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A Refinement to Aït-Sahalia's (2002) "Maximum Likelihood Estimation of Discretely Sampled Diffusions: A Closed-Form Approximation Approach"*

This paper provides a closed-form density approximation when the underlying state variable is a one-dimensional diffusion. Building on Aït-Sahalia (2002), we show that our refinement is applicable under a wide class of drift and diffusion functions. In addition, it facilitates the maximum likelihood estimation of discretely sampled diffusion models of short interest-rate or stock volatility with unknown conditional densities. Our interest-rate examples demonstrate that the analytical approximation is sufficiently accurate.

I. Introduction and the Approach in Aït-Sahalia (2002)

In a recent contribution, Aït-Sahalia (2002) proposed a general method to derive the density function of a one-dimensional diffusion process. To describe his closed-form approximation approach, we follow his notation and consider a one-dimensional diffusion process for some state variable X_t:

$$dX_t = \mu(X_t;\theta)dt + \sigma(X_t;\theta)dW_t, \tag{1}$$

where W_t is a standard Brownian motion. The functions $\mu(X_t;\theta)$ and $\sigma(X_t;\theta)$, respectively, are the coefficients of drift and diffusion, and θ represents the parameter vector in an open bounded set $\Theta \subset R^K$. Under the assumption that X_t is observed at dates $\{t = i\Delta | i = 0, \ldots, n\}$, Aït-Sahalia (2002) is interested in characterizing the conditional density

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function of $X_{t+\Delta}$ given X_t. Let this conditional density be denoted by $p_x(\Delta, x|x_0; \theta)$. The maximum likelihood estimator of θ is obtained by maximizing the function

$$\ell_n(\theta) = \sum_{i=1}^{n} \log \{p_X(\Delta, X_{i\Delta}|X_{(i-\Delta)}; \theta)\}. \tag{2}$$

Until the work of Aït-Sahalia (2002), the density function $p_x(\Delta, x|x_0; \theta)$ remained intractable for a sufficiently wide class of $\mu(X_t; \theta)$ and $\sigma(X_t; \theta)$.

The intent of this article is to modify and refine the methodology in Aït-Sahalia (2002). Building on Aït-Sahalia (2002), we offer one primary contribution. Specifically, it is shown that our closed-form approximation approach for determining the transition density expands the class of $\mu(X; \theta)$ and $\sigma(X; \theta)$ to which the method can be applied (in a manner to be made precise shortly).

To highlight the general difficulty in determining the transition density of X_t and understand the innovations in Aït-Sahalia (2002), let us outline the basic steps underlying his approach. First, he constructs a transformed process Y defined next:

$$Y \equiv \gamma(X; \theta) = \int^{X} \frac{du}{\sigma(u; \theta)}, \tag{3}$$

where $\sigma(u; \theta)$ is prespecified from (1). Second, applying Ito's lemma to Y_t, he derives the unit diffusion process for Y_t:

$$dY_t \equiv \mu_Y(Y_t; \theta)dt + dW_t, \tag{4}$$

where

$$\mu_Y(y; \theta) \equiv \frac{\mu[\gamma^{-1}(y; \theta); \theta]}{\sigma[\gamma^{-1}(y; \theta); \theta]} - \frac{1}{2}\frac{\partial \sigma}{\partial x}\left[\gamma^{-1}(y; \theta); \theta\right]. \tag{5}$$

Note that $\gamma(X; \theta)$ is derived by solving the integral in (3) and $\gamma^{-1}(y; \theta)$ represents its inverse function. It is assumed that $\gamma(X; \theta)$ and $\gamma^{-1}(y; \theta)$ are known analytically. Suppose, as in Aït-Sahalia (1999), $\sigma(X; \theta) = X^{\xi}$ with $0 < \xi < 1$, then $\gamma(X; \theta) = (1-\xi)^{-1}X^{1-\xi}$ and $\gamma^{-1}(y; \theta) = [y(1-\xi)]^{1/(1-\xi)}$.

Proceeding to the next step, he rescales and standardizes Y via

$$Z = \Delta^{-1/2}(Y - y_0), \tag{6}$$

where y_0 is the initial value of $y_{t+\Delta}$. For small Δ, y_0 is approximately the mean of $y_{t+\Delta}$, and Δ its variance. Aït-Sahalia (2002) shows that Z is close to a standard normal random variable. Accordingly, its density,

$p_z(\Delta, z|y_0;\theta)$, can be expanded around a standard normal density using Hermite polynomials (up to the Jth term),

$$p_Z^{(J)}(\Delta, z|y_0;\theta) \equiv \phi(z)\sum_{j=0}^{J}\eta_Z^{(j)}(\Delta, y_0;\theta)H_j(z), \tag{7}$$

where $\phi(z) = e^{-z^2/2}/\sqrt{2\pi}$, and $H_j(z)$ are the Hermite polynomials,

$$H_j(z) \equiv e^{z^2/2}\frac{d^j}{dz^j}\left[e^{-z^2/2}\right], \quad j \geq 0. \tag{8}$$

As is standard, the expansion coefficients, $\eta_Z^{(j)}(\Delta, y_0;\theta)$, are recovered through the orthogonal conditions of the Hermites:

$$\eta_Z^{(j)}(\Delta, y_0;\theta) \equiv \frac{1}{j!}\int_{-\infty}^{\infty}H_j(z)p_Z(\Delta, z|y_0;\theta)dz. \tag{9}$$

Fourth, the expectation for the Hermites ($j = 1, 2, \ldots$) are approximated using Taylor theorem up to the Kth term,

$$\frac{1}{j!}\int_{-\infty}^{\infty}H_j(z)p_Z(\Delta, z|y_0;\theta)dz \approx \frac{1}{j!}\sum_{k=0}^{K}\frac{\Delta^k}{k!}A^k(\theta)\cdot H_j\left(\frac{y-y_0}{\Delta^{1/2}}\right)\Bigg|_{y\to y_0}, \tag{10}$$

where $A(\theta) = ½(\partial^2/\partial y^2) + \mu_Y(y;\theta)(\partial/\partial y)$ is the infinitesimal generator of the diffusion Y_t. The resulting density approximation of Z is termed as $p_Z^{(J,K)}$.

Finally, the approximate density of $X_{t+\Delta}$ conditional on $X_t = x_0$ is obtained from the density of Z in (7) and by the Jacobian formula

$$p_X^{(J,K)}(\Delta, x|x_0;\theta) = \frac{1}{\sigma(x;\theta)\Delta^{1/2}} \times p_Z^{(J,K)}\left(\Delta, \Delta^{-1/2}[\gamma(x;\theta) - \gamma(x_0;\theta)]|\gamma(x_0;\theta), \theta\right). \tag{11}$$

Under mild regularity conditions, $p_X^{(J,K)}(\Delta, x|x;\theta)$ converges to the true density function as $J \to \infty$ and $K \to \infty$. The regularity conditions and the convergence properties of the approximations are available in Aït-Sahalia (2002).

Aït-Sahalia (1999, 2002) argues that there are different ways of collecting the terms in (11). One appealing alternative is to collect all terms with different orders of the Hermite polynomials j but keep the expansion coefficients to the same order K in Δ. In other words, let $J \to \infty$ in (7) and leave K the same for all values of j in (10). The resulting density approximation, $p_Y^{(K)} \equiv p_Y^{(\infty,K)}$, of $Y_{t+\Delta}$ becomes (up to the Δ^Kth term;

consult [11] and [12] in Aït-Sahalia 1999 and [4.10] and [4.11] in Aït-Sahalia 2002)

$$p_Y^{(K)}(\Delta, y|y_0;\theta) = \Delta^{-1/2}\phi\left(\frac{y-y_0}{\Delta^{1/2}}\right)\exp\left[\int_{y_0}^{y}\mu_Y(w;\theta)dw\right] \times \sum_{k=0}^{K} c_k(y|y_0;\theta)\frac{\Delta^k}{k!}, \tag{12}$$

where $c_0(y|y_0;\theta) = 1$, and for $j \geq 1$,

$$c_j(y|y_0;\theta) = j(y-y_0)^{-j}\int_{y_0}^{y}(w-y_0)^{j-1} \times \left[\lambda(w)c_{j-1}(w) + \frac{\partial^2 c_{j-1}(w|y_0;\theta)}{\partial w^2}\right]dw, \tag{13}$$

and

$$\lambda_Y(y;\theta) \equiv -\frac{1}{2}\left[\mu_Y^2(y;\theta) + \frac{\partial\mu_Y(y;\theta)}{\partial y}\right]. \tag{14}$$

Again, the approximate density of $X_{t+\Delta}$ is obtained by invoking the Jacobian formula

$$p_X^{(K)}(\Delta, x|x_0;\theta) = \frac{1}{\sigma(x;\theta)}p_Y^{(K)}[\Delta, \gamma(x;\theta) - \gamma(x_0;\theta)|\gamma(x_0;\theta), \theta]. \tag{15}$$

For the later accuracy and comparisons tests, we refer to the approximation (15) as *Aït-Sahalia enhanced* and to (11) as *Ait-Sahalia basic*.

II. A Class of Closed-Form Density Approximations

While Aït-Sahalia (2002) showed that his method is accurate when applied to commonly adopted interest-rate models and works, in theory, for any $\mu(X;\theta)$ and $\sigma(X;\theta)$, there are potential difficulties with implementing his approach. First, the method requires that the integration in (3) be done analytically, which is not always feasible in a general setting. Second, even when the integration in (3) is analytical, the density approximation assumes that the inverse of $\gamma(X;\theta)$ is known in exact closed form (see equation [5]). The following volatility functions illustrate the practical limitations of the method and show that it precludes an important collection of continuous-time models.

Case 1. Consider the volatility function (for the short interest-rate) studied in Aït-Sahalia (1996):

$$\sigma(X;\theta) = \sqrt{\beta_0 + \beta_1 X + \beta_2 X^{\beta_3}}, \tag{16}$$

for some constants $\beta_0, \beta_1, \beta_2,$ and β_3. For this volatility specification, the integration $\int^X du/\sqrt{\beta_0 + \beta_1 u + \beta_2 u^{\beta_3}}$ cannot be performed analytically.

CASE 2. Now consider a stochastic volatility model for the stock price, S_t, of the type: $dS_t = \mu_S(S;\theta)S_t dt + (\beta_0 + \beta_1 S_t^{-\beta_2})S_t dW_t$. This process nests the constant elasticity of the variance model. Letting $X_t \equiv \log(S_t)$, the stock return volatility is

$$\sigma(X_t;\theta) = \beta_0 + \beta_1 e^{-\beta_2 X_t}, \tag{17}$$

with $\beta_0 > 0, \beta_1 > 0$, and $\beta_3 > 0$. The behavior of return volatility outlined in (17) is theoretically plausible with volatility bounded between β_0 and ∞. Specifically, $\sigma(-\infty;\theta) = \infty$ and $\sigma(\infty;\theta) = \beta_0$. While $\gamma(X;\theta) = \int^X du/\sqrt{\beta_0 + \beta_1 e^{-\beta_2 u}} = (X/\beta_0) + (1/\beta_0\beta_2)\log(\beta_0 + \beta_1 e^{-\beta_2 X})$, the function $\gamma(X;\theta)$ is not invertible in closed form.

We now propose a methodological refinement that circumvents the aforementioned difficulties and makes the method applicable to a wider class of one-dimensional diffusions. There are two aspects to our refinement. First, as opposed to Aït-Sahalia (2002), who constructs Z in (6), we use the true mean and variance of $\int^X du/\sigma(u;\theta)$ in our standardization. We show that this modification enhances the accuracy of the method. Second, when computing the moments of Z (or equivalently Y) to obtain the expansion coefficients, we regard Z as a function of X (see equation [23]). For any given $\mu(X_t;\theta)$ and $\sigma(X_t;\theta)$, this allows us to obtain the approximate moments of Z by applying the Taylor theorem to a function of X. As formalized in the following three subsections, the resulting closed-form approximation is less demanding because it avoids using the drift of $\int^X du/\sigma(u;\theta)$.

A. Standardization of Y_t

As articulated already, we keep the transformation from X to Y in (3) but standardize Y as

$$\hat{Z} = \Sigma^{-1/2}(Y - \bar{Y}), \tag{18}$$

where $\bar{Y}$ and Σ are, respectively, the true mean and variance of Y. For future use, rewrite (18) as

$$\begin{aligned}\hat{Z} &= \Sigma^{-1/2}\Delta^{1/2}\left(\frac{Y - y_0}{\Delta^{1/2}} - \frac{\bar{Y} - y_0}{\Delta^{1/2}}\right)\\ &\equiv \rho(Z - \bar{Z}),\end{aligned} \tag{19}$$

where $\bar{Z}$ is the mean of Z, and

$$Z \equiv \frac{Y - y_0}{\Delta^{1/2}}, \tag{20}$$

$$\rho \equiv \Sigma^{-1/2}\Delta^{1/2} = \left(E[Z^2] - \bar{Z}^2\right)^{-1/2}. \tag{21}$$

Here, E[·]denotes expectation operator throughout. Equation (21) follows from noting that $\Sigma = E[(Y - \bar{Y})^2] = \Delta E[(Z - \bar{Z})^2] = \Delta(E[Z^2] - \bar{Z}^2)$. In model implementations, the true mean $\bar{Y}$ and variance Σ of $Y$are approximated by Taylor expanding in Δ. Suppressing this dependence of $\bar{Y}$ and Σ on Δ for brevity of presentation, we now proceed to characterizing the density of X_t.

B. Characterizing the Density of X_t

Guided by Aït-Sahalia (2002), the density of $\hat{Z}$ is approximated by appealing to (7) with expansion coefficients $\eta_j(\Delta, x_0; \theta)$ (shorthand corresponding to $\eta_{\hat{Z}}^{(j)}(\Delta, x_0; \theta)$) determined via (9) and (10). By a straightforward application of the Jacobian formula to the density of $\hat{Z}$, we then arrive at the closed-form density of X:

$$p_X^{(J)}(\Delta, x|x_0; \theta) \equiv \frac{\rho}{\sigma(x; \theta)\Delta^{1/2}} \varphi[\rho(Z - \bar{Z})] \sum_{j=0}^{J} \eta_j(\Delta, x_0; \theta) H_j[\rho(Z - \bar{Z})], \tag{22}$$

where we may write Z as a function of X, as shown follows:

$$Z = \frac{1}{\Delta^{1/2}} \int_{x_0}^{X} \frac{du}{\sigma(u; \theta)}. \tag{23}$$

Several comments are in order regarding the density approximation presented in (22). At the outset, note that the density approximation (22) requires three inputs for implementation: (1) the value of ρ to be determined from (21); (2) the Hermite polynomials, $H_j(\cdot)$, evaluated at $\rho(Z - \bar{Z})$: and (3) the expansion coefficients $\eta_j(\Delta, x_0; \theta)$ in terms of the initial x_0. Second, even though we treat Z as an integral of x in (23), only a numerical value of Z is needed in (22). Third, although the densities (11) and (22) are observationally similar, we depart from Aït-Sahalia (2002) by obtaining $\eta_j(\Delta, x_0; \theta)$ in terms of the moments of $(1/\Delta^{1/2}) \int_{x_0}^{X} du/\sigma(u; \theta)$. This is done by directly using the primitive $\mu(X; \theta)$ and $\sigma(X; \theta)$, thereby circumventing the use of $\mu_Y(y; \theta)$ to approximate the required moments. In this sense, the $p_X^{(J)}(\Delta, x|x_0; \theta)$ so approximated enables the maximum likelihood estimation for a wider class of one-dimensional diffusion models.

C. Recovering the Expansion Coefficients $\eta_j(\Delta, x_0; \theta)$

This subsection focuses on recovering the expansion coefficients $\eta_j(\Delta, x_0; \theta)$ in terms of the moments of Z. First, we fix $J = 6$ and determine the desired coefficients in terms of the moments of $\hat{Z}$ (suppressing the arguments of $\eta_j(\cdot,\cdot;\theta)$ for compactness and using [9]):

$$\eta_0 = 1, \tag{24}$$

$$\eta_1 = E[H_1(\hat{z})] = -E[\hat{z}] = 0, \tag{25}$$

$$\eta_2 = \frac{1}{2}E[H_2(\hat{z})] = \frac{1}{2}E[\hat{z}^2 - 1] = 0, \tag{26}$$

$$\eta_3 = \frac{1}{6}E[H_3(\hat{z})] = \frac{1}{6}E[-\hat{z}^3 + 3\hat{z}] = -\frac{1}{6}E[\hat{z}^3], \tag{27}$$

$$\eta_4 = \frac{1}{24}E[H_4(\hat{z})] = \frac{1}{24}E[-\hat{z}^4 - 6\hat{z}^2 + 3] = \frac{1}{24}E[\hat{z}^4] - \frac{1}{8}, \tag{28}$$

$$\eta_5 = \frac{1}{120}E[H_5(\hat{z})] = \frac{1}{120}E[-\hat{z}^5 + 10\hat{z}^3 - 15\hat{z}] = -\frac{1}{120}E[\hat{z}^5] + \frac{1}{12}E[\hat{z}^3], \tag{29}$$

$$\begin{aligned}\eta_6 &= \frac{1}{720}E[H_6(\hat{z})] = \frac{1}{720}E[\hat{z}^6 - 15\hat{z}^4 + 45\hat{z}^2 - 15] \\ &= \frac{1}{720}E[\hat{z}^6] - \frac{1}{48}E[\hat{z}^4] + \frac{1}{24}.\end{aligned} \tag{30}$$

In the next step, the moments of $\hat{Z}$ are expressed in those of Z. From (19), it is easy to see that the higher-moments of $\hat{Z}$ and Z are linked as

$$E[\hat{Z}^3] = \rho^3(E[Z^3] - 3\bar{Z}E[Z^2] + 2\bar{Z}^3), \tag{31}$$

$$E[\hat{Z}^4] = \rho^4(E[Z^4] - 4\bar{Z}E[Z^3] + 6\bar{Z}^2E[Z^2] - 3\bar{Z}^4), \tag{32}$$

$$E[\hat{Z}^5] = \rho^5(E[Z^5] - 5\bar{Z}E[Z^4] + 10\bar{Z}^2E[Z^3] - 10\bar{Z}^3E[Z^2] + 4\bar{Z}^5), \tag{33}$$

$$\begin{aligned}E[\hat{Z}^6] = \rho^6(&E[Z^6] - 6\bar{Z}E[Z^5] + 15\bar{Z}^2E[Z^4] - 20\bar{Z}^3E[Z^3] \\ &+ 15\bar{Z}^4E[Z^2] - 5\bar{Z}^6).\end{aligned} \tag{34}$$

Thus, the remaining task is to compute $E[Z^i] = E\{[(1/\Delta^{1/2})\int_{x_0}^{X} du/\sigma(u;\theta)]^i\}$, where i = 1, 2, 3, 4, 5, 6. For this, we rely on a result on the conditional expectation of a function of X.

LEMMA 1. For any generic one-dimensional diffusion process X_t governed by

$$dX_t = \mu(X_t;\theta)dt + \sigma(X_t;\theta)dW_t,$$

the conditional expectation $W(t,x) = \int G(w)p_w(t,w|x)dw$ satisfies the partial differential equation

$$\frac{1}{2}\sigma^2(x;\theta)\frac{\partial^2 W(t,x)}{\partial x^2} + \mu(x;\theta)\frac{\partial W(t,x)}{\partial x} = \frac{\partial W(t,x)}{\partial t}, \tag{35}$$

subject to the boundary condition $W(0,x) = G(x)$. Let $B(\theta) = \tfrac{1}{2}\sigma^2(x;\theta)(\partial^2/\partial^2 x) + \mu(x;\theta)(\partial/\partial x)$ represent the infinitesimal generator of the diffusion X_t. Then, the formal solution to equation (35) is given by

$$W(t,x) = e^{Bt}G(x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} B^j \cdot G(x), \tag{36}$$

for any function $G(x)$. Because the same moments are computed, the sufficient conditions on $[\mu(X_t;\theta), \sigma(X_t;\theta)]$ under which the series $\Sigma_{j=0}^{\infty}(t^j/j!)B^j \cdot G(x)$ converges coincide with those in proposition 4 of Aït-Sahalia (2002). Restrictions on the drift of $Y \equiv \int^X du/\sigma(u;\theta)$ can be mapped back into those on $\mu(X_t;\theta)$ and $\sigma(X_t;\theta)$.[1]

Finally, with the aid of lemma 1, we can show that the first six moments can be derived as (using *Mathematica* and with $K = 3$ for the expectation of the Hermites):

$$\begin{aligned} E[Z^1] \approx\; & \varphi_0\Delta^{1/2} + (2\varphi_0\varphi_1 + \varphi_2)\Delta^{3/2}/4 \\ & + [4\varphi_0^2\varphi_2 + 6\varphi_1\varphi_2 + 4\varphi_0(\varphi_1^2 + \varphi_3) + \varphi_4]\Delta^{5/2}/24, \end{aligned} \tag{37}$$

$$\begin{aligned} E[Z^2] \approx\; & 1 + (\varphi_0^2 + \varphi_1)\Delta + (6\varphi_0^2\varphi_1 + 4\varphi_1^2 + 7\varphi_0\varphi_2 + 2\varphi_3)\Delta^2/6 \\ & + [16\varphi_1^3 + 16\varphi_0^3\varphi_2 + 32\varphi_1\varphi_3 + 28\varphi_0^2(\varphi_1^2 + \varphi_3) \\ & + 8\varphi_0(11\varphi_1\varphi_2 + 2\varphi_4) + 3(7\varphi_2^2 + \varphi_5)]\Delta^3/48, \end{aligned} \tag{38}$$

$$\begin{aligned} E[Z^3] \approx\; & 3\varphi_0\Delta^{1/2} + (4\varphi_0^3 + 18\varphi_0\varphi_1 + 7\varphi_2)\Delta^{3/2}/4 + [6\varphi_0^3\varphi_1 + 13\varphi_0^2\varphi_2 \\ & + 15\varphi_1\varphi_2 + \varphi_0(16\varphi_1^2 + 9\varphi_3) + 2\varphi_4]\Delta^{5/2}/4, \end{aligned} \tag{39}$$

$$\begin{aligned} E[Z^4] \approx\; & 3 + 6(\varphi_0^2 + \varphi_1)\Delta + (\varphi_0^4 + 12\varphi_0^2\varphi_1 + 7\varphi_1^2 + 11\varphi_0\varphi_2 + 3\varphi_3)\Delta^2 \\ & + [80\varphi_0^4\varphi_1 + 240\varphi_1^3 + 280\varphi_0^3\varphi_2 + 241\varphi_2^2 + 368\varphi_1\varphi_3 \\ & + 20\varphi_0^2(27\varphi_1^2 + 17\varphi_3) + 4\varphi_0(290\varphi_1\varphi_2 + 43\varphi_4) + 31\varphi_5]\Delta^3/40, \end{aligned} \tag{40}$$

1. To avoid unnecessary duplication, we focus on how to map one such condition: near $\bar{y} = +\infty, \mu_Y(y) \le -Ky^{\beta^*}$ for some $K > 0$ and $\beta^* > 1$, which is analogous to the statement that, near $\bar{x}$, $[\mu(x)/\sigma(x)] - \tfrac{1}{2}[\partial\sigma(x)/\partial x] \le -K[\int^X du/\sigma(u)]^{\beta^*}$. Since the desired infinite integral may not be available, we restate this condition as a definite integral so that numerical integration can be employed to check its validity. We write $Z = \int_{X_0}^X du/\sigma(u) = \Gamma(x) - \Gamma(x_0)$, where $\Gamma(x) = \int_{X^*}^X du/\sigma(u)$ and x^* is some appropriately chosen lower limit. Hence, for $Z^* = \Gamma(x) = Z + \Gamma(x_0)$, Ito's lemma implies $dZ_t^* = (\mu[\Gamma^{-1}(Z^*)]/\sigma[\Gamma^{-1}(Z^*)] - \tfrac{1}{2}(\partial\sigma/\partial x)[\Gamma^{-1}(Z^*)])dt + dW_t$. Realize that these dynamics for Z^* coincide with that for Y. Therefore, if we want to compute the moments of Z^*, the desired condition $\mu_Y(y) \le -Ky^{\beta^*}$ for Y gets translated into $\mu_{Z^*}(z^*) \le -K'z^{*\beta^*}$ for Z^* provided $K' > 0$ and $\beta' > 1$. In terms of x, the technical condition takes the form $\mu(x)/\sigma(x) - \tfrac{1}{2}[\partial\sigma(x)/\partial x] \le -K'[\int_{X^*}^X du/\sigma(u)]^{\beta'}$.

$$E[Z^5] \approx 15\varphi_0\Delta^{1/2} + 5(8\varphi_0^3 + 30\varphi_0\varphi_1 + 11\varphi_2)\Delta^{3/2}/4 + [8\varphi_0^5 + 200\varphi_0^3\varphi_1 + 320\varphi_0^2\varphi_2 + 350\varphi_1\varphi_2 + 20\varphi_0(21\varphi_1^2 + 10\varphi_3) + 43\varphi_4]\Delta^{5/2}/8, \tag{41}$$

$$E[Z^6] \approx 15 + 45(\varphi_0^2 + \varphi_1)\Delta + 15(2\varphi_0^4 + 18\varphi_0^2\varphi_1 + 10\varphi_1^2 + 15\varphi_0\varphi_2 + 4\varphi_3)\Delta^2/2 + [16\varphi_0^6 + 720\varphi_0^4\varphi_1 + 1440\varphi_1^3 + 1760\varphi_0^3\varphi_2 + 1291\varphi_2^2 + 1968\varphi_1\varphi_3 + 60\varphi_0^2(59\varphi_1^2 + 31\varphi_3) + 24\varphi_0(275\varphi_1\varphi_2 + 37\varphi_4) + 157\varphi_5]\Delta^3/16, \tag{42}$$

where

$$\varphi_0 = \mu_0/\sigma_0 - \sigma_1/2, \tag{43}$$

$$\varphi_1 = \mu_1 - \mu_0\sigma_1/\sigma_0 - \sigma_0\sigma_2/2, \tag{44}$$

$$\varphi_2 = -\mu_1\sigma_1 + \mu_0(\sigma_1^2/\sigma_0 - \sigma_2) - \sigma_0(-2\mu_2 + \sigma_1\sigma_2 + \sigma_0\sigma_3)/2, \tag{45}$$

$$\varphi_3 = -\mu_0\sigma_1^3/\sigma_0 + \sigma_1(\mu_1\sigma_1 + 2\mu_0\sigma_2) - \sigma_0(2\mu_1\sigma_2 + \sigma_1^2\sigma_2/2 + \mu_0\sigma_3) - \sigma_0^2(\sigma_2^2/2 - \mu_3 + 3\sigma_1\sigma_3/2) - \sigma_0^3\sigma_4/2, \tag{46}$$

$$\varphi_4 = \mu_0\sigma_1^4/\sigma_0 - \sigma_1^2(\mu_1\sigma_1 + 3\mu_0\sigma_2) + \sigma_0[\sigma_1^2\mu_2 - \sigma_1^3\sigma_2/2 + 2\mu_0\sigma_2^2 + \sigma_1(2\mu_1\sigma_2 + \mu_0\sigma_3)] - \sigma_0^2(2\mu_2\sigma_2 + 2\sigma_1(\sigma_2^2 - \mu_3) + 3\mu_1\sigma_3 + 7\sigma_1^2\sigma_3/2 + \mu_0\sigma_4) + \sigma_0^3(-5\sigma_2\sigma_3/2 + \mu_4 - 3\sigma_1\sigma_4) - \sigma_0^4\sigma_5/2, \tag{47}$$

$$\varphi_5 = -\mu_0\sigma_1^5/\sigma_0 + \sigma_1^3(\mu_1\sigma_1 + 4\mu_0\sigma_2) - \sigma_0\sigma_1(4\mu_1\sigma_1\sigma_2 + \sigma_1^3\sigma_2/2 + 4\mu_0\sigma_2^2 + 2\mu_0\sigma_1\sigma_3) - \sigma_0^2[\sigma_1^2(11\sigma_2^2/2 - 5\mu_3) + 15\sigma_1^3\sigma_3/2 - 5\mu_0\sigma_2\sigma_3 + \mu_1(-4\sigma_2^2 + 3\sigma_1\sigma_3) + \mu_0\sigma_1\sigma_4] - \sigma_0^3(2\sigma_2^3 + 5\mu_2\sigma_3 + 37\sigma_1\sigma_2\sigma_3/2 - 5\sigma_1\mu_4 + 4\mu_1\sigma_4 + 25\sigma_1^2\sigma_4/2 + \mu_0\sigma_5) - \sigma_0^4(5\sigma_3^2/2 + 11\sigma_2\sigma_4/2 - \mu_5 + 5\sigma_1\sigma_5) - \sigma_0^5\sigma_6/2, \tag{48}$$

and $\mu_i \equiv \partial^i\mu(x_0;\theta)/\partial x_0^i$ and $\sigma_i \equiv \partial^i\sigma(x_0;\theta)/\partial x_0^i$, for $i = 1, 2, 3, 4, 5, 6$.

Our construction for determining the expansion coefficients $\eta_j(\Delta, x_0;\theta)$ deserves several remarks. One, each expansion coefficient is obtained in closed-form in terms of $\mu(X;\theta)$ and $\sigma(X;\theta)$ and their partial derivatives. As such, this method affords convenience and can be applied without having to construct $\mu_Y(y;\theta)$ in (5). Two, while the derived expressions appear cumbersome, they are straightforward to program. For implementation, $\eta_j(\Delta, x_0;\theta)$ are obtained by substituting (43)–(48) into (37)–(42) and (31)–(34) to get the moments of $\hat{Z}$. Allowing for possible modifications in μ_i and σ_i, the same computer code can be used for any $\mu(X;\theta)$ and $\sigma(X;\theta)$.

The method of this paper that allows $1/\Delta^{1/2}\int_{x_0}^{x} du/\sigma(u;\theta)$ to be determined numerically at the last stage of the procedure is related to the multivariate expansion for the "irreducible" case explored in Aït-Sahalia (2003). In this multivariate extension, the expansion coefficients satisfy a cascade of differential equations. To obtain analytical solutions, the expansion coefficients are Taylor expanded in $(x - x_0)$, which reduces the differential equations to a system of linear equations. The multivariate procedure thus relies on a double Taylor series expansion in $(x - x_0)$ and Δ. Both our approach and that of Aït-Sahalia (2002), however, obtain Hermite coefficients η_j by a Taylor expansion in Δ. The dependence of the density function of x is through the Hermite polynomials.

III. Application to Models of Interest-Rate and Stochastic Volatility

As emphasized earlier, when (3) and (5) can be done analytically, the Aït-Sahalia (2002) method is elegant and accurate. Examples in this class include, among others, the Vasicek process, the CIR square-root process, the inverse square-root process of Ahn and Gao (1999), and certain members in the constant elasticity of variance class. To show the promise of the proposed refinement, we now present specific examples where the true density function is not analytically known but can nonetheless be approximated using our approach.

Example 1. Suppose the short interest-rate is modeled as in Aït-Sahalia (1996), where the drift and diffusion are

$$\mu(X;\theta) = \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \alpha_3/X, \tag{49}$$

$$\sigma(X;\theta) = \sqrt{\beta_0 + \beta_1 X + \beta_2 X^{\beta_3}}. \tag{50}$$

The density function can easily be approximated based on (22), and the parameter vector $\theta \equiv (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3)$ can be estimated using maximum likelihood. We merely require the successive derivatives of $\mu(X;\theta)$ and $\sigma(X;\theta)$ with respect to X. For convenience, they are provided next (for initial value x_0):

$$\mu_0 = \mu(x_0) = \alpha_0 + \alpha_1 x_0 + \alpha_2 x_0^2 + \alpha_3/x_0, \tag{51}$$

$$\mu_1 = \mu'(x_0) = \alpha_1 + 2\alpha_2 x_0 - \alpha_3/x_0^2, \tag{52}$$

$$\mu_2 = \mu''(x_0) = 2\alpha_2 + 2\alpha_3/x_0^3, \tag{53}$$

$$\mu_3 = \mu^{(3)}(x_0) = -6\alpha_3/x_0^4, \tag{54}$$

$$\mu_4 = \mu^{(4)}(x_0) = 24\alpha_3/x_0^5, \tag{55}$$

$$\mu_5 = \mu^{(5)}(x_0) = -120\alpha_3/x_0^6. \tag{56}$$

and

$$\sigma_0 = \sigma(x_0) = \sqrt{\beta_0 + \beta_1 x_0 + \beta_2 x_0^{\beta_3}}, \tag{57}$$

$$\sigma_1 = \sigma'(x_0) = (\beta_1 + \beta_2\beta_3 x_0^{\beta_3 - 1})/(2\sigma_0), \tag{58}$$

$$\sigma_2 = \sigma''(x_0) = [\beta_2\beta_3(\beta_3 - 1)x_0^{\beta_3 - 2} - 2\sigma_1^2]/(2\sigma_0), \tag{59}$$

$$\sigma_3 = \sigma^{(3)}(x_0) = [\beta_2\beta_3(\beta_3 - 1)(\beta_3 - 2)x_0^{\beta_3 - 3} - 6\sigma_1\sigma_2]/(2\sigma_0), \tag{60}$$

$$\sigma_4 = \sigma^{(4)}(x_0) = [\beta_2\beta_3(\beta_3 - 1)(\beta_3 - 2)(\beta_3 - 3)x_0^{\beta_3 - 4} - 8\sigma_1\sigma_3 - 6\sigma_2^2]/(2\sigma_0), \tag{61}$$

$$\begin{aligned}\sigma_5 = \sigma^{(5)}(x_0) = [&\beta_2\beta_3(\beta_3 - 1)(\beta_3 - 2)(\beta_3 - 3)(\beta_3 - 4)x_0^{\beta_3 - 5} \\ &- 10\sigma_1\sigma_4 - 20\sigma_2\sigma_3]/(2\sigma_0),\end{aligned} \tag{62}$$

$$\begin{aligned}\sigma_6 = \sigma^{(6)}(x_0) = [&\beta_2\beta_3(\beta_3 - 1)(\beta_3 - 2)(\beta_3 - 3)(\beta_3 - 4)(\beta_3 - 5)x_0^{\beta_3 - 6} \\ &- 12\sigma_1\sigma_5 - 30\sigma_2\sigma_4 - 20\sigma_3^2]/(2\sigma_0).\end{aligned} \tag{63}$$

Substituting the expressions for the derivatives of $\mu(X;\theta)$ in (51)–(56) and the derivatives of $\sigma(X;\theta)$ in (57)–(63) into (43)–(48), we get the required moments for Z.

Example 2. As another parametric example outside of the Bessel class, suppose the stock price, S_t, is driven by a generalized constant elasticity of variance process (as outlined in case 2). The drift and diffusion of $X_t \equiv \log(S_t)$ are, respectively,

$$\mu(X;\theta) = \mu_S(e^X;\theta) - \frac{1}{2}\sigma^2(X;\theta), \tag{64}$$

$$\sigma(X;\theta) = \beta_0 + \beta_1 e^{-\beta_2 X}. \tag{65}$$

Consistent with our procedure, the derivatives of $\sigma(X;\theta)$ and $\mu(X;\theta)$ with respect to X are given by

$$\sigma_0 = \sigma(x_0) = \beta_0 + \beta_1 e^{-\beta_2 x_0}, \tag{66}$$

$$\sigma_1 = \sigma'(x_0) = -\beta_1\beta_2 e^{-\beta_2 x_0}, \tag{67}$$

$$\sigma_2 = \sigma''(x_0) = \beta_1\beta_2^2 e^{-\beta_2 x_0}, \tag{68}$$

$$\sigma_3 = \sigma^{(3)}(x_0) = -\beta_1\beta_2^3 e^{-\beta_2 x_0}, \tag{69}$$

$$\sigma_4 = \sigma^{(4)}(x_0) = \beta_1\beta_2^4 e^{-\beta_2 x_0}, \tag{70}$$

$$\sigma_5 = \sigma^{(5)}(x_0) = -\beta_1\beta_2^5 e^{-\beta_2 x_0}, \tag{71}$$

$$\sigma_6 = \sigma^{(6)}(x_0) = \beta_1\beta_2^6 e^{-\beta_2 x_0}, \tag{72}$$

and

$$\mu_0 = \mu(x_0) = \mu_S(e^{x_0}; \theta) - \sigma_0^2/2, \tag{73}$$

$$\mu_1 = \mu'(x_0) = \mu'_S(e^{x_0}; \theta) - \sigma_0\sigma_1, \tag{74}$$

$$\mu_2 = \mu''(x_0) = \mu''_S(e^{x_0}; \theta) - \sigma_1^2 - \sigma_0\sigma_2, \tag{75}$$

$$\mu_3 = \mu^{(3)}(x_0) = \mu_S^{(3)}(e^{x_0}; \theta) - 3\sigma_1\sigma_2 - \sigma_0\sigma_3, \tag{76}$$

$$\mu_4 = \mu^{(4)}(x_0) = \mu_S^{(4)}(e^{x_0}; \theta) - 3\sigma_2^2 - 4\sigma_1\sigma_3 - \sigma_0\sigma_4, \tag{77}$$

$$\mu_5 = \mu^{(5)}(x_0) = \mu_S^{(5)}(e^{x_0}; \theta) - 10\sigma_2\sigma_3 - 5\sigma_1\sigma_4 - \sigma_0\sigma_5, \tag{78}$$

for any stock price drift function $\mu_S(S; \theta)$. Armed with these expressions, we arrive at the conditional density of the log stock price by appealing to the series approximation in (22). In sum, the refinement offers flexibility and renders the basic methodology applicable to an expanded class of one-dimensional diffusion processes.

IV. Accuracy of the Method and Comparison of Expansion Coefficients $\eta_j(\Delta, x_0; \theta)$

One remaining central issue concerns the accuracy of the refinement and whether using our $\hat{Z} = \Sigma^{-1/2}(Y - \bar{Y})$ reduces the magnitude of the expansion coefficients $\eta_j(\Delta, x_0; \theta)$. To explore these points, we first compare the approximate density using Aït-Sahalia basic (i.e., based on [11]) and Aït-Sahalia enhanced (i.e., based on [15]), with that derived in (22).

We consider three candidate interest-rate models for which the conditional density function is known in closed-form:

$$dX_t = \begin{cases} \kappa(\mu - X_t)dt + \sigma dW_t & \text{Vasicek (1977)}, \\ \kappa(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t & \text{Cox-Ingersoll-Ross (1985)}, \\ \kappa(\mu - X_t)X_t dt + \sigma\sqrt{X_t^3}dW_t & \text{Ahn-Gao (1999)}. \end{cases} \tag{79}$$

As recommended by Aït-Sahalia (1999), our yardstick for testing accuracy relies on comparing the maximum absolute error of the approximate density relative to its exact counterpart. That is, in what follows, we set $J = 6$ and $K = 3$ and compute $\text{MAXE} = \max|p^{\text{exact}}(\Delta, x|x_0) - p^{\text{approx}}(\Delta, x|x_0)|$. Given an initial value, this criterion gauges the worst possible approximation error. In essence, a superior approximation method produces a lower value of maximum absolute error. For benchmarking this error, we also present the maximum exact conditional density, denoted as $\max(p|x_0)$.

Several points can be made based on the accuracy results shown in table 1. First, it highlights the finding in Aït-Sahalia (1999, 2002) that approximating the density of the transformed variable Y around a standard normal distribution using the Hermites is indeed good. With $\Delta = 1/12$, the tests indicate that the proposed method is likely to be very accurate for many conditional densities.

Second, our refinement reduces maximum absolute errors several-fold with respect to Aït-Sahalia basic but not with respect to the enhanced formulae (15). To illustrate this point, we note that, for the Cox-Ingersoll-Ross model and $x_0 = 10\%$, MAXE using the approximation in Aït-Sahalia basic is $5.25(\times 10^{-5})$, $0.14(\times 10^{-5})$ using (22) and $0.31(\times 10^{-8})$ using Aït-Sahalia enhanced. The fact that the limit $J \to \infty$ is taken in the enhanced formula (15) explains why this method is far more accurate. Overall, this exercise demonstrates a potential trade-off between the methodology in Aït-Sahalia enhanced and the refinement in (22): the former is more precise but the latter provides additional applicability by not requiring the closed-form integration of $1/\sigma(u;\theta)$.

To understand the impact of adopting different standardization schemes for Z, in table 2, we report and compare the magnitude of the Hermite coefficients η_1 through η_6 in (25)–(30) with the counterparts in Aït-Sahalia basic. Observe that using the true mean and variance in the calculation of $\hat{Z} = \Sigma^{-1/2}(Y - \bar{Y})$ reduces the coefficients η_3 through η_6 in comparison to $Z = \Delta^{-1/2}(Y - y_0)$ in Aït-Sahalia basic. This improvement pattern in η_j is observed regardless of the level of x_0 (by construction our $\eta_1 = 0$ and $\eta_2 = 0$).

V. Concluding Remarks

In theory, the density approximation developed in Aït-Sahalia (2002) holds for any general diffusion process $dX_t = \mu(X_t;\theta)dt + \sigma(X;\theta)dW_t$.

TABLE 1 Maximum Absolute Errors for Interest Rate Models

	Initial Interest Rate, x_0	.02	.04	.06	.08	.10	.12	.14	.16	.18
Vasicek	$\max(p\|x_0)(\times 10^1)$	6.31	6.31	6.31	6.31	6.31	6.31	6.31	6.31	6.31
	MAXE:									
	Aït-Sahalia basic [$\times 10^{-4}$]	9.49	1.14	5.10	5.65	.21	8.17	13.17	5.16	28.70
	This article [$\times 10^{-5}$]	.86	3.18	2.04	1.51	3.31	.59	7.24	10.82	13.41
	Aït-Sahalia enhanced [$\times 10^{-7}$]	1.28	1.23	.15	.10	1.09	.49	10.49	17.27	124.99
Cox-Ingersoll-Ross	$\max(p\|x_0)(\times 10^1)$	15.0	10.7	8.71	7.55	6.75	6.17	5.71	5.34	5.04
	MAXE:									
	Aït-Sahalia basic [$\times 10^{-5}$]	58.77	21.51	11.32	1.11	5.25	8.27	8.60	6.75	3.07
	This article [$\times 10^{-6}$]	5.12	4.24	2.81	3.14	1.45	.59	2.24	3.00	2.75
	Aït-Sahalia enhanced [$\times 10^{-8}$]	89.65	4.11	1.33	.22	0.31	.11	1.36	2.83	3.26

NOTE.—All reported calculations are based on $\Delta = 1/12$. For each initial value of the short interest rate (denoted x_0), the entries corresponding to $\max(p|x_0)$ represent the maximum exact conditional density. That is, $\max(p|x_0) = \max$ of $p^{\text{exact}}(\Delta, x|x_0)$ over x. We also report the maximum absolute errors (MAXE), which is the maximum value of $|p^{\text{exact}}(\Delta, x|x_0) - p^{\text{approx}}(\Delta, x|x_0)|$ over x. Setting $J = 6$ and $K = 3$, this calculation is reported for (i) the density approximation in Aït-Sahalia basic, (ii) the density approximation presented in equation (22), and (iii) the density approximation based on Aït-Sahalia enhanced (i.e., equations [11] and [12] in Aït-Sahalia 1999 or [4.11] and [4.12] in Aït-Sahalia 2002). The results are presented separately for two interest-rate processes: (1) $dX_t = \kappa(\mu - X_t)dt + \sigma dW_t$ (Vasicek 1977), and (2) $dX_t = \kappa(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t$ (Cox, Ingersoll, and Ross 1985). The structural parameters used in the density calculations for the Vasicek and the Cox-Ingersoll-Ross model are adopted from Aït-Sahalia (1999). Specifically, for the Vasicek model, we set $\kappa - 0.258$, $\mu = 0.0717$, and $\sigma = 1.02213$. For the CIR model, we set $\kappa = 0.145$, $\mu = 0.732$, and $\sigma = 0.0621$. The results for $dX_t = \kappa(\mu - X_t)X_t dt + \sigma\sqrt{X_t^3}dW_t$ (Ahn and Gao 1999) are similar and not presented to save on space. For the Ahn-Gao model, we take $\kappa = 3.4387$, $\mu = 0.0828$, and $\sigma = 1.1920$. The final expressions for the approximate densities for each interest-rate model are rather lengthy and omitted (but available from the authors).

TABLE 2 Comparison of $\eta_j(\Delta, x_0; \theta)$ Coefficients for Interest Rate Models: Aït-Sahalia Basic versus the Refinement in (22)

	Initial Interest Rate, x_0	.02	.04	.06	.08	.10	.12	.14	.16	.18
Vasicek	η_1, Aït-Sahalia [$\times 10^{-1}$]	−1.72	−1.06	−.39	.27	.94	1.61	2.27	2.94	3.61
	η_2, Aït-Sahalia [$\times 10^{-3}$]	4.22	−5.03	−9.84	−10.22	−6.16	2.33	15.26	32.62	54.42
	η_3, Aït-Sahalia [$\times 10^{-4}$]	9.74	9.22	4.03	−2.89	−8.59	−10.11	−4.50	11.18	39.90
	η_3, this article [$\times 10^{-7}$]	−2.16	−3.33	−1.61	1.16	3.19	2.64	−2.29	−13.42	−32.57
	η_4, Aït-Sahalia [$\times 10^{-5}$]	−6.93	.17	5.09	5.51	1.25	−5.73	−11.34	−9.33	8.68
	η_4, this article [$\times 10^{-8}$]	1.51	−2.55	−3.08	−3.08	−2.77	.36	11.58	38.93	93.18
	η_5, Aït-Sahalia [$\times 10^{-6}$]	−1.94	−4.10	−2.17	1.57	4.02	2.64	−3.13	−10.54	−12.09
	η_5, this article [$\times 10^{-7}$]	−.26	−1.57	−.90	.66	1.58	.64	−2.31	−5.61	−5.05
	η_6, Aït-Sahalia [$\times 10^{-7}$]	2.96	.66	−1.63	−1.85	.19	2.69	2.74	−1.92	−10.18
	η_6, this article [$\times 10^{-8}$]	1.01	−1.20	−1.08	−1.02	−1.27	.35	6.08	13.54	8.39
Cox-Ingersoll-Ross	η_1, Aït-Sahalia [$\times 10 - ^{2}$]	−20.62	−8.26	−1.54	3.18	6.89	9.98	12.66	15.04	17.19
	η_2, Aït-Sahalia [$\times 10^{-3}$]	8.60	−4.50	−6.17	−4.97	−2.61	.32	3.59	7.06	10.67
	η_4, Aït-Sahalia [$\times 10^{-4}$]	7.37	4.08	.13	−2.23	−3.28	−3.29	−2.46	−.91	1.24
	η_3, this article [$\times 10^{-5}$]	−42.81	−15.44	−8.46	−5.51	−3.95	−3.01	−2.39	−1.96	−1.65
	η_4, Aït-Sahalia [$\times 10^{-5}$]	−4.95	1.42	1.84	.93	−.23	−1.22	−1.89	−2.14	−1.92
	η_4, this article [$\times 10^{-7}$]	−205.27	−53.47	−24.09	−13.65	−8.77	−6.09	−4.47	−3.40	−2.66
	η_5, Aït-Sahalia [$\times 10^{-7}$]	−16.91	−9.64	1.86	6.51	6.26	3.10	−1.44	−6.19	−10.20
	η_5, this article [$\times 10^{-8}$]	−110.86	−20.62	−6.35	−2.44	−1.42	−1.44	−1.82	−2.21	−2.40
	η_6, Aït-Sahalia [$\times 10^{-8}$]	11.38	−3.28	−3.07	−.46	1.71	2.73	2.524	1.28	−.69
	η_6, this article [$\times 10^{-9}$]	23.95	2.61	1.04	.36	.30	.75	1.51	2.28	2.76

NOTE.—All calculations are based on $\Delta = 1/2$. For each initial value of the short interest rate (denoted x_0), we report the magnitude of $\eta_j(\Delta, x_0; \theta)$ used in (i) the density approximation in Aït-Sahalia basic and (ii) the density approximation presented in equation (22). The structural parameters for the interest rate models are the same as in table 1. As seen from equations (25) and (26), $\eta_1 = 0$ and $\eta_2 = 0$ in our approach and therefore are suppressed in the tabulations.

However, in practice, the method can be applied only in the narrower diffusion class for which $1/\sigma(u;\theta)$ can be analytically integrated. Even when this integration is possible, the method demands the closed-form inversion of the function $\int^X du/\sigma(u;\theta)$. In either scenario, the likelihood function of the observations cannot be explicitly computed.

In this article, we developed a methodological refinement to Aït-Sahalia (2002). We show how to determine the density function of X_t in a less restrictive environment. A key feature of this refinement is that it renders the basic methodology applicable in a setting where the integral transformation in (3) and the inverse function (5) cannot be performed analytically. Our density characterization is based on two building blocks. First, we adopt a standardization that uses the true mean and variance of $\int^X du/\sigma(u;\theta)$. Next, for any given $\mu(X_t;\theta)$ and $\sigma(X_t;\theta)$, we show how to obtain the approximate moments of $\int^X du/\sigma(u;\theta)$ by applying the Taylor theorem to a generic function of X.

Our examples establish that the refinement generates closed-form density approximations for a wider class of $\mu(X_t;\theta)$ and $\sigma(X_t;\theta)$. The numerical implementation suggests that this refinement produces approximate densities that are sufficiently accurate but nonetheless less precise than the enhanced formulae when $J \to \infty$ in the original density approximation (11). Among various applications, the refinement can be employed to empirically compare alternative models of short interest-rates (or stochastic volatility models describing equity returns) for which the exact conditional density is generally unavailable. This approximation method not only rests on sound theoretical foundations but has promising applications in the field of financial economics.

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