g\sigma_t z_{g,t+1}$$
$$+ \sqrt{\Lambda_t}W_{g,t+1} + \kappa_1 A_x\sigma_x\sigma_t z_{x,t+1}$$
$$+ \kappa_1 A_\sigma\sigma_\sigma z_{\sigma,t+1},$$
(C.6)

$$E_t(r_{M,t+1}) = \kappa_0 + \kappa_1(A_0+A_x E_t(x_{t+1})+A_\sigma E_t(\sigma^2_{t+1})$$
$$+ A_\Lambda E_t(\Lambda_{t+1})) - wg_t + E_t(g_{t+1}),$$
(C.7)

where

$$E_t(x_{t+1}) = \rho_x x_t, \quad E_t(\sigma^2_{t+1}) = \overline{\sigma}^2(1-\rho_\sigma)+\rho_\sigma\sigma_t^2,$$
(C.8)

$$E_t(\Lambda_{t+1}) = \overline{\Lambda}(1-\rho_\Lambda)+\rho_\Lambda\Lambda_t, \quad E_t(g_{t+1}) = \mu+x_t.$$
(C.9)

Using the moment generating function of the Laplace variable

$$Var_t[r_{M,t+1}] = \sigma^2_{M,t+1} = \kappa_1^2(A_\Lambda^2\sigma_\Lambda^2+A_\sigma^2\sigma_\sigma^2)$$
$$+ (\sigma_g^2+\kappa_1^2 A_x^2\sigma_x^2)\sigma^2_{t+1}+2\Lambda_{t+1}.$$
(C.10)

Then the innovation in market variance is

$$\sigma^2_{M,t+1}-E_t(\sigma^2_{M,t+1}) = (\sigma_g^2+\kappa_1^2 A_x^2\sigma_x^2)\sigma_\sigma z_{\sigma,t+1}+2\sigma_\Lambda z_{\Lambda,t+1}.$$
(C.11)

Moving to the transitory component of the SDF, define $\mathbf{X}_t = (x_t,\sigma_t^2-\overline{\sigma}^2,\Lambda_t-\overline{\Lambda})'$ for notational simplicity. The aim is to solve the eigenfunction problem

$$E_t\left(\frac{M_{t+1}}{M_t}\frac{1}{M^e_{t+1}}\right) = \frac{\upsilon}{M^e_t},$$

or equivalently, $E_t((M_{t+1}/M_t)e[\mathbf{X}_{t+1}]) = \upsilon e[\mathbf{X}_t]$. Under our assumptions, we conjecture that the solution is of the form

$$e[\mathbf{X}_{t+1}] = \exp(\mathbf{c}'\mathbf{X}_{t+1}), \quad \mathbf{c} = (c_1,c_2,c_3)'.$$
(C.12)

Skipping tedious algebra, we deduce that

$$\upsilon = \exp\begin{pmatrix} \theta\log(\beta)+(\theta-1)\kappa_0+(\theta-1)\kappa_1 A_0-\frac{\theta}{\psi}\mu \\ +(\theta-1)\kappa_1 A_\sigma\overline{\sigma}^2(1-\rho_\sigma) \\ +(\theta-1)\mu+(\theta-1)\kappa_1 A_\Lambda\overline{\Lambda}(1-\rho_\Lambda) \\ -(\theta-1)(A_0+A_\sigma\overline{\sigma}^2+A_\Lambda\overline{\Lambda}) \\ +(\theta-1)A_\Lambda\kappa_1\rho_\Lambda\overline{\Lambda}+(\theta-1)A_\sigma\kappa_1\rho_\sigma\overline{\sigma}^2 \\ +\frac{1}{2}(\lambda_\sigma^2\sigma_\sigma^2+\lambda_\Lambda^2\sigma_\Lambda^2)-(\lambda_\sigma\sigma_\sigma^2 c_2+\lambda_\Lambda\sigma_\Lambda^2 c_3) \\ +\frac{1}{2}(c_2^2\sigma_\sigma^2+c_3^2\sigma_\Lambda^2)+\lambda_g^2\overline{\Lambda} \\ +(-\lambda_x\sigma_x^2 c_1+\frac{1}{2}c_1^2\sigma_x^2+\frac{1}{2}\lambda_g^2\sigma_g^2+\frac{1}{2}\lambda_x^2\sigma_x^2)\overline{\sigma}^2 \end{pmatrix},$$
(C.13)

and

$$c_1 = \frac{-\lambda_g+(\theta-1)A_x(\kappa_1\rho_x-1)}{1-\rho_x},$$

$$c_2 = \frac{-\lambda_x\sigma_x^2 c_1+\frac{1}{2}c_1^2\sigma_x^2+\frac{1}{2}\lambda_g^2\sigma_g^2+\frac{1}{2}\lambda_x^2\sigma_x^2+(\theta-1)A_\sigma(\kappa_1\rho_\sigma-1)}{1-\rho_\sigma},$$

$$c_3 = \frac{\lambda_g^2+(\theta-1)A_\Lambda(\kappa_1\rho_\Lambda-1)}{1-\rho_\Lambda}.$$
(C.14)

The transitory component of the SDF is $M^T_{t+1}/M^T_t = \upsilon\exp(-\mathbf{c}'(\mathbf{X}_{t+1}-\mathbf{X}_t))$, and the permanent component is determined accordingly. □

Appendix D. Solution to the eigenfunction problem in Bekaert and Engstrom (2010)

Write $q_{t+1}-\delta_q = \rho_q(q_t-\delta_q)+\sigma_{qp}\omega_{p,t+1}+\sigma_{qn}\omega_{n,t+1}$, with $\mu_q/(1-\rho_q)\equiv\delta_q$. Denote the state vector as $\mathbf{Z}_t = (q_t-\delta_q,x_t,p_t-\overline{p},n_t-\overline{n})'$. In the eigenfunction problem $E_t((M_{t+1}/M_t)e[\mathbf{Z}_{t+1}]) = \upsilon e[\mathbf{Z}_t]$, conjecture that $e[\mathbf{Z}_t] = \exp(\mathbf{c}'\mathbf{Z}_t)$. Because ge_{t+1} and be_{t+1} are independent and follow a gamma distribution,

$$E_t(\exp(\lambda\,ge_{t+1})) = \exp(-p_t\log(1-\lambda)), \quad \text{and}$$
$$E_t(\exp(\lambda\,be_{t+1})) = \exp(-n_t\log(1-\lambda)),$$
(D.1)

for some real number λ. Then we manipulate a set of equations to determine the eigenvalue as

$$\upsilon = \exp(\log(\beta)-\gamma\overline{g}-a_0\overline{p}-b_0\overline{n}-\overline{p}\log(1-a_0)-\overline{n}\log(1-b_0)),$$
(D.2)

and

$$c_1 = -\gamma, \quad c_2 = -\frac{\gamma}{1-\rho_x}, \quad c_3(1-\rho_p) + a_0 + \log(1-a_0) = 0,$$

$$c_4(1-\rho_n) + b_0 + \log(1-b_0) = 0, \tag{D.3}$$

where setting $a_0 \equiv -\gamma\sigma_{gp} - (\gamma\sigma_{xp}/(1-\rho_x)) + c_3\sigma_{pp}$ and $b_0 \equiv \gamma\sigma_{gn} - (\gamma\sigma_{xn}/(1-\rho_x)) + c_4\sigma_{nn}$. This concludes our proof. □

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