Deducing the Implications of Jump Models for the Structure of Stock Market Crashes, Rallies, Jump Arrival Rates, and Extremes

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This article studies the structure of stock market crashes, rallies, their jump arrival rates, and extremes. Large market moves are characterized in a pure-jump modeling framework. Based on both raw and devolatized returns, it is shown empirically that crashes are more severe in intensity than rallies, and have higher arrival rates. At the same time, both left-tail and right-tail extreme events conform with Fréchet limit laws. Pure-jump models which describe well the tail properties of market returns are identified via their Lévy measures. The distribution of extreme events implied by our model's Lévy measure is closer to the actual realization of extremes than those of competing models. Finally, there is information content in the Lévy measure of pure-jump models for forward arrival rate of jumps.

KEY WORDS: Arrival rates; Crashes; Extremes; Jump structure; Lévy measure; Limit laws; Pure-jump price processes; Rallies.

1. INTRODUCTION

The purpose of this article is to examine the structure of stock market crashes, rallies, their arrival rates, and extremes. We characterize large moves in a pure-jump modeling framework, and we show empirically that pure-jump models can more aptly capture the tail properties of market returns. Compared to the extant literature, we make contributions in several dimensions by addressing four questions: Have equity markets experienced a higher number of crashes than rallies? How distinct are the left- and right-tails of market returns? Are they governed by different limit laws? What properties must be shared by a theoretical model class to match the patterns of observed market extremes and the arrival rate of jumps?

At the center of financial economics is the model of Merton (1976), who treated stock prices as jump-diffusions with Poisson intensity of jumps and Gaussian jump distribution. Jump-diffusions possess the feature that their path is continuous except for occasional discontinuities. While jump-diffusions have proved flexible in modeling large perturbations, they are susceptible to the drawback that the densities of the diffusion component, that surrogates small moves, and of the jump component, that surrogates large moves, are analytically detached. As a possible remedy, we exploit a parsimonious one-dimensional Lévy pure-jump model for market returns. Such pure-jump models are suited for our study since they can generate asymmetric jump arrival rates and jump sizes, which allow for a better differentiation between the left- and the right-tail. Economic and statistical considerations that argue for pure-jump stock price models can be found in Madan and Seneta (1990), Eberlein and Keller (1995), Barndorff-Nielsen (1998), Madan, Carr, and Chang (1998), Barndorff-Nielsen and Shephard (2001), Eberlein (2001), Carr et al. (2002), Huang and Wu (2004), Cont and Tankov (2004), Wu (2006), Bakshi, Carr, and Wu (2008), Aït-Sahalia and Jacod (2008), Li, Wells, and Yu (2008), Jacod and Todorov (2009), and Todorov (2009).

In contrast to classic models, the source of randomness in our model is a Brownian motion evaluated at a gamma directing process (e.g., see, among others, Madan and Seneta 1990; Conley et al. 1997; Madan, Carr, and Chang 1998; and Carr and Wu 2004). The directing process can be motivated by information arrival, represented by some measure of economic activity. For instance, it is volume in Clark (1973), number of trades in Ané and Geman (2000), and volatility in Carr et al. (2003) and Barndorff-Nielsen and Shephard (2006). Intuitively, a Brownian motion law in economic time instead of calender time provides the economic underpinnings for the model. The resulting price process (i) has non-Gaussian local increments, (ii) is pure-jump, devoid of any continuous martingale components, and (iii) possesses a tractable return characteristic function with finite moments of all orders. More distinctively, by appealing to the Lévy–Khintchine theorem, the Lévy measure is derivable in analytical closed-form. Special to our theoretical framework, the Lévy measure controls the arrival rate of jumps over the entire continuum; whether large or small and whether negative or positive. Conforming with the observed dynamics of crashes and rallies, the higher the jump size, the lower are the respective jump arrival rates. Based on the parametric form of the Lévy

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measure, the distribution of the largest percentage price fluctuation is derived analytically. Furthermore, it is shown that the returns process in our theoretical model is in the domain of attraction of the fat-tailed Fréchet limit law.

Our empirical investigation employs daily data on the Dow Jones Industrial Average (DJIA) from the beginning of 1897 to the end of 2007, and we use both raw and devolatized returns. Return devolatization can be motivated by the many studies that show strongly time-varying volatility (e.g., Nelson 1991; Bollerslev, Chou, and Kroner 1992; and Engle 2004), and allows us to reconcile our focus on Lévy return models, implying independent and time-homogeneous increments, with the observed properties of stock returns data. To see that devolatization can highlight tail events, consider the significance of the -3.35% drop in the DJIA on March 27, 2007, in terms of devolatization. Given that the post 1946 daily volatility is around 0.9%, this single-day drop materializes into a 3.8-sigma event in the raw returns data. At the same time, devolatization accentuates the raw move into a -7.56% drop, which translates into a 8.4-sigma event. In fact, it is one of the five largest drops in the devolatized time series post 1946.

Based on raw returns, we find that the probability of a daily stock market decline in excess of 5% is nonnegligible; about 0.25%. There are 69 days on which the stock market has dropped by more than 5%. But a market rally of 5% or higher is observed only 52 times. Moreover, market crashes are not only more likely to occur than rallies with higher crash arrival rates, but are substantially more severe. The pre 1946 crash valuation measures depart radically from the post 1946 counterpart with the left-tail decaying to zero much slower than the right-tail. To emphasize further the distinction between the two tails, we construct a time series of left- and right-tail events measured by the maximum daily absolute percentage decline and the maximum daily percentage rise, respectively, over fixed block sizes. We find a positive spread between the left-tail and right-tail extremes when the block size is 42, 84, and 126 days. Many of the features of raw returns are more pronounced in devolatized returns.

Next we examine whether the structure of jumps implied by our model is consistent with the arrival rates of jumps of various sizes, as observed in the data. When the log arrival rate of jumps is regressed on a constant, the jump size, and the log jump size, as specified by the functional form of the log Lévy measure in our model, there are no violations of the restriction that each estimated coefficient be negative. In contrast, the Cox and Ross (1976) jump-model with log Lévy measure that is quadratic in the jump size, and the Das and Foresi (1996) and Kou (2002) models with log Lévy measure that is linear in the jump size are both rejected in our performance horse-races. Further test of empirical specification uncovers the finding that our model is consistent with the Carr et al. (2002) model.

Relying on the Fisher–Tippett theorem we investigate the limit laws of extremes. Estimation results show that both left-tail and right-tail extremes have limiting Fréchet distributions. Consistent with the observed pattern of the extremes, the estimated parameters of the Fréchet densities imply that crashes are more probable than rallies, and post 1946 markets are less inclined to extreme fluctuations. The Fréchet tail-indexes reveal that the distribution of right-tail events has thinner tails than that of the left-tail events. Supporting the viability of our pure-jump modeling framework from another perspective, we also find that returns simulated from the model exhibit limit law properties that bear resemblance to actual return data.

Finally, motivated by the notion that Lévy measures are sufficient (in theory) to pin down the arrival rate of jumps, we pursue an empirical specification to assess whether there is information content in the Lévy measure, estimated in an earlier period for jump arrival rates in a later period. Predictive regressions reveal that the current Lévy measure contains information for forward arrival rates.

This article is organized as follows. Section 2 develops a pure-jump representation of the price process. We present the Lévy measure of the model and all other models used in the empirical investigation. Section 3 describes the procedure applied to devolatize returns. Section 4 highlights features of crashes and extremes in the stock market and the structure of jump arrival rates. The purpose of Section 5 is to validate models based on their consistency with the observed jump arrival rates and jump sizes. Section 6 reports our findings on the limit laws of extremes constructed from left-tail and right-tail events based on devolatized returns as well as those from simulated returns. Plausibility of the tail probability model is evaluated through a predictive exercise in Section 7. Conclusions are in Section 8.

2. MODEL OF MARKET CRASHES, RALLIES, JUMP ARRIVAL RATES, AND EXTREMES

Fix the probability space as $(\Omega, \mathcal{F}, \mathbb{P})$. Unless stated otherwise, all conditional expectations, $\mathbb{E}_t(\cdot)$, are taken under the objective probability measure and according to the filtration generated by $\mathcal{F}_t$. Denote the per share price of the market index by $\{S(t), t \in [0, \Upsilon]\}$ and the logarithmic rate of return as

$$R(0,t) \equiv \ln S(t) - \ln S(0). \tag{1}$$

To preserve tractability of theoretical analysis, we maintain the assumption of stationary and independently distributed returns and, hence, adopt a Levy process to model index returns. However, in the empirical investigation of model implications we adjust our data for variations in return volatility.

Our treatment of market crashes and rallies essentially incorporates jumps in market index returns. Assuming that the continuous-time price process has a left limit and is right continuous, we formally define a jump of any size in the market index as in Merton (1976):

$$z(u) \equiv \ln S(u) - \ln S(u_-), \qquad z(u) \in (-\infty, +\infty). \tag{2}$$

We note that in our modeling setup jumps occur at inaccessible (surprise) times and the probability of a jump at any fixed time is zero.

Furthermore, the biggest positive or negative jump that occurs over any fixed interval $(0, b)$ is a well-defined object. Define the entities

$$M_b^- \equiv \left|\min_{s\in(0,b)} z(s)\right|, \qquad M_b^- \in [0, +\infty), \tag{3}$$

$$M_b^+ \equiv \max_{s\in(0,b)} z(s), \qquad M_b^+ \in [0, +\infty), \tag{4}$$

which, respectively, represent the absolute value of the largest instantaneous percentage price decline and the largest percentage rise in the stock price. M_b^- captures the worst possible instantaneous loss for a long investor and M_b^+ captures the worst possible instantaneous loss for a short investor.

The entities defined in Equations (3) and (4) allow us to formalize a framework for modeling the extreme return fluctuation, whether positive or negative. Our theoretical interest lies in studying the laws $\mathrm{Prob}(M_b^- \geq K)$ and $\mathrm{Prob}(M_b^+ \geq K)$ for $K > 0$. Abstracting from the specific parent distribution governing local fluctuations in returns, limit laws of the extremes will be obtained in the sense of Fisher and Tippett (1928) (see also Kendall and Stuart 1977 and Embrechts, Kluppelberg, and Mikosch 1997).

2.1 Pure-Jump Return Dynamics

We focus on one-dimensional Lévy (pure-jump) processes in modeling market returns. The hallmark of a pure-jump model is that its path is nowhere continuous and every move constitutes a jump. Our contention is that pure-jump models generate asymmetric jump arrival rates and jump sizes, which allow for a better differentiation between the left- and the right-tail. That is, they offer the versatility to capture diverse tail properties of market returns. Furthermore, they present a parsimonious alternative to jump-diffusions in modeling extreme market moves. Note that the proposed pure-jump model shares an important property with diffusions, which, at the same time is not exhibited by jump-diffusions. Diffusions are associated with Gaussian limit laws. In a similar way, pure-jump models of the class considered are associated with infinitely divisible limit laws, hence, they generalize over the Gaussian local law of motion, but preserve the limit law property. In contrast, it is provable that jump-diffusions do not satisfy the limit law. Specifically, the generic property of limit law that $|z|\Pi(z)$, for Lévy measure $\Pi(z)$, be decreasing for $z > 0$ and $|z|\Pi(z)$ be increasing for $z < 0$, does not hold for the jump diffusion model of Merton (1976).

Following Madan and Seneta (1990), Madan, Carr, and Chang (1998), and Carr et al. (2002), let the market index evolve as

$$\ln S(t) - \ln S(0) = (\mu + \omega)t + g(t), \qquad \omega \equiv \frac{1}{\kappa}\ln\left(1 - \theta\kappa - \frac{1}{2}\kappa\sigma^2\right), \tag{5}$$

$$g(t) = \theta y(t) + \sigma B(y(t)), \qquad g(t) \mid y(t) \sim \mathcal{N}(\theta y(t), \sigma^2 y(t)), \tag{6}$$

$$y(t) \sim \mathrm{Gamma}(t, \kappa t), \quad \text{with density} \quad \Phi(y) = \frac{\kappa^{-t/\kappa}}{\Gamma(t/\kappa)} y^{t/\kappa - 1} e^{-y/\kappa}, \tag{7}$$

where $B(t)$ represents a standard Brownian motion, and $B(y(t))$ a standard Brownian motion evaluated at a random (gamma) time $y(t)$. Realize that $B(t_1)$ and $B(t_2)$ are correlated, but that does not imply return predictability, as return is modeled as the difference and, hence, is not autocorrelated. While the level of $B(t_2)$ is predictable, given $B(t_1)$, the increment $B(t_2) - B(t_1)$ is unpredictable. Here $y(t)$ governs the evolution of the time-change or the directing process. Refining the diffusion paradigm, the second source of randomness is obtained by superimposing stochastic time-changes on a standard Brownian motion.

Intuitively, the random time $y(t)$ can be abstractly thought as representing aggregate economic activity and is, thus, distinct from calender time. However, the attributes of subordination in our model are different from others: it is lognormally distributed volume in Clark (1973), it is number of trades in Ané and Geman (2000), and it is volatility in Carr et al. (2003) and Barndorff-Nielsen and Shephard (2006). Absent economic activity, there is no time change. Thus, we have the Brownian motion law in economic time rather than in calender time.

In our approach, $g(t)$ is a stand-in for the variance-gamma process (e.g., Madan, Carr, and Chang 1998). Let $B(t)$ and $y(t)$ be independent for all t. Conditional on $y(t)$, the stochastic process $g(t)$ is distributed normal with mean $\theta y(t)$ and variance $\sigma^2 y(t)$. This model structure facilitates simulation of the increments of the process over interval Δt: $g(t + \Delta t) - g(t) = (\theta y)\Delta t + \sigma\sqrt{y}\tilde{z}\sqrt{\Delta t}$, where $\tilde{z}$ is $\mathcal{N}(0, 1)$ and $y \sim$ Gamma (see Ribeiro and Webber 2003). The characteristic function of $g(t)$ is

$$F(0, t; \phi) \equiv \mathbb{E}_0\left(e^{i\phi g(t)}\right) = \left(\frac{1}{1 - i\phi\theta\kappa + \phi^2\kappa\sigma^2/2}\right)^{t/\kappa}. \tag{8}$$

It can be shown that $\sigma > 0$, θ, and $\kappa > 0$ parameterize higher moments of the return distribution and the sign of skewness in the market return is the same as that of θ. The compensator $\omega = \frac{1}{\kappa}\ln(1 - \theta\kappa - \frac{1}{2}\kappa\sigma^2)$ in Equation (5) is there to guarantee that the drift of the market index μ satisfies $\mathbb{E}_0(S(t)) = S_0 e^{\mu t}$.

The continuous-time price process in Equation (5) is conceptually and theoretically attractive. First, the price process posited in Equation (5) is a semimartingale and consequently arbitrage-free. It inherits from traditional models the trait that the return distribution has well-defined algebraic moments up to all orders. Additionally, as the time change is distributed gamma, the return distribution is stable under additivity.

By relying on the return specification Equation (5) and the characteristic function in Equation (8), the return characteristic function $F_R(0, t; \phi) \equiv \mathbb{E}_0(e^{i\phi R(0,t)})$ takes the form

$$F_R(0, t; \phi) = \exp\left(i\phi\mu t + \frac{i\phi t}{\kappa}\ln\left(1 - \theta\kappa - \frac{1}{2}\sigma^2\kappa\right)\right) \times \left(\frac{1}{1 - i\phi\theta\kappa + \phi^2\kappa\sigma^2/2}\right)^{t/\kappa}, \tag{9}$$

which is crucial for deriving the structure of jumps by applying the Lévy–Khintchine integral representation of the log characteristic function. It follows from Equation (9) that $F_R(0, t; \phi) = \exp(-t\Psi(\phi))$, where $\Psi(\phi)$ is the characteristic exponent, hence, the returns process also has independent and time-homogeneous increments.

Denoting the adjusted return as $\bar{R}(0, t) \equiv R(0, t) - \mu t - \frac{t}{\kappa}\ln(1 - \theta\kappa - \frac{1}{2}\kappa\sigma^2)$ and recognizing that conditional on $y(t)$, $\bar{R}(0, t)$ is normal, it follows that the transition density function, $\Phi(R(0, t))$, is

$$\Phi(R(0, t)) = \frac{2\exp(\theta\bar{R}(0, t)/\sigma^2)\bar{R}(0, t)^{\upsilon}}{\sqrt{2\pi}\sigma\Gamma(\upsilon + 1/2)\kappa^{1/2+\upsilon}\beta^{\upsilon/2}}\mathbb{K}_{\upsilon}\left(\frac{\sqrt{\beta}}{\sigma^2}\bar{R}(0, t)\right), \tag{10}$$

where $\mathbb{K}_a(b)$ represents the modified Bessel function of the second kind, $\upsilon \equiv \frac{t}{\kappa} - \frac{1}{2}$, and $\beta \equiv \theta^2 + \frac{2\sigma^2}{\kappa}$. Based on Equation (10) it may be verified from Johnson, Kotz, and Balakrishnan (1994) that the stock price relative,

$$\exp(R(0,t)) - 1 \equiv r(0,t) \sim \text{Loggamma}. \tag{11}$$

In principle, one can rely on the density function in Equation (10) or the transformed density for $r(0,t)$ for implementing the pure-jump model.

The price process in Equation (5) is (i) locally non-Gaussian and (ii) a pure-jump process with no continuous martingale components. This can be verified by observing that any continuous price process with a finite variance rate is locally Gaussian (e.g., Revuz and Yor 1991), which the price process in Equation (5) fails to satisfy. The absence of a continuous martingale component is not critical as small jumps of highly active jump processes can closely mimic diffusion dynamics (Aït-Sahalia and Jacod 2009). In fact, such small jumps are often simulated by a Gaussian component.

2.2 The Lévy Measure and the Structure of Jumps

Associated with each continuous-time stochastic jump process of homogeneous independent increments is a Lévy measure that governs the instantaneous arrival rate of jumps of all sizes, whether negative or positive. As previously defined, let z denote the size of the instantaneous jump in the log price. Since $z > 0$ or $z < 0$, denote the Lévy measure of positive jumps by $\Pi^+(z)$ and the negative counterpart by $\Pi^-(z)$.

2.2.1 Parametric Form of the Lévy Measure. Applying the Lévy–Khintchine Theorem (Jacod and Shiryaev 1987; Bertoin 1996; and Sato 1999) to the present pure-jump model, the Lévy measures $\Pi^+(z)$ and $\Pi^-(z)$ satisfy

$$\ln F_R(0,t;\phi) - \left(i\phi\varrho t - t\int_{-\infty}^{0} (e^{i\phi z} - 1)\Pi^-(z)\,dz - t\int_{0}^{+\infty} (e^{i\phi z} - 1)\Pi^+(z)\,dz \right) = 0. \tag{12}$$

Substituting the return characteristic function in Equation (9) into Equation (12) and setting $\varrho = \mu + \frac{1}{\kappa}\ln(1 - \theta\kappa - \frac{1}{2}\kappa\sigma^2)$, we obtain the Lévy measure (hereby, LM)

$$\Pi(z) = \begin{cases} \Pi^+(z) \equiv \dfrac{e^{-\lambda^+ z}}{\kappa z} & \text{if } z > 0 \\[2ex] \Pi^-(z) \equiv \dfrac{e^{-\lambda^- |z|}}{\kappa\,|\,z\,|} & \text{if } z < 0, \end{cases} \tag{13}$$

where the derived parameters $\lambda^+ > 0$ and $\lambda^- > 0$ are, respectively,

$$\lambda^+ \equiv -\frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\kappa\sigma^2}} \quad \text{and} \quad \lambda^- \equiv \frac{\theta}{\sigma^2} + \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\kappa\sigma^2}}. \tag{14}$$

The LM postulated in Equation (13) is central to our framework. Since $\int_0^{+\infty}\Pi^+(z)\,dz = \int_{-\infty}^{0}\Pi^-(z)\,dz = +\infty$, the LM is consistent with infinitely many jumps per unit time. But if jumps of size less than ϵ (say, the minimum tick size) are excluded, both $\int_{\epsilon}^{+\infty}\Pi^+(z)\,dz$ and $\int_{-\infty}^{-\epsilon}\Pi^-(z)\,dz$ stay finite. One may then interpret $\Pi^+(z)$ as the instantaneous arrival rate of positive jumps of size z with conditional density $\Pi^+(z)/\int_{\epsilon}^{+\infty}\Pi^+(z)\,dz$ and a jump having occurrence probability $\int_{\epsilon}^{+\infty}\Pi^+(z)\,dz$ (properly normalized). Clearly, the arrival rate of positive (negative) jumps of size greater than K must be: $\int_K^{+\infty}\Pi^+(z)\,dz$ $[\int_{-\infty}^{-K}\Pi^-(z)\,dz]$. Furthermore, as the stochastic process of the arrival of jumps of size $z > K$ is Poisson, the probability of no (instantaneous) arrival of the jump is $\exp(-\int_K^{+\infty}\Pi(z)\,dz)$.

In short, the LM embodies complete probabilistic information about the arrival rate of jumps, both positive and negative. There is a relationship between the arrival rates of small and large moves, but not in the occurrence of events as the moves are occurring at inaccessible times. Thus, there is no causal relation between small and large events.

The LM in Equation (13) is downward-sloping with $\frac{\partial\Pi(z)}{\partial z} = -\Pi(z)(\lambda + \frac{1}{z}) < 0$ and $\frac{\partial^2\Pi(z)}{\partial z^2} = \Pi(z)((\lambda + \frac{1}{z})^2 + \frac{1}{z^2}) > 0$. In fact, the LM is *completely monotone* with $(-1)^j\frac{\partial^j\Pi(z)}{\partial z^j} > 0$ for all z and for all j, and implies the coexistence of high arrival rates of small jumps (positive or negative) and the low arrival rates of large jumps.

When $\theta \neq 0$, the LM assigns an unequal weight to the arrival of negative versus positive moves, i.e., $\int_{\epsilon}^{+\infty}\Pi^+(z)\,dz$ can be higher or lower than $\int_{\infty}^{-\epsilon}\Pi^-(z)\,dz$. Thus, it is possible to reconcile that (i) positive jumps overall have higher arrival rates than negative jumps, and (ii) big positive jumps possess lower arrival rates compared to big negative jumps (the crash probability is higher).

Before proceeding, recall from theorem 5 in Clark (1973) that the return characteristic function is not in analytical closed-form. Consequently, the Lévy–Khintchine integral representation is not solvable, and hence the Lévy measure and the structure of price jumps cannot be recovered. For this reason, Clark's model of time-changes is of limited value in studying jump arrival rates and extremes.

2.2.2 Lévy Measures for Competing Models. To put the main ideas in perspective, consider first the classic Merton (1976) jump-diffusion model where uncertainty is driven by a diffusion component and an orthogonal jump component:

$$\frac{dS(t)}{S(t)} = \sigma\,dB(t) + J(t)\,dq(t), \qquad \ln(1 + J(t)) \sim \mathcal{N}\big(\ln(1+\mu_J) - \tfrac{1}{2}\sigma_J^2, \sigma_J^2\big). \tag{15}$$

$J(t)$ is the jump amplitude and $q(t)$ is a Poisson jump counter with intensity λ_J. Setting diffusion volatility $\sigma = 0$ results in the Cox and Ross (1976) jump model: $\frac{dS(t)}{S(t)} = J(t)\,dq(t)$. Given that its return characteristic function is $\exp(\lambda_J t[\exp(i\phi\ln(1 + \mu_J) - \frac{1}{2}i\phi\sigma_J^2 + \frac{1}{2}(i\phi)^2\sigma_J^2) - 1])$, we apply the Lévy–Khintchine representation in Equation (12) and deduce the Lévy measure

$$\Pi^{\text{CR}}(z) = \lambda_J\frac{1}{\sqrt{2\pi}\sigma_J}\exp\left(-\frac{(z - \ln(1+\mu_J) + 1/2\sigma_J^2)^2}{2\sigma_J^2}\right), \quad -\infty < z < +\infty, \tag{16}$$

which is the probability of a Poisson jump multiplied by the density of the percentage jump size. It also represents the Lévy measure of the Merton (1976) model.

Departing from Merton (1976) and Cox and Ross (1976), the model of Das and Foresi (1996) considered an exponential distribution alternative for $J(t)$. With Bernoulli probability of positive (negative) jumps being $\vartheta(1-\vartheta)$, the analytical form of the Lévy measure is

$$\Pi^{\mathrm{DF}}(z) = \begin{cases} \Pi^{+}(z) \equiv \vartheta \varsigma e^{-\varsigma z} & \text{if } z > 0 \\ \Pi^{-}(z) \equiv (1-\vartheta)\varsigma e^{-\varsigma |z|} & \text{if } z < 0, \end{cases} \tag{17}$$

for some constant ς of the exponential distribution. The model of Kou (2002) shares the same lineage as Equation (17), and the Lévy measure of both models is nested within Equation (13).

Of possible interest in our context are models where the arrival rate of jumps are not only linked to jump density, but are also regulated by a random diffusion component. To be precise, suppose one takes the model of Merton (1976) and Cox and Ross (1976) and adopts a modification where the Poisson intensity of jumps is a stochastic process (maintaining Gaussian jump distribution). In the option model of Bates (2000), for instance, $dq(t) = 1$, with Probability$(dq(t) = 1) = \lambda(t)\,dt = (\lambda_0 + \lambda_v V(t))\,dt$, where $V(t)$ follows a square-root diffusion process with linear drift. Hence, $\lambda(t)$, which is a constant in Merton (1976) and Cox and Ross (1976), is linear in diffusive volatility. The arrival rate of jumps takes the form: $(\lambda_0 + \lambda_v V(t))\exp(-(z - \ln(1+\mu_J) + \frac{1}{2}\sigma_J^2)^2/(2\sigma_J^2))$. A likewise attribute is shared by the model of Santa-Clara and Yan (2009), where $\lambda(t)$ is quadratic in latent volatility. See Carr et al. (2003) for another model class where the arrival rate of jumps of size z is represented by $(\lambda_0 + \lambda_v V(t))\widehat{\Pi}(z)$, and $\widehat{\Pi}(z)$ represents Lévy jump density. The overall impact of such models is to enhance jump activity for both small and large jump sizes in volatile markets. These models, thus, introduce another layer of complexity where the volatility process must be parameterized, and are outside of our present empirical scope.

Finally, Carr et al. (2002) have generalized the Lévy measure in Equation (13) to

$$\Pi^{\mathrm{CGMY}}(z) = \begin{cases} \Pi^{+}(z) \equiv \dfrac{e^{-\lambda^{+} z}}{\kappa z^{1+\xi}} & \text{if } z > 0 \\ \Pi^{-}(z) \equiv \dfrac{e^{-\lambda^{-}|z|}}{\kappa \mid z \mid^{1+\xi}} & \text{if } z < 0. \end{cases} \tag{18}$$

If $\xi < 0$, it supports finite activity. On the other hand $\xi > 0$ implies infinite activity, and $\xi > 1$ implies infinite variation. If $\xi = 0$, we recover the Lévy measure of Equation (13).

Although appealing in their own right, we choose not to pursue three other pure-jump models in the interest of brevity. The first model is the generalized hyperbolic model (Eberlein, Keller, and Prause 1998). Its Lévy density is expressed in modified Bessel functions of the first and second kind. Second, from equation (2.9) in Barndorff-Nielsen (1998), the Lévy measure of the Normal Inverse Gaussian model is: $\Pi^{NIG}(x) = \frac{\delta\zeta \exp(\beta z)\mathbb{K}_1(\zeta|z|)}{\pi|z|}$, for some parameters ζ, $\delta > 0$, and β and $\mathbb{K}_\nu(\cdot)$ is the modified Bessel function of the second kind. The third model is the infinite variation model of Schoutens (2003), which has Lévy measure: $\Pi^{Meixner}(z) = \delta \frac{\exp(\beta z/\zeta)}{z\sinh(\pi z/\zeta)}$, for parameters $\zeta > 0$, $|\beta| < \pi$, and $\delta > 0$.

Lévy measures for the candidate models derived in Equations (13), (16), (17), and (18) induce important testable restrictions on the jump arrival rates. The Lévy measure of Equation (13) is a core building block and we determine which model can mimic the jump arrival rates of size z in equity markets.

2.3 Distribution of the Extremes

The analyticity of the Lévy measure for the pure-jump model in Equation (13) is crucial for deriving the probabilistic properties of return extremes. We now present the following result:

Theorem 1. For the pure-jump stock price process proposed in Equations (5) to (7), let $M_b^- = |\min_{s\in(0,b)} z(s)|$ denote the absolute value of the largest instantaneous logarithmic price decline, and let $M_b^+ = \max_{s\in(0,b)} z(s)$ denote the largest instantaneous logarithmic price rise. Then,

$$\mathrm{Prob}(M_b^- \geq K) = 1 - \exp\left(-\frac{b}{\kappa}\,\mathrm{Ei}(K\lambda^-)\right), \tag{19}$$

$$\mathrm{Prob}(M_b^+ \geq K) = 1 - \exp\left(-\frac{b}{\kappa}\,\mathrm{Ei}(K\lambda^+)\right), \tag{20}$$

where $\mathrm{Ei}(\ell) = \int_\ell^\infty \frac{e^{-w}}{w}\,dw$ denotes the exponential integral function, and λ^+ and λ^- are the previously defined parameters of the Lévy measure found in Equation (14).

Proof. Consider the distribution of the maximum absolute logarithmic price decline. The arrival rate of a drop larger (in absolute value) than $K > 0$ is given by $\int_{-\infty}^{-K} \Pi^-(z)\,dz$, where $\Pi^-(z) = \frac{e^{-\lambda^-|z|}}{\kappa|z|}$. By a standard argument, the process of arrivals of drops larger than K is Poisson and the probability of no arrival of such drops is thus:

$$\begin{aligned} \mathrm{Prob}(M_b^- < K) &= \exp\left(-b\int_{-\infty}^{-K} \Pi^-(z)\,dz\right) \\ &= \exp\left(-b\int_{K}^{\infty} \frac{e^{-\lambda^- z}}{\kappa z}\,dz\right), \end{aligned} \tag{21}$$

which is the distribution function of M_b^-. The complementary function is $1 - \mathrm{Prob}(M_b^- < K)$, and Equation (19) follows by a change-of-variable $\lambda^- z = w$ in Equation (21). The distribution of the maximum logarithmic price rise in Equation (20) is derived in the same way.

Theorem 1 derives the exact distribution of M_b^- and M_b^+ over $s \in (0, b)$. Higher return moments are the sole determinants of tail event probabilities, which are (i) increasing in σ and κ, and (ii) decreasing and convex in θ. If parameters σ, θ, and κ can be estimated from equity markets, one could infer the exact probability of M_b^- and M_b^+. We hypothesize tail asymmetry with $\mathrm{Prob}(M_b^- \geq K) > \mathrm{Prob}(M_b^+ \geq K)$. The exact distribution of extremes for Cox and Ross (1976), Das and Foresi (1996), and Carr et al. (2002) models can also be deduced from their respective Lévy measure. Details are omitted here.

2.4 Limit Law of Extremes and Connections With the Fisher–Tippett Theorem

According to the Fisher–Tippett theorem (e.g., Embrechts, Kluppelberg, and Mikosch 1997, p. 121), the limit distribution of the greatest value among n independent variables each having the same continuous distribution, as $n \to +\infty$, must be in the collection of either (i) the Gumbell (defined on $\Re$), or (ii) the Fréchet (defined on $\Re^+$), or (iii) the Weibull distribution (defined on $\Re^+$). Since $M_b^- \in \Re^+$ and $M_b^+ \in \Re^+$, the Gumbell distribution is excluded as a limit law for extremes in our setup.

To enable our log return based extreme-value theory to conform with the counterpart in Embrechts, Kluppelberg, and Mikosch (1997), consider the transformations

$$m_b^- = \exp(M_b^-) - 1, \qquad m_b^+ = \exp(M_b^+) - 1, \tag{22}$$

so m_b^+ reflects the maximum relative price rise instead of the maximum logarithmic price rise. Then, regardless of the time change and irrespective of the parametric form of the Lévy measure, by adopting the model-free central limit theorem result for extremes we have either $\eta^\alpha m_b \xrightarrow{d}$ Fréchet with limit density:

$$\Phi_F[m] = \alpha\eta^\alpha \exp\left(-\left(\frac{m}{\eta}\right)^{-\alpha}\right) m^{-\alpha-1}, \qquad m = \{m_b^-, m_b^+\}, m \in \Re^+, \tag{23}$$

or, alternatively, $\eta^{-\alpha} m_b \xrightarrow{d}$ Weibull with limit density:

$$\Phi_W[m] = \alpha\eta^{-\alpha} \exp\left(-\left(\frac{m}{\eta}\right)^{\alpha}\right) m^{\alpha-1}, \qquad m = \{m_b^-, m_b^+\}, m \in \Re^+. \tag{24}$$

In both limit laws the tail behavior is captured by the shape parameter α (the tail-index).

It must be borne in mind that the limit laws for m_b^- and m_b^+ can differ and the tails can be inhomogeneous. Specifically one tail can converge to zero at a high (lower) speed than the other. Or, one tail can be in the domain of attraction of the Fréchet and the other tail in the domain of attraction of the Weibull.

Since the parent distribution of price relative $r(0, t)$ is loggamma from Equation (11), it is provable from example 3.3.11 (p. 134) in Embrechts, Kluppelberg, and Mikosch (1997) that loggamma is in the domain of attraction of (fat-tailed) Fréchet distribution. This is a testable implication that can be potentially verified through the simulation of the pure-jump return process. The novelty of this implication is that we can solely concentrate on the behavior of large stock price movements.

Benchmarking the appropriate limit law for the local price fluctuation can be potentially useful in model assessment: it can help direct the search for the class of pricing models that can ultimately be consistent with a theory of the stock market in the tails. The distribution function of the largest movement in Equations (19) and (20), and the associated limit theory, can also assist investors in managing the risk of extreme events (e.g., Longin 1996 and Kearns and Pagan 1997).

3. EMPIRICAL APPROACH TO RETURN DEVOLATIZATION

To assess model implications and study tail properties, we choose a time series of index returns and focus on 111 years of daily price observations on the Dow Jones Industrial Average (hereby, DJIA). Data on DJIA is available from the first day of 1897 to the last trading day in 2007, with 30,150 observations.

When raw returns are adopted in the empirical analysis a possible difficulty arises. Specifically, if returns are posited to be a pure-jump Lévy process driven by stationary and independent increments with constant volatility, the assumptions may refute certain empirical facts. It is possible that return data over protracted periods is inconsistent with the assumption of identical distribution, given the recognition of at least time-varying volatility. One avenue for modeling refinement clearly would be to introduce stochastic volatility, which is a pervasive feature of stock returns (e.g., Nelson 1991; Bollerslev, Chou, and Kroner 1992; Barndorff-Nielsen and Shephard 2001; Maheu and McCurdy 2004; and references therein).

To mitigate the effect of time-varying volatility on raw returns, each observation is devolatized. With the view to internalizing moving volatility in the context of returns driven by a Lévy process (e.g., Sato 1999), we adopt the specification for the returns process $X = (X(t))_{t\geq 0}$ below:

$$dX(t) = v(t)\, dL(t), \tag{25}$$

where $v(t)$ denotes the process of volatility, and $L(t)$ is a pure-jump Lévy process with stationary and independent increments, independent from $v(t)$. Such a decomposition potentially purges the influence of volatility and allows us to focus on the properties of extremes related to return jumps, without incorporating Lévy building blocks that admit stochastic volatility (as in Carr et al. 2003).

To exploit estimation methods in discrete time consider the return process for the price series:

$$\begin{aligned} S(\Delta t) &= S(0)\exp(X(\Delta t)), \\ S(2\Delta t) &= S(0)\exp(X(2\Delta t)), \\ &\vdots \\ S(T) &= S(0)\exp(X(T)), \end{aligned} \tag{26}$$

where $S(t)$ is the level of the DJIA at date t. Under the representation in Equation (26), $\Delta X(t) = X(t + \Delta t) - X(t)$ is the logarithmic return over $(t, t + \Delta t)$. Accordingly, the characterization in Equation (25) is

$$\begin{aligned} \Delta X(t) &= v(t)\Delta L(t), \qquad \text{and hence,} \\ \ln(\Delta X(t)^2) &= \ln(v(t)^2) + \ln(\Delta L(t)^2), \end{aligned} \tag{27}$$

which allows us to interpret log squared returns as signal $[\ln v(t)^2]$ plus additive noise, as advocated in the approach of Eberlein, Kallsen, and Kristen (2002). Applying nonparametric smoothing techniques to extract the signal as in Hastie and Tibshirani (1990), we use a running-mean smoother to obtain

$$\ln \widehat{v^2(t)} = \frac{1}{I}\sum_{i=0}^{I-1} \ln\big((\Delta X(t-i))^2\big), \tag{28}$$

Table 1. Optimal window to devolatize returns

I	15	20	25	30	35	40	45	50	55	60	65	70	75	80
$\widehat{\mathrm{CV}}$	0.41	0.46	0.48	0.47	0.45	0.43	0.40	0.42	0.46	0.51	0.51	0.57	0.61	0.64

where the smoothing parameter I is chosen through cross-validation (see also Andreou and Ghysels 2002; Barndorff-Nielsen and Shephard 2002; and Andersen et al. 2003). To implement this scheme we minimize

$$\widehat{\mathrm{CV}} = \frac{1}{T}\sum_{t=1}^{T}\left|\ln\big((\Delta X(t))^2\big) - \frac{1}{I}\sum_{i=1}^{I}\ln\big((\Delta X(t-i))^2\big)\right|. \quad (29)$$

We estimate $\widehat{\mathrm{CV}}$ on a discrete set of values for $I = \{15, 20, 25, \ldots, 80\}$ and choose the smallest $\widehat{\mathrm{CV}}$ among the set of values. Based on daily DJIA returns over the full sample, we find in Table 1.

Hence the optimal window to devolatize our returns data is 45 days. The estimates $v(t)$ are scaled to ensure that the variance of $\Delta L(t)$ is unity, as suggested in Eberlein, Kallsen, and Kristen (2002).

4. EMPIRICAL PROPERTIES OF JUMP ARRIVAL RATES AND EXTREMES

Before we can examine the theoretical predictions of Lévy return models with respect to the observed jump structure, we first describe the empirical properties of the tails. At the outset, we present nonparametric characterizations of daily returns and devolatized returns. Then, we examine the attributes of realized stock market extremes M_b^- and M_b^+. In the final subsection, we document asymmetries in jump arrival rates and then shed light on the empirical probabilities of crashes and rallies.

4.1 Basic Features of Raw Returns and Devolatized Returns

The focal point of Table 2 is daily raw returns (Panel A) and devolatized returns (Panel B) and their empirical attributes. To make comparisons meaningful, the devolatized series is standardized to match the volatility of raw returns. Such an approach merely scales the devolatized return distribution by a fixed constant. The point that needs to be emphasized here is that accounting for return volatility substantially reduces the kurtosis of the devolatized returns. The order of reduction is 50% over 1897–2007 sample period and 70% over the 1946–2007 sample period.

Table 2 reveals an inherently puzzling empirical regularity of the stock markets with respect to price movements: the daily stock market crashes are harsh relative to stock market rallies. The largest daily percentage DJIA price decline of 25.63% (on October 19, 1987), for instance, is of substantially higher magnitude relative to the maximum daily percentage DJIA rise of 13.86% (on October 6, 1931). Over the 1897–2007 sample, the average across the ten largest crashes is −11.35% (cross-sectional standard deviation of 5.38%) compared to 9.94% (cross-sectional standard deviation of 1.68%) across the ten largest rallies. Another asymmetry exists between crashes and rallies in the 1946–2007 sample with average across the ten largest crashes (rallies) being −8.65% (5.76%).

Furthermore, the post 1946 equity market is more resilient to external shocks: with the exception of the 1987 crash, the amplitude of crashes and rallies are markedly different between the pre and post 1946 stock markets. The largest 10 single-day rallies pre 1946 range between 8.35% and 13.86%, while the corresponding range of single-day rallies is 4.78% to 9.67% in the post 1946 sample. From the beginning of 1946 through the end of 2007, there are only 10 large downward movements beyond 5%.

Even though the impact of devolatization is to generate a markedly lower kurtosis, the amplitude asymmetry between crashes and rallies is clearly detected in devolatized returns as well. Other than magnifying the dichotomy between the tails, the devolatized return distribution shares qualitatively similar features of crashes and rallies explicit in the raw return data. Furthermore, based on conventional criteria, the autocorrelations of raw returns and devolatized returns are negligible and do not exceed 0.06 and 0.09 in absolute value for raw and devolatized returns, respectively.

Why are daily market price declines much larger in absolute value than daily price rises? The divergence between the intensities of crashes and rallies presents a challenge for theoretical models of market return dynamics. We will revisit this theme in the ensuing discussion.

4.2 Comparative Behavior of Extremes M_b^- and M_b^+

Even though in the real world there may be time-variation in the neck of the return distribution, we explore the hypothesis that if one looks away from the center of the empirical distribution and puts aside small movements, it can be conjectured that tail movements are close to iid. That is, the operation of taking maximum on daily movements over a block of observations focuses attention to the tails, and allows us to investigate large movements in either direction. There may be value in looking at the laws of tail movements as it exemplifies features that can be used to build more realistic models of local motion.

Thus, guided by our findings on the amplitude of daily crashes and rallies, we are prompted to ask: Is the comparative behavior of the left-tail unique? What time series evidence can be brought to bear on the behavior of extremes and tail events?

To go to the heart of these questions using extreme value theory, we, henceforth, proxy jumps with daily return moves and consider a sequence of iid variables $\{z(i)\}_{i=1}^N$. By dividing the entire dataset into n nonoverlapping subsamples and taking the maximum, $M^-(j)$ or $M^+(j)$, from every subsample, we end up with a sequence of maxima, $\{M^-(j)\}_{j=1}^n$ and $\{M^+(j)\}_{j=1}^n$ whose limit law is characterized by the Fisher–Tippett theorem.

With the view to balance concerns with respect to maxima obtained over shorter block sizes versus longer block sizes, we experimented with block size, b, of 42 days (2 months), 84 days

Table 2. Raw and devolatized Dow Jones industrial average daily returns

	Panel A: Raw returns			Panel B: Devolatized returns		
	1897–2007 (full)	1897–1945	1946–2007 (post WW–II)	1897–2007 (full)	1897–1945	1946–2007 (post WW–II)
Average	4.70%	2.50%	6.70%	6.34%	4.34%	7.41%
Std. dev.	16.89%	19.34%	14.29%	16.89%	19.34%	14.29%
Skewness	−0.70	−0.27	−1.57	−0.84	−0.84	−0.84
Kurtosis	24.58	13.60	50.13	12.14	9.76	14.60
NOBS	30,150	14,389	15,761	30,150	14,389	15,761
min(1)	−25.63%	−13.72%	−25.63%	−19.06%	−14.24%	−16.39%
min(2)	−13.72%	−12.48%	−8.38%	−12.67%	−14.20%	−10.90%
min(3)	−12.48%	−10.44%	−7.45%	−12.66%	−11.38%	−9.17%
min(4)	−10.44%	−9.13%	−7.16%	−12.61%	−11.03%	−7.57%
min(5)	−9.13%	−8.78%	−7.10%	−10.67%	−9.48%	−6.52%
min(6)	−8.78%	−8.65%	−6.77%	−10.11%	−9.31%	−5.94%
min(7)	−8.65%	−8.17%	−6.58%	−9.80%	−8.84%	−5.72%
min(8)	−8.38%	−8.07%	−5.88%	−8.80%	−8.77%	−5.34%
min(9)	−8.17%	−7.52%	−5.82%	−8.43%	−8.73%	−5.19%
min(10)	−8.07%	−7.42%	−5.72%	−8.27%	−8.14%	−5.00%
Average	−11.35%	−9.44%	−8.65%	−11.31%	−10.41%	−7.77%
Std. dev.	5.38%	2.13%	6.02%	3.22%	2.25%	3.58%
max(1)	13.86%	13.86%	9.67%	8.08%	7.09%	6.95%
max(2)	11.64%	11.64%	6.53%	6.30%	6.74%	5.42%
max(3)	10.76%	10.76%	6.15%	6.30%	6.44%	5.36%
max(4)	9.67%	9.09%	5.72%	6.24%	6.37%	5.34%
max(5)	9.09%	9.05%	5.27%	6.21%	5.94%	4.62%
max(6)	9.05%	8.94%	4.95%	5.99%	5.93%	4.40%
max(7)	8.94%	8.94%	4.86%	5.72%	5.89%	4.27%
max(8)	8.94%	8.79%	4.84%	5.66%	5.62%	4.21%
max(9)	8.79%	8.69%	4.81%	5.37%	5.50%	4.12%
max(10)	8.69%	8.35%	4.78%	5.27%	5.21%	3.99%
Average	9.94%	9.81%	5.76%	6.11%	6.07%	4.87%
Std. dev.	1.68%	1.75%	1.51%	0.79%	0.58%	0.91%
$\rho(1)$	0.03	0.01	0.06	0.07	0.05	0.09
$\rho(2)$	−0.03	−0.02	−0.04	−0.02	−0.02	−0.03
$\rho(3)$	0.00	0.01	−0.01	0.02	0.04	0.01
$\rho(4)$	0.03	0.06	−0.01	0.03	0.06	0.01
$\rho(12)$	0.01	0.00	0.02	0.02	0.01	0.02

NOTE: Reported are average, standard deviation, skewness, kurtosis, minimum (calculated as the largest percentage daily price drop), and maximum (calculated as the largest percentage daily price rise). The average return and standard deviation are annualized by respectively scaling the daily counterparts by 252 and $\sqrt{252}$. The autocorrelation coefficient at lag j is denoted by $\rho(j)$. Here $\{\min(j)\}_{j=1}^{10}$ ($\{\max(j)\}_{j=1}^{10}$) are the ordered largest negative (positive) daily moves. NOBS denotes the number of observations. The first trading day for DJIA is January 2, 1897, and the last day is December 31, 2007 (111 years of daily data). Devolatized returns are calculated as $\Delta X_t/\widehat{\sigma}_t$, where $\ln \widehat{\sigma}_t^2 = \frac{1}{I}\sum_{i=0}^{I-1}\log((\Delta X_{t-i})^2)$, with an optimally chosen I set to 45 days. Devolatized returns are scaled to equalize the variance of raw returns and devolatized returns in each sample period.

(4 months), and 126 days (6 months) resulting in n equal to 717, 358, and 239, respectively.

The results reported in Table 3 merit some remarks. First, an investor with a long (short) position in the DJIA can be expected to experience a maximum daily loss of 3.20% (2.87%) every six months. Second the series of left-tail extremes $\{M^-(j)\}_{j=1}^n$ is far more volatile with kurtosis many times that of the right-tail extremes $\{M^+(j)\}_{j=1}^n$. Third, in raw returns, the right-tail extremes are substantially more autocorrelated compared to the left-tail extremes and show slow decay even up to a longer lag. In other words, right-tail extremes have longer memories with large movements followed by movements of similar size (and the reverse), while left-tail extremes tend to be more idiosyncratic, which may reconcile why such events are traditionally difficult to hedge a priori. In contrast, both right-tail extremes and left-tail extremes in devolatized returns show little evidence for autocorrelation.

Next, we apply the Kolmogorov–Smirnov statistic to test the null hypothesis that the left- and the right- tail events belong to the same distribution.

	42 days	84 days	126 days
Raw returns, p-value	0.46	0.09	0.11
Devolatized returns, p-value	0.00	0.00	0.00

The p-values of the Kolmogorov–Smirnov statistic reported above indicate that the null hypothesis cannot be rejected for raw returns, but is rejected on devolatized returns. In sum, accounting for time-varying return volatility in our devolatiza-

Table 3. Behavior of extremes, M_b^- and M_b^+

	Raw returns						Devolatized returns					
	Left-tail extreme (M_b^-)			Right-tail extreme (M_b^+)			Left-tail extreme (M_b^-)			Right-tail extreme, (M_b^+)		
Block size, b (days)	42	84	126	42	84	126	42	84	126	42	84	126
n	717	358	239	717	358	239	717	358	239	717	358	239
Average	2.27%	2.80%	3.20%	2.19%	2.59%	2.87%	2.36%	2.92%	3.34%	2.15%	2.52%	2.78%
Std. dev.	1.67%	2.04%	2.30%	1.43%	1.61%	1.78%	1.35%	1.64%	1.82%	0.77%	0.83%	0.89%
Skewness	5.30	5.00	4.78	3.12	2.85	2.69	4.25	3.79	3.52	1.95	1.93	1.75
Kurtosis	59.61	47.86	41.4	17.6	15.16	13.28	34.5	25.63	21.41	9.49	8.49	7.12
Minimum	0.50%	0.78%	0.78%	0.52%	0.74%	0.82%	0.72%	1.10%	1.41%	0.78%	1.29%	1.54%
Maximum	25.63%	25.63%	25.63%	13.86%	13.86%	13.86%	17.37%	17.37%	17.37%	7.36%	7.36%	7.36%
$\rho(1)$	0.35	0.26	0.27	0.59	0.48	0.46	0.05	0.05	0.10	0.01	−0.10	−0.03
$\rho(2)$	0.24	0.24	0.19	0.45	0.43	0.40	0.03	0.05	0.01	−0.05	−0.01	0.03
$\rho(3)$	0.25	0.19	0.14	0.47	0.42	0.38	0.06	0.01	0.05	0.03	0.02	0.03
$\rho(4)$	0.20	0.14	0.21	0.42	0.33	0.33	0.03	0.04	0.22	−0.01	0.04	0.13
$\rho(12)$	0.17	0.05	−0.03	0.26	0.07	−0.04	0.11	0.01	0.03	0.01	−0.05	0.00

NOTE: For this exercise we fix a block size b for daily returns and set it equal to 42 days, 84 days, and 126 days. By dividing the entire dataset into n nonoverlapping subsamples of length b and taking the maximum, $M^-(j)$ or $M^+(j)$, from every subsample, we obtain a series of maxima, $\{M^-(j)\}_{j=1}^n$ and $\{M^+(j)\}_{j=1}^n$. Reported are the average, standard deviation, skewness, kurtosis, minimum, and maximum of the respective series. The autocorrelation coefficient at lag j is denoted by $\rho(j)$.

tion procedure accentuates the distinction between the left- and right-tails.

4.3 Historical Probabilities of Crashes and Rallies

In view of our goal to examine whether the observed arrival rates of negative and positive price jumps conform to various Lévy measures, we decompose daily price fluctuations into their positive and negative constituents. We subdivide the universe of possible negative jump sizes into 21 classifications ranging from <0%, ≤−0.5%, ≤−1.0%, ..., <−10% (in increments of −0.5%) and the same for positive return jumps. The zero return observations are excluded and hence the total number of observations drops to 29,980.

Concentrate on the heading marked "Negative Jumps" in Table 4. For this jump classification, we calculate the number of instances a stock price jump of size less than or equal to, say, 5% has occurred. Record this statistic as "Count." Then, the *probability* of the stock price jump of the same size is "Count" divided by the universe of all jump sizes. We record this statistic under the heading "Prob."

Tables 4 and 5 offer a coherent picture of the arrival rate of crashes to rallies and also their historical probabilities of occurrence. The main empirical findings are as follows.

First, the probability of a positive jump in the DJIA of all sizes, whether raw or devolatized, surpasses the negative counterpart by about 5%. For example, over the entire 111 years, the market declined on 14,215 days and rose on 15,765 days. This outcome translates into a 47.41% probability of a decline and a 52.59% of a stock market rise. The decline probability is 47.65% during the 1946–1997 DJIA period.

Second, rally and crash probabilities of the same magnitude exhibit pronounced asymmetries in both raw returns and devolatized returns. Considering the full period, as often as 321 (251) times, the raw DJIA declined (rose) more than 3%. More fundamentally, on any given day, the market declined (rose) by more than 5% on 69 (52) occasions. Overall, this amounts to a 0.23% probability for a daily decline of 5% or higher, and 0.17% for a surge of 5% or more. Along the same lines, a daily catastrophic drop of 10% or higher has been observed four times in the entire period while surges exceeding this magnitude have occurred three times. The higher probability of crashes poses a puzzle: why have equity markets experienced a higher number of crashes than rallies?

Third, crash and rally frequencies differ radically depending on whether one is considering the post or pre 1946 stock markets. Out of the total 69 crashes in the DJIA of 5% or higher, 59 crashes (or 86%) were confined to pre 1946 period, and only 10 to the post 1946 period. Out of 52 rallies, only 5 are attributable to the post 1946 period. Not only has the probability of a crash decreased dramatically in the post 1946 period, the probability of a rally has also declined.

Fourth, we can extract the *jump arrival rates* and normalized jump arrival rates directly from Table 4. We report these under "Arrival rate" and "Norm. arrival" in Table 5. The normalized arrival rate of, e.g., a negative jump size −2% to −2.5% can be recovered by counting all jumps in this range and then dividing the count by the number of negative jumps of all sizes.

Based on Table 5, several points can be established. One, jumps of small magnitude have strictly higher arrival rates than jumps of larger magnitude. This pattern is particularly clear in devolatized returns. Two, during the post 1946 sample, the majority of the jumps (positive or negative) are of relatively smaller magnitude. For example, 80.41% (81.44%) of the negative (positive) jumps are concentrated between 0 to −1%. The arrival rates of positive and negative jumps are also sparse beyond 10%. Three, there does not appear to be any association between the arrival rates of positive and negative jumps for any jump size.

Table 4. Probabilities of stock market declines and rises of all sizes

	Negative jumps									Positive jumps							
	Raw returns				Devolatized returns					Raw returns				Devolatized returns			
	1897–2007		1946–2007		1897–2007		1946–2007			1897–2007		1946–2007		1897–2007		1946–2007	
Jump size	Count	Prob	Count	Prob	Count	Prob	Count	Prob	Jump size	Count	Prob	Count	Prob	Count	Prob	Count	Prob
< 0.0%	14,215	0.4741	7,473	0.4765	14,215	0.4741	7,473	0.4765	>0.0%	15,765	0.5259	8,211	0.5235	15,765	0.5259	8,211	0.5235
≤−0.5%	7,068	0.2358	3,441	0.2194	7,963	0.2656	3,764	0.24	≥0.5%	7,940	0.2648	3,894	0.2483	9,240	0.3082	4,379	0.2792
≤−1.0%	3,370	0.1124	1464	0.0933	4,044	0.1349	1629	0.1039	≥1.0%	3,452	0.1151	1524	0.0972	4,478	0.1494	1799	0.1147
≤−1.5%	1640	0.0547	608	0.0388	1944	0.0648	646	0.0412	≥1.5%	1533	0.0511	607	0.0387	1900	0.0634	632	0.0403
≤−2.0%	888	0.0296	265	0.0169	963	0.0321	267	0.0170	≥2.0%	751	0.0251	269	0.0172	736	0.0245	206	0.0131
≤−2.5%	504	0.0168	121	0.0077	491	0.0164	121	0.0077	≥2.5%	411	0.0137	128	0.0082	319	0.0106	76	0.0048
≤−3.0%	321	0.0107	61	0.0039	269	0.0090	62	0.0040	≥3.0%	251	0.0084	69	0.0044	125	0.0042	29	0.0018
≤−3.5%	216	0.0072	36	0.0023	144	0.0048	37	0.0024	≥3.5%	156	0.0052	42	0.0027	59	0.0020	14	0.0009
≤−4.0%	149	0.0050	23	0.0015	95	0.0032	23	0.0015	≥4.0%	107	0.0036	25	0.0016	33	0.0011	9	0.0006
≤−4.5%	94	0.0031	17	0.0011	65	0.0022	19	0.0012	≥4.5%	78	0.0026	14	0.0009	21	0.0007	5	0.0003
≤−5.0%	69	0.0023	10	0.0006	42	0.0014	10	0.0006	≥5.0%	52	0.0017	5	0.0003	13	0.0004	4	0.0003
≤−5.5%	47	0.0016	10	0.0006	31	0.0010	7	0.0004	≥5.5%	31	0.0010	4	0.0003	8	0.0003	1	0.0001
≤−6.0%	33	0.0011	7	0.0004	23	0.0008	5	0.0003	≥6.0%	26	0.0009	3	0.0002	5	0.0002	1	0.0001
≤−6.5%	26	0.0009	7	0.0004	20	0.0007	5	0.0003	≥6.5%	17	0.0006	2	0.0001	1	0	1	0.0001
≤−7.0%	22	0.0007	5	0.0003	16	0.0005	4	0.0003	≥7.0%	13	0.0004	1	0.0001	1	0	0	0
≤−7.5%	11	0.0004	2	0.0001	14	0.0005	4	0.0003	≥7.5%	12	0.0004	1	0.0001	1	0	0	0
≤−8.0%	10	0.0003	2	0.0001	10	0.0003	3	0.0002	≥8.0%	11	0.0004	1	0.0001	1	0	0	0
≤−8.5%	7	0.0002	1	0.0001	8	0.0003	3	0.0002	≥8.5%	10	0.0003	1	0.0001	0	0	0	0
≤−9.0%	5	0.0002	1	0.0001	7	0.0002	3	0.0002	≥9.0%	6	0.0002	1	0.0001	0	0	0	0
≤−9.5%	4	0.0001	1	0.0001	7	0.0002	2	0.0001	≥9.5%	4	0.0001	1	0.0001	0	0	0	0
≤−10%	4	0.0001	1	0.0001	6	0.0002	2	0.0001	≥10%	3	0.0001	0	0	0	0	0	0

NOTE: Each for raw returns and devolatized returns, the daily returns are initially divided into negative movements and positive movements. Tracking the negative and positive movements separately, we compute the distribution of negative movements as $<0.0\%$, $\leq -0.50\%, \ldots, \leq -10.0\%$ and for positive movements as $>0.0\%$, $\geq 0.5\%, \ldots, \geq 10.0\%$. In what is reported, the probability (denoted “Prob”) of a stock market decline or a rise in certain size range is then computed by normalizing the number of moves in this range (denoted “Count”) by the total number of trading days in the respective sample. The notation of NOBS is the number of trading days. The number of observations in the 1897–2007 (1946–2007) sample period is 29,980 (15,684), whereby days with no price change are excluded.

Table 5. Lévy measure and the arrival rates of negative and positive jumps

	Arrival of negative jumps									Arrival of positive jumps							
	Raw returns				Devolatized returns					Raw returns				Devolatized returns			
	1897–2007		1946–2007		1897–2007		1946–2007			1897–2007		1946–2007		1897–2007		1946–2007	
Jump size	Arrival rate	Norm. arrival	Arrival rate	Norm. arrival	Arrival rate	Norm. arrival	Arrival rate	Norm. arrival	Jump size	Arrival rate	Norm. arrival	Arrival rate	Norm. arrival	Arrival rate	Norm. arrival	Arrival rate	Norm. arrival
[−0.5%, 0%)	7,147	0.5028	4,032	0.5395	6,252	0.4398	3,709	0.4963	(0%, 0.5%]	7,825	0.4964	4,317	0.5258	6,525	0.4139	3,832	0.4667
[−1%, −0.5%)	3,698	0.2601	1977	0.2646	3,919	0.2757	2135	0.2857	(0.5%, 1%]	4,488	0.2847	2370	0.2886	4,762	0.3021	2580	0.3142
[−1.5%, −1%)	1730	0.1217	856	0.1145	2100	0.1477	983	0.1315	(1%, 1.5%]	1919	0.1217	917	0.1117	2578	0.1635	1167	0.1421
[−2%, −1.5%)	752	0.0529	343	0.0459	981	0.0690	379	0.0507	(1.5%, 2%]	782	0.0496	338	0.0412	1164	0.0738	426	0.0519
[−2.5%, −2%)	384	0.0270	144	0.0193	472	0.0332	146	0.0195	(2%, 2.5%]	340	0.0216	141	0.0172	417	0.0265	130	0.0158
[−3%, −2.5%)	183	0.0129	60	0.0080	222	0.0156	59	0.0079	(2.5%, 3%]	160	0.0101	59	0.0072	194	0.0123	47	0.0057
[−3.5%, −3%)	105	0.0074	25	0.0033	125	0.0088	25	0.0033	(3%, 3.5%]	95	0.0060	27	0.0033	66	0.0042	15	0.0018
[−4%, −3.5%)	67	0.0047	13	0.0017	49	0.0034	14	0.0019	(3.5%, 4%]	49	0.0031	17	0.0021	26	0.0016	5	0.0006
[−4.5%, −4%)	55	0.0039	6	0.0008	30	0.0021	4	0.0005	(4%, 4.5%]	29	0.0018	11	0.0013	12	0.0008	4	0.0005
[−5%, −4.5%)	25	0.0018	7	0.0009	23	0.0016	9	0.0012	(4.5%, 5%]	26	0.0016	9	0.0011	8	0.0005	1	0.0001
[−5.5%, −5%)	22	0.0015	0	0	11	0.0008	3	0.0004	(5%, 5.5%]	21	0.0013	1	0.0001	5	0.0003	3	0.0004
[−6%, −5.5%)	14	0.0010	3	0.0004	8	0.0006	2	0.0003	(5.5%, 6%]	5	0.0003	1	0.0001	3	0.0002	0	0
[−6.5%, −6%)	7	0.0005	0	0	3	0.0002	0	0	(6%, 6.5%]	9	0.0006	1	0.0001	4	0.0003	0	0
[−7%, −6.5%)	4	0.0003	2	0.0003	4	0.0003	1	0.0001	(6.5%, 7%]	4	0.0003	1	0.0001	0	0	1	0.0001
[−7.5%, −7%)	11	0.0008	3	0.0004	2	0.0001	0	0	(7%, 7.5%]	1	0.0001	0	0	0	0	0	0
[−8%, −7.5%)	1	0.0001	0	0	4	0.0003	1	0.0001	(7.5%, 8%]	1	0.0001	0	0	0	0	0	0
[−8.5%, −8%)	3	0.0002	1	0.0001	2	0.0001	0	0	(8%, 8.5%]	1	0.0001	0	0	1	0.0001	0	0
[−9%, −8.5%)	2	0.0001	0	0	1	0.0001	0	0	(8.5%, 9%]	4	0.0003	0	0	0	0	0	0
[−9.5%, −9%)	1	0.0001	0	0	0	0	1	0.0001	(9%, 9.5%]	2	0.0001	0	0	0	0	0	0
[−10%, −9.5%)	0	0	0	0	1	0.0001	0	0	(9.5%, 10%]	1	0.0001	1	0.0001	0	0	0	0

NOTE: In order to conform with the definition of Lévy measure for jump size $z < 0$ and $z > 0$, the daily returns are initially divided into negative movements and positive movements. For $z < 0$ we compute the frequency of movements in 21 buckets as $[-0.5\%, 0\%)$, $[-1\%, -0.5\%)$, …, $[-10\%, -9.5\%)$, and for $z > 0$ also in 21 buckets as $(0, 0.5\%]$, $(0.50\%, 1\%]$, …, $[9.5\%, 10.0\%)$. Respectively for the stock market declines (rises), the empirical "Arrival rate" is obtained by computing the number of negative (positive) jumps in each size bucket. On the other hand, "Norm. arrival" is the arrival rate divided by the total number of negative (positive) jumps in the respective period.

5. DISENTANGLING THE STRUCTURE OF JUMPS

Since every jump model can be characterized by its Lévy measure, we can ask the following important question using devolatized returns: Which Lévy measure, and accordingly, which theoretical model best matches the pattern of jump arrival rates observed in the stock market?

To describe the rationale for the empirical specifications and the associated testable restrictions, consider the Lévy measures $\Pi(z)$ for the jump models in Equations (13), (16), (17), and (18). Since Lévy measures link arrival rate of jumps to jump size, we can regress $\ln \Pi(z)$ on model-specific functions of the jump size z and determine the internal consistency of the resulting regression coefficients. In our implementation, we surrogate $\Pi(z)$ by the arrival rate of jumps as shown in Table 5, and z by the jump size interval midpoint.

Germane to the jump model in Equation (13) is the empirical specification in log-form of the type

$$\ln \Pi[z] = \Omega_0 + \Omega_1 |z| 1_{z<0} + \Omega_2 z 1_{z>0} + \Omega_3 \ln(|z|), \qquad (30)$$

which generates the testable restrictions $\Omega_1 = -\lambda^- < 0$ and $\Omega_2 = -\lambda^+ < 0$. Equation (30) is amenable to casting $\Omega_3 = -(1 + \xi)$, where ξ corresponds to the exponent in Equation (18). The wider interest in ξ stems from the fact that it governs the departure from Equation (13), and hence Ω_3 regulates the nature of small activity. If inferences regarding small moves are to be drawn based on estimated Ω_3, then small movements should not be discarded since they are an integral part of the Lévy measure.

Estimated magnitude of Ω_3 is key to validating finite activity (i.e., $\xi < 0$ and hence $\Omega_3 > -1$), infinite activity (i.e., $\xi > 0$ and hence $\Omega_3 < -1$), or infinite variation (i.e., $\xi > 1$ and hence $\Omega_3 < -2$). The exponent on the Lévy measure in Equation (13) is exactly unity when $\xi = 0$, a testable hypothesis.

The results shown in Table 6 confirm the plausibility of the jump model in Equation (13) in explaining the jump structure in market returns observed since 1897. First, in line with theory, $\Omega_1 < 0$ and $\Omega_2 < 0$, which supports the completely monotone property of the Lévy measure. Second, the estimated coefficients imply $\lambda^- = -\Omega_1 = 69.60$, $\lambda^+ = -\Omega_2 = 86.00$, and crucially $\lambda^- < \lambda^+$. Statistical significance of Ω_1 and Ω_2 is not a concern as the minimum absolute t-statistic is 4.06. Moreover, it is the distinction between λ^- and λ^+ that epitomizes the asymmetry between the arrival rate of downward jumps versus positive jumps. Often this feature is difficult to identify in raw returns, but devolatization has sharpened return asymmetries embedded in the Lévy measures (see also Barndorff-Nielsen 1998).

Returning to Ω_3, we infer that it is -1.01 (t-statistic of -4.10) and -1.45 (t-statistic of -3.98), respectively, over 1897–2007 and 1946–2007. Furthermore the null hypothesis $\Omega_3 = -1$ is not rejected. The conventional F-test statistic for

Table 6. Testing the restrictions on the Lévy measures based on devolatized data

	$\ln \Pi[z] = \Omega_0 + \Omega_1 \vert z\vert 1_{z<0} + \Omega_2 z 1_{z>0} + \Omega_3 \ln(\vert z\vert)$ (General specification)			$\ln \Pi(z) = \Omega_0^* + \Omega_1^* z + \Omega_2^* z^2$ (Cox–Ross specification)	
	1897–2007	1946–2007		1897–2007	1946–2007
Ω_0	−5.98	−8.40	Ω_0^*	−3.19	−3.39
	(−4.96)	(−4.58)		(−8.82)	(−7.21)
Ω_1	−69.60	−58.37	Ω_1^*	−11.23	−15.25
	(−8.11)	(−4.06)		(−1.94)	(−1.64)
Ω_2	−86.00	−78.72	Ω_2^*	−1062.39	−1222.58
	(−8.54)	(−4.57)		(−9.53)	(−6.43)
Ω_3	−1.01	−1.45			
	(−4.10)	(−3.98)			
Adj. R^2	96.1%	92.7%		74.6%	61.3%
NOBS	33	27		33	27
Das–Foresi					
Ho: $\Omega_3 = 0$	16.84	15.86			
	{0.000}	{0.000}			
CGMY					
Ho: $\Omega_3 = -1$	0.00	1.54			
	{0.982}	{0.226}			
Encompassing				17.94	6.51
$\Omega_1^{**} = \Omega_2^{**}$, $\Omega_3^{**} = 0$				{0.000}	{0.005}

NOTE: In the regression analysis, the log of the Lévy measure, $\ln \Pi(z)$ is the dependent variable where $\Pi(z)$ is surrogated by the arrival rate of jumps and z by the jump size interval midpoint, as shown in Table 5. The t-statistics are reported in parenthesis. We investigate the empirical specification,

$$\ln \Pi[z] = \Omega_0 + \Omega_1 |z| 1_{z<0} + \Omega_2 z 1_{z>0} + \Omega_3 \ln(|z|).$$

First, if $\Omega_3 = 0$, we get the Lévy measure in Das and Foresi (1996). Second, if $\Omega_3 = -1$, we cannot reject the jump model in Equations (5)–(7) that gives rise to the Lévy measure in Equation (13). The parameter transformation are $\Omega_0 = -\ln(\kappa)$, $\Omega_1 = -\lambda^- < 0$, $\Omega_2 = -\lambda^+ < 0$, and $\Omega_3 = -(1 + \xi)$. The null hypothesis are tested using the standard F-test with p-value in curly brackets. For the Cox and Ross (1976) model, we examine $\ln \Pi(z) = \Omega_0^* + \Omega_1^* z + \Omega_2^* z^2$. The model imposes the testable restriction that the log Lévy measure is quadratic in the jump size with the sign of Ω_1^* being the sign of μ_J, and $\Omega_2^* < 0$. The comparison between the pure-jump model and the Cox–Ross model is examined in the row "Encompassing," which represents an artificial encompassing specification of the type:

$$\ln \Pi[z] = \Omega_0 + \Omega_1^{**} |z| 1_{z<0} + \Omega_2^{**} z 1_{z>0} + \Omega_3^{**} \ln(|z|) + \Omega_4^{**} z^2.$$

The joint restriction is $\Omega_1^{**} = \Omega_2^{**}$, and $\Omega_3^{**} = 0$. Reported is the value of the F-statistic along with the p-value in curly brackets.

this hypothesis does not exceed 1.54 (p-value 0.226). Based on this test, ξ is indistinguishable from zero.

An advantage of adopting specification in Equation (30) is that it also nests the log Lévy measure for the Das–Foresi jump-model. The F-test reported in Table 6 examines the exclusion restriction $\Omega_3 = 0$. The p-values indicate an overwhelming rejection of the Das and Foresi (1996) and Kou (2002) jump models.

How does the quadratic arrival rate model of Cox and Ross (1976) fare with respect to the purely discontinuous counterpart in capturing jump arrival rates? The model imposes the testable restriction that the log Lévy measure is quadratic in the jump size with the sign of Ω_1^* being the sign of μ_J and $\Omega_2^* < 0$:

$$\ln \Pi(z) = \Omega_0^* + \Omega_1^* z + \Omega_2^* z^2, \tag{31}$$

where $\Omega_0^* \equiv \ln(\frac{\lambda_J}{\sqrt{2\pi}\sigma_J}) - \frac{1}{2\sigma_J^2}(\ln(1+\mu_J) - \frac{1}{2}\sigma_J^2)^2$, $\Omega_1^* \equiv \frac{\ln(1+\mu_J)-1/2\sigma_J^2}{\sigma_J^2}$, and $\Omega_2^* \equiv -\frac{1}{2\sigma_J^2}$. The values of Ω_1^* and Ω_2^* reported in Table 6 imply a $\sigma_J = 2.169\%$ (2.022%), $\mu_J = -0.504\%$ (-0.601%), and $\lambda_J \approx 0$ over 1897–2007 (1946–2007). Although $\mu_J < 0$ and $\Omega_2^* < 0$ hold in the data, the parameters of the jump distribution are unreasonable, based on what is known from Bakshi, Cao, and Chen (2000), Bates (2000), and Eraker, Johannes, and Polson (2003). To examine model failure from a different angle, we take an artificial encompassing regression $\ln \Pi[z] = \Omega_0^{**} + \Omega_1^{**}|z|1_{z<0} + \Omega_2^{**} z 1_{z>0} + \Omega_3^{**}\ln(|z|) + \Omega_4^{**} z^2$ and test $\Omega_1^{**} = \Omega_2^{**}$ and $\Omega_3^{**} = 0$. The reported p-value indicates the inadequacy of the Cox and Ross (1976) model.

Overall, these results support the view that the structure of large movements has a fatter left-tail relative to the Gaussian distribution of jump sizes. Thus, the generalized LM in Equation (13) may be needed for a better performing theory of stock market in the tails.

We should emphasize that the results in Table 6 should not be interpreted as an exact estimation of the Lévy measure. Suppose one generated (daily) returns from the geometric Brownian motion model and then binned them according to size as done in Table 5. At an empirical level the regression of the frequency of movements on size is valid even though there is no Lévy measure for a geometric Brownian motion (the path is continuous). This observation bears analogy with the fact that a Lévy measure of a process (when it exists), is the theoretical limit of the density divided by a small time interval Δt as $\Delta t \to 0$, while at the same time this limit can be estimated even for processes with no Lévy measure. Therefore, Table 6 only presents a possibly crude attempt to differentiate the tail behavior across models. To rigorously extract the Lévy measure, one must estimate the structural parameters (say, σ, θ, κ) through maximum likelihood of the return density and then recover the Lévy measure through λ^- and λ^+, a task we turn to later.

6. DEDUCING THE LIMIT LAWS OF EXTREMES AND THE THICKNESS OF TAILS

Still three questions remain unanswered. First, are extreme fluctuations constructed from devolatized returns consistent with Fréchet or Weibull limit laws? Second, if a large scale simulation is performed to approximate $n \to +\infty$ on the pure-jump dynamics postulated in Equations (5) to (7), which limit law is supported? Each metric imposes a distinct barrier on the purely discontinuous price dynamics and the tail probability model. Finally, is the right-tail thinner than the left-tail based on the estimate of the tail-index α?

6.1 Limit Laws of Left-Tail Event Extremes and Right-Tail Event Extremes

To answer the first question, we fix, as before, the block size to 42 days, 84 days, and 126 days. Thus, we have a set of six block maximas $\{m^-(j)\}_{j=1}^n$ and $\{m^+(j)\}_{j=1}^n$ for devolatized returns, where $m^- = \exp(M^-) - 1$ and $m^+ = \exp(M^+) - 1$. Then, in the spirit of Pesaran and Deaton (1978), the examination of the Weibull versus Fréchet limit laws can be conducted by maximizing the log-likelihood function,

$$\max \Gamma\left(\frac{1}{n}\sum_{j=1}^{n} \ln \Phi_F[m(j)]\right) + (1-\Gamma)\left(\frac{1}{n}\sum_{j=1}^{n} \ln \Phi_W[m(j)]\right),$$
$$\Gamma \in (0,1), m = \{m^-, m^+\}, \tag{32}$$

where the functional form of the Fréchet density, denoted $\Phi_F[\cdot]$, and the Weibull density, denoted $\Phi_W[\cdot]$, are as presented in Equations (23) and (24). In the artificially nested log-likelihood function in Equation (32), the null of Weibull limit law versus Fréchet is equivalent to testing whether $\Gamma = 0$ and is a hypothesis on the boundary.

When data are uncertain about the parametric form of the underlying density, our maximum likelihood estimations reveal that Weibull is being rejected in favor of the Fréchet for both m^- or m^+. In particular, the estimated Γ is virtually unity and the null hypothesis $\Gamma = 0$ is rejected. Thus, the goodness-of-fit diagnostic is suggesting the Fréchet distribution as the limit law for both the left-tail and right-tail events.

Fréchet distribution asserts a power law tail behavior, that is, $\lim_{m\to\infty} \Phi_F[m] \to \frac{\alpha}{m^{1+\alpha}}$, and, hence, heavy tailed extremes. It must be appreciated that the tail index α is directly linked to the tail heaviness and the number of bounded moments of the extremes (Feller 1971), with $1 + \alpha = \sup_{j>1} \int m^j \Phi_F[m]\, dm < \infty$.

The fundamental observation that can be garnered from Table 7 is that the maximum likelihood estimations are stipulating that the tail-index, α, is substantially different for left-tail extremes m^- versus right-tail extremes m^+. For negative extremes, α is in the range of 2.593 and 3.040, while for positive extremes, α is in the range of 3.448 to 4.221. The t-statistics reported in parenthesis are large.

Consider block size of 126 days over 1897–2007. The entry of $\alpha = 3.04$ implies finite moments up to order 4 for the distribution of m^- whereas the entry of $\alpha = 4.221$ implies finite moments up to order 5 for the distribution of m^+. Essentially the distribution of right-tail events has thinner tail and gravitates to zero at a faster rate. The distribution of left-tail events has an even heavier tail in the post 1946 period. Longin (1996) arrived at the Fréchet limit law for the S&P 500 index. The work here differs from Longin (1996) and a related study by Jansen and Vries (1991) in two ways. First, we develop and empirically examine a model of stock market extremes, their arrival

Table 7. Analyzing the limit law of extremes, m_b^- and m_b^+, for devolatized returns

		Left-tail extreme, m^- (Fréchet)			Right-tail extreme, m^+ (Fréchet)		
Sample	Block size, b	42	84	126	42	84	126
1897–2007	α	2.778	2.933	3.04	3.488	4.139	4.221
	t-stat	(32.94)	(24.05)	(19.14)	(36.72)	(23.98)	(19.09)
	η	0.0194	0.0243	0.0282	0.0195	0.0235	0.026
	t-stat	(69.19)	(52.31)	(44.52)	(81.88)	(73.10)	(60.98)
	$\mathcal{L}/n$	3.207	3.038	2.927	3.503	3.481	3.400
	n	717	358	239	717	358	239
1946–2007	α	2.955	2.973	2.593	3.586	4.217	4.167
	t-stat	25.79	18.07	13.42	28.73	17.3	13.38
	η	0.0190	0.0232	0.0267	0.0194	0.0230	0.0255
	t-stat	53.76	38.57	31.35	61.36	54.56	43.92
	$\mathcal{L}/n$	3.285	3.084	2.928	3.542	3.510	3.398
	n	375	187	125	375	187	125

NOTE: Throughout, we fix the block size to 42 days, 84 days, and 126 days and take the block maxima $\{m^-(j)\}_{j=1}^n$ and $\{m^+(j)\}_{j=1}^n$, where $m^- = \exp(M^-) - 1$ and $m^+ = \exp(M^+) - 1$. We arrive at the maximum likelihood estimation results in two steps. First, we do a constrained maximization for $m = \{m^-, m^+\}$: $\max \Gamma(\frac{1}{n}\sum_{j=1}^n \ln \Phi_F[m(j)]) + (1-\Gamma)(\frac{1}{n}\sum_{j=1}^n \ln \Phi_W[m(j)])$, where the parametric forms of the Fréchet density, denoted $\Phi_F(\cdot)$, and the Weibull density, denoted $\Phi_W(\cdot)$, are

$$\Phi_F[m] = \alpha\eta^\alpha \exp\left(-\left(\frac{m}{\eta}\right)^{-\alpha}\right) m^{-\alpha-1}, \qquad \Phi_W[m] = \alpha\eta^{-\alpha} \exp\left(-\left(\frac{m}{\eta}\right)^{\alpha}\right) m^{\alpha-1}.$$

The data strongly reject Weibull distribution in favor of Fréchet based on the value of Γ. Second, once the data have validated the Fréchet, we estimate parameters of the Fréchet density through maximum likelihood. Reported are the tail-index α, the scale parameter η, and the log-likelihood $\mathcal{L}$. The maximum likelihood estimation relies on the BHHH algorithm. The t-statistics are reported in parenthesis.

rates, and the probability of extremes. Thus, our focus is not restricted to the examination of limit laws. Second, our analysis employs devolatized returns instead of raw returns which show dependence (Kearns and Pagan 1997).

6.2 Simulation Confirmation

We now exploit the fact that the distribution of the local motion for $S(t)/S(0)$ is loggamma under the posited Lévy model, as shown in Equation (11), and loggamma is in the domain of attraction of the Fréchet (e.g., Embrechts, Kluppelberg, and Mikosch 1997, p. 135). This feature permits us to construct a diagnostic test to confirm the most appropriate parent distribution governing the local price motion. If generated data on the extremes rejects the Weibull limiting law, then models with local motions in the domain of attraction of the Fréchet get a push, since such models are compatible with extreme-value theory. Besides, one can further verify whether the examined model is able to reproduce specific tail features of the data.

Building on these themes, we perform such a diagnostic test on the price process in Equations (5) to (7) and simulate 10,000 paths of block size of either 42 days or 126 days. Thus, we select block maxima $\{m^-(j)\}_{j=1}^n$ and $\{m^+(j)\}_{j=1}^n$, where $m^- = \exp(M^-) - 1$, $m^+ = \exp(M^+) - 1$, and $n = 10{,}000$. We simulate the process with parameters obtained by maximum likelihood estimation on the entire sample of devolatized data.

When maximum likelihood estimation is conducted on simulated returns and extremes, Table 8 shows that it mirrors the tail features of devolatized data reported in Table 7. Specifically, the estimated tail-index of the left-tail event distribution is less than the tail-index of the right-tail event distribution. Further, we confirm the robust result that the limit law is the heavy-tailed Fréchet and not Weibull. We conclude that Equations (5) to (7) make a good candidate model for the data at hand, reproducing closely their tail behavior.

7. QUANTITATIVE ASSESSMENTS OF ARRIVAL RATES AND PROBABILITY OF EXTREMES

Led by empirical realities, we devolatized daily returns to obtain our results in Sections 4 through 6, but we recognize that it does not preclude time-variation in other aspects of the return distribution. However, locally the Lévy process may still be relevant if departures from independence and time-homogeneity occur slowly in the data, and the estimate of homogeneity is time-varying only over longer periods. With this motivation, our approach is to estimate the return distribution in a rolling manner to extract the Lévy measure and consequently work out the probability of extremes. Specifically we examine whether there is information content in the Lévy measure extracted from a rolling estimation. This is done by investigating whether realized forward arrival rates of extreme moves are predicted by the current Lévy measure.

To preserve the large move focus in devolatized returns we take $|z| \in (2\%, 3\%]$, $|z| \in (3\%, 4\%]$, and $|z| \in (4\%, 5\%]$ and $|z^*|$ as the midpoint. For our purposes, we employ the following empirical specification:

$$\begin{aligned} \Pi_t^{-,\text{actual}}(z) &= a^- + b^- \Pi_{t-1}^{-,\text{model}}(z^*) + \epsilon_t^-, \\ \Pi_t^{+,\text{actual}}(z) &= a^+ + b^+ \Pi_{t-1}^{+,\text{model}}(z^*) + \epsilon_t^+. \end{aligned} \qquad (33)$$

The dependent variable in Equation (33) is $\Pi_t^{-,\text{actual}}(z)$, which is the actual arrival rate of size $z < 0$ observed at date t. The

Table 8. Simulation of the pure-jump stock price model specified in Equations (5)–(7)

Block size	n	Fréchet density α	Fréchet density η	Fréchet density $\mathcal{L}/n$	Log relative statistics for maxima Mean	Std. dev.	Min	Max
m^-, 42 days	10,000	3.312	0.0217	3.373	0.0261	0.0081	0.0073	0.0802
		(163)	(279)					
m^+, 42 days	10,000	3.605	0.0199	3.549	0.0236	0.0069	0.0084	0.0677
		(151)	(303)					
m^-, 126 days	10,000	4.543	0.0292	3.418	0.0330	0.0078	0.0140	0.0925
		(146)	(390)					
m^+, 126 days	10,000	4.843	0.0263	3.581	0.0296	0.0068	0.0147	0.0725
		(136)	(422)					

NOTE: Returns are simulated according to the model in Equations (5)–(7) using $\sigma = 0.1626$, $\theta = -0.4426$, and $\kappa = 0.0019$ for a block size equal to either 42 days or 126 days. For this simulation we discretize the process as $g(t+\Delta t) - g(t) = \theta y + \sigma \tilde{z}\sqrt{y}$, where $\tilde{z}$ is $\mathcal{N}(0,1)$, $y \sim Gamma(\Delta t, \kappa\Delta t)$ and the interval Δt is 1 day. In each trial we store the absolute value of the maximum daily decline and the value of the maximum daily rise over the respective block size. The process is repeated for $n = 10{,}000$, yielding the block maxima $\{m^-(j)\}_{j=1}^n$ and $\{m^+(j)\}_{j=1}^n$, where $m^- = \exp(M^-) - 1$ and $m^+ = \exp(M^+) - 1$. Reported are the tail-index α and the scale parameter η, and the log-likelihood $\mathcal{L}$ from maximum likelihood of the Fréchet density $\Phi_F[m] = \alpha\eta^\alpha \exp(-(\frac{m}{\eta})^{-\alpha}) m^{-\alpha-1}$. Estimation relies on the BHHH algorithm. The t-statistics are shown in parenthesis below the parameters. To maintain consistency with Table 3, we report the log relative statistics for the maxima (i.e., for M^-, M^+).

explanatory variable is the theoretical arrival rate measured by $\Pi_{t-1}^{-,\text{model}}(z^*) = \frac{e^{-\lambda^-|z^*|}}{\kappa|z^*|}$ at date $t-1$. The hypothesis is $b^- > 0$ and $b^+ > 0$, which translates into the statement that there is a correspondence between forward arrival rates and the estimated theoretical arrival rates.

To alter the estimate of homogeneity in the Lévy measure we obtain σ, θ, and κ via maximum likelihood estimation of Equation (10) employing a trailing window of 1000 days (4 years). Here σ, θ, and κ are connected to return volatility, skewness, and kurtosis, whose estimation requires a reasonably long time series (Kim and White 2004). Hence we concentrate on 1000 days, but in our pretrial estimations we examined other trailing windows and obtained similar results. The instruments $\Pi_{t-1}^{-,\text{model}}(z^*) = \frac{e^{-\lambda^-|z^*|}}{\kappa|z^*|}$ and $\Pi_{t-1}^{+,\text{model}}(z^*) = \frac{e^{-\lambda^+ z^*}}{\kappa z^*}$ are determined in advance of date t realization of the arrival rates, where we compute λ^- and λ^+ according to Equation (14).

Through the aforementioned, we examine whether estimated model arrival rates can predict the realized forward arrival rate observed over the next 2 months (42 days) and over the next 6 months (i.e., 126 days). At the core of Table 9 is the finding that both the slope coefficients b^- and b^+ are positive. The entries for robust t-statistics (p-values in curly brackets) on b^- and b^+ show that the slope coefficients are statistically significant in the majority of the regressions. Our results are stronger for down-moves with p-values on b^- not exceeding 0.10 and p-values less than 0.01 in four out of six estimations. In general, the empirical specification is supportive of the conjecture that the theoretical arrival rates contain information that is useful to anticipate forward arrival rates. Estimation results also imply that it is more difficult to predict large moves occurring between 4% and 5%, reinforcing the viability of the tail probability model. To pull it all together, the positive and significant

Table 9. Forward arrival rates of large movements (devolatized returns)

	$\Pi_t^{-,\text{actual}}(z) = a^- + b^-\Pi_{t-1}^{-,\text{model}}(z^*) + \epsilon_t^-$						$\Pi_t^{+,\text{actual}}(z) = a^+ + b^+\Pi_{t-1}^{+,\text{model}}(z^*) + \epsilon_t^+$					
	126 days			42 days			126 days			42 days		
	a^-	b^-	Adj-R^2 [DW]	a^-	b^-	Adj-R^2 [DW]	a^+	b^+	Adj-R^2 [DW]	a^+	b^+	Adj-R^2 [DW]
$\|z\| \in (2\%, 3\%]$	2.30	0.154	3.44%	0.79	0.049	0.8%	2.26	0.173	1.45%	0.71	0.07	0.8%
$\|z^*\| = 2.5\%$	(11.73)	(3.35)	[1.86]	(9.91)	(2.75)	[1.81]	(11.07)	(1.99)	[1.74]	(11.06)	(2.63)	[1.77]
	{0.000}	{0.000}		{0.000}	{0.005}		{0.000}	{0.046}		{0.000}	{0.000}	
$\|z\| \in (3\%, 4\%]$	0.51	0.198	4.64%	0.19	0.057	1.1%	0.41	−0.00	0.50%	0.12	0.025	0.2%
$\|z^*\| = 3.5\%$	(6.59)	(3.12)	[1.68]	(6.63)	(2.78)	[1.73]	(6.60)	(−0.12)	[1.90]	(6.32)	(0.79)	[1.70]
	{0.000}	{0.000}		{0.000}	{0.005}		{0.000}	{0.905}		{0.000}	{0.420}	
$\|z\| \in (4\%, 5\%]$	0.16	0.123	1.18%	0.059	0.039	0.3%	0.048	0.279	3.06%	0.019	0.062	0.4%
$\|z^*\| = 4.5\%$	(4.45)	(1.80)	[1.95]	(4.64)	(1.62)	[1.57]	(2.32)	(1.87)	[2.00]	(2.72)	(1.45)	[1.80]
	{0.000}	{0.071}		{0.000}	{0.100}		{0.000}	{0.061}		{0.000}	{0.140}	

NOTE: Date t is taken to be 126 or 42 days from date $t-1$ and the daily return data is devolatized. We take $|z| \in (2\%, 3\%]$, $|z| \in (3\%, 4\%]$, and $|z| \in (4\%, 5\%]$ and the midpoint of the interval as $|z^*|$. The dependent variables $\Pi_t^{-,\text{actual}}(z)$ and $\Pi_t^{+,\text{actual}}(z)$ are, respectively, the count of negative and positive jumps that transpire over $(t-1, t)$. Model determined $\Pi_{t-1}^{-,\text{model}}(z^*) = \frac{e^{-\lambda^-|z^*|}}{\kappa|z^*|}$ and $\Pi_{t-1}^{+,\text{model}}(z^*) = \frac{e^{-\lambda^+ z^*}}{\kappa z^*}$ are measured as of date $t-1$. The coefficients λ^- and λ^+ are estimated via maximum likelihood on a backward window of 1000 days based on Equation (14). The t-statistics, in parenthesis, are based on robust standard errors with p-values in curly brackets. Adj-R^2 is the adjusted R^2 and DW is the Durbin–Watson statistic (in square brackets).

slope coefficient is a noteworthy result that is indicative of information content embedded in the Lévy measure.

8. CONCLUDING REMARKS

This article studies, both theoretically and empirically, stock market tail events. We show that the pre 1946 jump arrival rate patterns depart from the post 1945 counterpart, and that overall the left-tail of market returns decays to zero much slower than the right-tail. Moreover, the left- and right-tail events both conform with Fréchet limit laws.

We identify a parsimoniously parameterized pure-jump model for market returns which can reconcile the empirical finding that crashes occur more often than rallies and are more severe in intensity. The empirical relevance of the model is confirmed both in raw and devolatized market returns. The model is not only consistent with the observed structure of return jumps, but also with the predictions of the extreme-value theory. The implications of our model's Lévy measure for the distribution of extreme events are closer to the actual realization of extremes than those of competing models. Finally, there is information content in our model's Lévy measure for forward arrival rates.

To address possible time-variation in return volatility, we perform our analysis on devolatized returns. However, we recognize that time-variation may remain in other aspects of the return distribution, and hence in the estimated Lévy measure. In this regard, theoretical refinements based on multidimensional Lévy measures can direct us to a finer understanding of arrival rate of jumps of all sizes. In these Lévy models the arrival rate of jumps will depend on variables other than jump size. Multidimensional Lévy measures may also prove to be fruitful in explaining why international stock markets are more correlated on the extreme downside moves than on the upside (Longin and Solnik 2001 and Poon, Rockinger, and Tawn 2002). Correlated arrival rates of large negative jumps across two financial markets can be accommodated using jump dependence. These are possible topics where future research efforts could be directed.

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