



First-passage probability, jump models, and intra-horizon risk[☆]

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ABSTRACT

This paper proposes a risk measure, based on first-passage probability, which reflects intra-horizon risk in jump models with finite or infinite jump activity. Our empirical investigation shows, first, that the proposed risk measure consistently exceeds the benchmark value-at-risk (VaR). Second, jump risk tends to amplify intra-horizon risk. Third, we find large variation in our risk measure across jump models, indicative of model risk. Fourth, among the jump models we consider, the finite-moment log-stable model provides the most conservative risk estimates. Fifth, imposing more stringent VaR levels accentuates the impact of intra-horizon risk in jump models. Finally, using an alternative benchmark VaR does not dilute the role of intra-horizon risk. Overall, we contribute by showing that ignoring intra-horizon risk can lead to underestimation of risk exposures.

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1. Introduction

To reflect the reality that trading portfolios of financial institutions can be adversely exposed to a multitude of risk factors, the Basel Capital Accord was amended in 1996 to include a capital charge for market risk. An important component of risk management efforts by bank regulatory bodies throughout the world, the capital charge is the

minimum required amount of capital that banks must set aside as protection against trading losses. While the Amendment gave banks freedom to choose their internal risk models in arriving at the capital charge, it stipulated that market risk be measured by value-at-risk (VaR), a quantile measure of the bank's trading profit and loss distribution at the end of a specified trading horizon.¹

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¹ Within the extensive VaR literature, see, for instance, the treatments in Aït-Sahalia and Lo (2000), Berkowitz and O'Brien (2002, 2005), Berkowitz, Christoffersen, and Pelletier (2006), Christoffersen and Pelletier (2004), Duffie and Pan (1997), Eberlein, Kallsen, and Kristen (2002), Eberlein, Kallsen, and Kristen (1998), Engle (2004), Engle and Manganelli (2004), Gouriéroux and Jasiak (2009), Gupta and Liang (2005), Jackson, Maude, and Perraudin (1997), Jorion (2002, 2006), Kupiec (1995), and Manganelli and Engle (2001).

While VaR has been widely accepted as a market risk measure, it suffers from a number of drawbacks, which are well understood. For example, the actual shape of the profit and loss distribution to the left of a specified quantile has no bearing on the VaR magnitude. It follows that VaR is only concerned with the probability of a loss and not with the loss size. One remedy for this drawback has been found in risk measures which focus on the expected loss at the end of the specified horizon, e.g., the tail conditional expectation and worst conditional expectation as in Artzner, Delbaen, Eber, and Heath (1997, 1999). These measures improve on VaR in the return space dimension, as they incorporate information about the tail distribution of profits and losses. Alternatively, VaR can be improved in the time dimension, by relaxing the end-of-horizon feature and focusing on the profit and loss distribution over the trading horizon, which would incorporate information about the dynamic path of possible losses. In fact, this alternative was explicitly acknowledged in the Overview of the Amendment (Basel Committee, 1996), where the Basel Committee on Banking Supervision points out, among its weaknesses, that VaR estimates do not take into consideration the magnitude of possible losses, incurred *before* the end of a specified trading horizon. This paper studies a risk measure reflecting such losses, and the associated risk is denoted “intra-horizon risk.”

In contrast to VaR measures based on expected end-of-horizon loss, which have attracted attention due to their coherence property,² intra-horizon risk remains a largely overlooked topic. Intra-horizon risk has been identified as an important risk dimension in Stulz (1996), Kritzman and Rich (2002), and Boudoukh, Richardson, Stanton, and Whitelaw (2004). However, these studies do not treat intra-horizon risk in the context of the current bank regulatory framework, which stipulates a 10-day horizon for VaR estimation. Furthermore, they model asset returns in the Brownian motion framework and ignore possible jumps in asset prices, whereas it is recognized that jumps are relevant at short horizons for risk management purposes (Basel Committee, 1996).

Building on the extensive VaR literature, the thrust of this paper is to ask two main questions and our aim is to fill particular gaps from the empirical perspective. First, to what extent can standard VaR be improved by a risk measure, based on first-passage probability for jump models, which takes into account intra-horizon risk over short horizons? In particular, how different are risk exposures under such a measure, compared with the standard VaR? Second, does intra-horizon risk depend on the structure of jumps in a model? In addition, we investigate whether the realism of intra-horizon risk in jump models can justify the multipliers which banks need to apply to their internal VaR in calculating the capital charge.³

² Coherent risk measures have been widely studied, and we mention, among many, Föllmer and Schied (2002), Jouini, Meddeb, and Touzi (2004), Rockafellar and Uryasev (2002), and Delbaen (2002).

³ In particular, the Amendment mandates banks to compute VaR using a regulatory horizon of 10 days and a 99% confidence interval. At

To accommodate concerns, both by market participants and bank regulators, for event risk and sudden and large trading losses, we consider three Lévy jump models for the underlying asset returns. To account for intra-horizon risk, we present methods to compute the first-passage probability for these models. Displaying diverse tail-behaviors, the return distribution under each model has a fatter left tail than the Normal distribution, and can generate higher VaR estimates than Normal VaR. Within the finite activity model class (i.e., relatively infrequent jumps), we consider the Merton (1976) jump-diffusion model with Normally distributed jump sizes. We also consider models with infinite activity, i.e., models with infinitely many jumps in any time interval.⁴ In this class we consider the finite-moment log-stable model (FMLS) (Carr and Wu, 2003). The left tail of the distribution of asset returns in this case is the fattest among all our models: it declines as a power law and can potentially accommodate highly skewed profit and loss profiles. In addition, we consider the two-sided pure-jump model of Carr, Geman, Madan, and Yor (2002).

In analogy with VaR, we measure intra-horizon risk as a quantile of the first-passage distribution over the regulatory 10-day horizon. We denote this risk measure VaR-I, which stands for VaR *with intra-horizon risk*. Unlike the VaR measure, VaR-I reflects the magnitude of losses *within* a trading horizon, and not just at the end of the horizon. Intra-horizon risk can be important when traders operate under mark-to-market constraints and, hence, sudden losses may trigger margin calls and otherwise adversely affect the trading position. We estimate our risk measures, i.e., VaR and VaR-I for models with jumps, for equity risk, interest rate risk, currency risk, and equity volatility risk as embedded in the price of at-the-money equity index options. In considering option returns, we seek to address concerns that typical VaR models do not adequately capture the risk inherent in complex instruments (Basel Committee, 1996).

Whereas the first-passage probability can be characterized in closed-form for Brownian motion (e.g., Karlin and Taylor, 1975), analytical expressions for the first-passage probability of processes with jumps are generally not available (e.g., Kyprianou, 2006). Here we contribute by presenting a tractable methodology based on partial integro-differential equations which can be adapted for calculation of first-passage probability for any Lévy jump process.

Based on the time series of asset returns, we estimate each jump model via maximum likelihood, and our

(footnote continued)

least one year of historical data should be used, updated at least once every quarter. The capital charge is the product of (i) the higher of the previous day's VaR and the average VaR over the preceding 60 days and (ii) a multiplier between three and four, called Basel multiplier. The value of the multiplier is determined following the Basel Committee rules for backtesting internal models (see also Jorion, 2006).

⁴ The advantages of these models have been studied, among others, in Eberlein (2001), Carr, Geman, Madan, and Yor (2002), Carr and Wu (2004), Huang and Wu (2004), Wu (2007), Ait-Sahalia and Jacod (2007, 2009), Bakshi, Carr, and Wu (2008), Li, Wells, and Yu (2008), and Bakshi, Madan, and Panayotov (2009).

empirical investigation provides a number of results about the structure of risk measures. These results are obtained by addressing time scaling issues that arise when models are estimated with high frequency return observations but risk measures must be calculated over the longer 10-day benchmark horizon.

First, model implementation shows that risk measures taking into account intra-horizon risk in jump models consistently exceed the benchmark VaR, and can be up to 2.64 times higher. Validating the uneasiness of regulators with respect to options, this divergence is most pronounced for equity volatility exposures. Second, our estimation shows that, allowing for an intra-horizon risk component can magnify the counterpart jump model VaR by at least 20%. The gist of these results is that omitting intra-horizon risk can lead to a significant underestimation of risk exposure in financial markets. Third, judging by the magnitudes of VaR and VaR-I, we infer that intra-horizon risk depends on the jump structure of a model, and that jump risk tends to amplify intra-horizon risk. Fourth, risk measures exhibit large variation across models, indicative of model risk. In particular, the FMLS model of Carr and Wu (2003) stands out by providing the most conservative risk multipliers, irrespective of the exposure. Since our diagnostic tests show that the FMLS performs empirically at par with other models, its ability to generate the highest VaR and VaR-I predominantly lies in its versatility to identify left tail behavior.

To reinforce our findings on the relevance of intra-horizon risk, we finally consider (i) a more stringent 99.9% VaR level and (ii) an alternative VaR benchmark based on the filtered historical simulation (as in Barone-Adesi, Giannopoulos, and Vosper, 1999; Barone-Adesi, Engle, and Mancini, 2008). We find that 99.9% VaR-I multiples are uniformly higher than the 99% counterpart, reflecting the impact of large jump activity. Furthermore, adopting an alternative VaR benchmark does not dilute the role of intra-horizon risk in jump models.

In what follows, Section 2 formalizes intra-horizon risk in terms of first-passage probability, and argues for its role in risk management. The aim of Section 3 is to present the approach for deriving first-passage probability. Section 4 is devoted to parametric jump models, and Section 5 outlines the estimation procedure. The empirical assessment results are in Section 6. Concluding remarks are offered in Section 7.

2. Rationale and definition of an intra-horizon risk measure

This section formalizes a risk measure which accounts for intra-horizon risk, denoted VaR-I for VaR with intra-horizon risk, and thus improves on the standard VaR that only considers risk at the end of an investment horizon.

Let X_t , for $t \in [0, T]$, be a real-valued random process, representing possible paths of the dollar return (i.e., profit or loss) on a position or portfolio. Note that $X_0 = 0$, and $X_T^{\min} := \min_{0 < t < T} \{X_t\}$ is a random variable.

Definition. VaR-I is the absolute value of a quantile of the distribution of the random variable X_T^{\min} . In particular, we

consider the 10-day, 99% VaR-I, which is the absolute value of the loss level exceeded at any point in time during the 10-day bank regulatory horizon with probability 1%. Formally, for $T = 10$ days,

$$\text{Prob}(X_T^{\min} \leq -\text{VaR-I}) = 1\%. \quad (1)$$

VaR-I is closely connected to the first-passage time \mathcal{T}_y of X_t to a lower level y :

$$\mathcal{T}_y := \inf\{t > 0 : X_t < y\} \quad \text{where } y < X_0, \quad (2)$$

which is the first time when X_t hits level y , and $\mathcal{T}_y = T$ if $X_t > y$ for all $t \in [0, T]$. Since, for any T ,

$$\text{Prob}(\mathcal{T}_y > T) = \text{Prob}(X_T^{\min} > y), \quad (3)$$

then the 1% quantile defining VaR-I can be found (numerically) as the level y , such that

$$1 - \text{Prob}(\mathcal{T}_y > 10 \text{ days}) = 1\%. \quad (4)$$

Exploiting this link between VaR-I and \mathcal{T}_y , the approach implemented in this paper relies on obtaining the probability distribution of the first-passage time, and then calculating VaR-I using Eq. (4).

VaR-I stands in contrast to the standard 10-day, 99% VaR, defined as the absolute value of the loss level exceeded at the end of the regulatory 10-day horizon with probability 1%. By design, VaR-I is larger than standard VaR and shares with VaR the property of being a quantile measure of risk. Among its various properties, Appendix A shows that VaR-I, like standard VaR, satisfies three coherence risk axioms of Artzner, Delbaen, Eber, and Heath (1997, 1999), namely positive homogeneity, monotonicity, and translation invariance, but lacks subadditivity.

Studies such as Stulz (1996), Kritzman and Rich (2002), and Boudoukh, Richardson, Stanton, and Whitelaw (2004) differ from our approach to addressing intra-horizon risk in three ways. First, they do not characterize intra-horizon risk in the context of market risk measurement under the current bank regulatory setting. Rather, they consider multi-year investment periods, while the relevant horizon for regulatory purposes is only 10 days. Second, their empirical results hinge on the drift of the returns process, which is of immaterial concern over the regulatory 10-day horizon. Third, they model asset returns as Brownian motion and ignore possible adverse jumps in asset prices. In contrast, our approach is, in general, grounded in the fact that jumps are relevant at short horizons for risk management and regulatory purposes (Basel Committee, 1996). In this study, we consider Lévy jump models for asset returns, and present methods to compute first-passage probability and estimate VaR-I over the regulatory horizon.

At a *short horizon*, which is the focus of this paper, there are grounds for considering intra-horizon risk. Specifically, when traders operate under mark-to-market constraints, sudden losses may trigger margin calls and otherwise adversely affect the trading position. Furthermore, when losses cause rebalancing of the trading position before the end of the intended holding period,

the standard VaR estimates are of limited value for risk management. Finally, intra-horizon risk matters when devising trading strategies by determining entry, exit, or stop-loss levels. In these examples, losses that exceed a specified level within a certain time period trigger events with potentially severe adverse impact.

Standard VaR, which does not reflect the *path properties* of asset prices, cannot capture such an impact on trading strategies, which motivates our interest in intra-horizon risk and first-passage probability for jump models. One goal of this paper is to give quantitative assessment of the importance of intra-horizon risk for risk management purposes.

3. General treatment of first-passage probability for Lévy jump models

The purpose of this section is to present our approach to estimating first-passage probability for jump models characterizing asset returns. Complete analytical solution to the first-passage probability problem for jump processes of interest is generally not available in closed-form. Therefore, to calculate first-passage probability, and hence, risk measures accounting for intra-horizon risk, we resort to a general approach based on partial integro-differential equations (PIDEs), which requires a simple boundary condition and is straightforward to implement. This calculation is equivalent to finding the value of a first-touch binary option, that is, an option paying \$1 if the price of an asset hits a certain level within a given time period, and 0 otherwise (Hirsa and Madan, 2003; Kudryavtsev and Levendorskii, 2006). However, VaR calculations are performed under the physical probability measure, so we also employ this measure throughout.

Recognize that the Laplace transform of the first-passage time for processes with two-sided, exponentially distributed jumps, and, more generally, with phase-type jumps, is known (Asmussen, Avram, and Pistorius, 2004; Kou and Wang, 2003). Efficient numerical procedures have also been developed for processes with one-sided jumps (see Rogers, 2000). Furthermore, certain analytical characterizations have been derived for stable processes, and in particular those with one-sided jumps (Kyprianou, 2006).

For a description of the PIDE approach to calculating first-passage probability, let S_t denote the asset price process, assumed to be right continuous with a left limit. Assume further that log-returns $X_t = \ln(S_t) - \ln(S_0)$ are governed by a one-dimensional Lévy process with a diffusion component with volatility σ , and a jump component with Lévy measure $k[x]$, where a jump $x_u \in (-\infty, +\infty)$ is defined as $x_u = \ln(S_u) - \ln(S_{u-})$. The PIDE approach is sufficiently general to cover candidate specifications, including those with double-sided jumps and infinite activity, where the integral of $k[x]$ is infinite.

Under the physical probability measure and for the short (10-day) horizon considered here, we assume that expected return is zero. Denote by

$$G[S, t, T] := \mathbb{E}[1_{S_u \leq H}, 0 < u < T] \tag{5}$$

the conditional expectation at time $t < T$ of the event that the asset price S_u hits the level $H < S_0$ within the time interval $u \in [0, T]$. If at any $u < t$ we have $S_u < H$, then $G[S, t, T] = 1$. Otherwise $0 < G[S, t, T] < 1$.

By construction, $G[S, t, T]$ is a martingale, since it is conditional expectation of a terminal random variable. Then $G[S, t, T]$ is the solution to the following PIDE:

$$\frac{1}{2} \sigma^2 S^2 G_{SS} - \frac{1}{2} \sigma^2 S G_S + G_t + \int_{-\infty}^{+\infty} \{G[S e^x, t, T] - G[S, t, T] - S G_S[S, t, T](e^x - 1)\} k[x] dx = 0, \tag{6}$$

with terminal and boundary conditions:

$$\begin{aligned} G[S, T, T] &= 0 \quad \text{for } S_T > H, \\ G[S, t, T] &= 1 \quad \text{for } S_t \leq H \text{ and } t \leq T. \end{aligned} \tag{7}$$

Eq. (6) follows from an application of Ito's lemma for semi-martingales (Jacod and Shiryaev, 1987). After changing variables to $s := \ln(S)$, $t := T - t$ and $g[s, t] := G[S, t, T]$, the PIDE (6) becomes

$$\frac{1}{2} \sigma^2 g_{ss} - \frac{1}{2} \sigma^2 g_s - g_t + \int_{-\infty}^{+\infty} \{g[s + x, t] - g[s, t] - g_s[s, t](e^x - 1)\} k[x] dx = 0, \tag{8}$$

with initial and boundary conditions:

$$\begin{aligned} g[s, 0] &= 0 \quad \text{for } s_T > \ln(H), \\ g[s, t] &= 1 \quad \text{for } s_t \leq \ln(H) \text{ and } t \leq T. \end{aligned} \tag{9}$$

It is important to emphasize that one should consider values of $g[s, t]$ for $s < \ln(H)$, as in (9), corresponding to asset prices below the level H . Downside-jumps can bring the log-price s below $\ln(H)$, hence $g[s, t]$ takes the value of one. For any s , this happens for jumps such that $x \leq \ln(H) - s$. Hence, jump models stand in contrast to diffusions, where, by continuity, values of $g[s, t]$ for $s < \ln(H)$ need not be considered.

Within the PIDE framework, we consider Merton's (1976) jump-diffusion model, the CGMY model by Carr, Geman, Madan, and Yor (2002), and the finite-moment log-stable (FMLS) model by Carr and Wu (2003). For these jump models, no Laplace transforms, or specific procedures, have been developed, while first-passage probability calculations under the PIDE approach are tractable. The models present a wide spectrum of jump features: one- and two-sided jumps, symmetric and asymmetric jump densities, jumps with finite and infinite activity, and jumps generating lighter or heavier tails of return densities. As it turns out, this variety of features accounts for quite different values of the risk measures, obtained across models in the empirical evaluation.

Staying in the class of Lévy models, we do not consider models that incorporate stochastic volatility. There are two reasons for this choice. First, for short horizons consistent with the regulatory convention, the jump structure is significantly more important than stochastic volatility (e.g., see Ait-Sahalia, 2004) in generating non-Normal returns. For applications of stochastic volatility in the VaR context, see Eberlein, Kallsen, and Kristen (2002). Second, the two-dimensional nature of a returns process with stochastic volatility makes the calculation of

first-passage probability more involved. Two-dimensional PIDEs for models with both jumps and stochastic volatility have been considered, for instance in Feng and Linetsky (2006), and can be applied in first-passage probability calculations, but are not pursued here.

4. Models of asset returns

This section outlines the theoretical models used in the empirical exercise. For each model we present the closed-form characteristic function used in time-series estimation, for goodness-of-fit tests, and to obtain VaR measures for processes with jumps, and we also present the first-passage probability PIDE. All expectations, $\mathbb{E}[\cdot]$, are taken under the physical probability measure.

4.1. Merton's jump-diffusion (JD) model

Merton (1976) suggested the following model for log-returns X_t :

$$dX_t = \mu dt + \sigma dW_t + dJ_t, \quad (10)$$

where $\sigma \in \mathbb{R}^+$ is the diffusion component of return volatility, $\{W_t, t \geq 0\}$ is standard Brownian motion, and $\{J_t, t \geq 0\}$ is a compound Poisson process with intensity $\lambda \in \mathbb{R}^+$ and Normally distributed jump sizes with mean μ_j and standard deviation σ_j . The Lévy measure is

$$k[x] = \frac{1}{\sigma_j \sqrt{2\pi}} \exp\left(-\frac{(x - \mu_j)^2}{2\sigma_j^2}\right), \quad \mu_j \in \mathbb{R}, \quad \sigma_j \in \mathbb{R}^+. \quad (11)$$

Under the physical measure and for short time intervals (up to 10 days), we assume that the expected return is equal to zero, so the drift in (10) is $\mu = -(\sigma^2/2) - \lambda \exp(\mu_j + \sigma_j^2/2)$.

The characteristic function of log-returns X_t (i.e., at horizon t) in this model is

$$\varphi_{X_t}[u] := \mathbb{E}[e^{iuX_t}] = \exp\left(iu\mu t - \frac{u^2\sigma^2 t}{2} + \lambda t \left(\exp\left(iu\mu_j - \frac{u^2\sigma_j^2}{2}\right) - 1 \right)\right). \quad (12)$$

Since the model has finite activity of jumps (the integral of $k[x]$ is finite), the PIDE (8) can be simplified:

$$\frac{1}{2}\sigma^2 g_{ss} + \mu g_s - g_t - \lambda g + \lambda \int_{-\infty}^{+\infty} g[s+x, t] k[x] dx = 0, \quad (13)$$

and needs to be solved with initial and boundary conditions (9). Various numerical procedures for solving this PIDE have been suggested in the literature (e.g., Andersen and Andreasen, 2000; Cont and Voltchkova, 2005; d'Halluin, Forsyth, and Vetzal, 2005; Feng and Linetsky, 2006). In the calculation of first-passage probability we implement the Feng and Linetsky (2006) approach, involving an implicit-explicit scheme with time-stepping extrapolation. The approach is robust and handles well the singularity in the initial condition at the first-passage level.

4.2. Exponentially dampened power law model of CGMY

The CGMY process, due to Carr, Geman, Madan, and Yor (2002), is a pure-jump Lévy process with no diffusion component and the following Lévy measure:

$$k[x] = \begin{cases} \lambda \frac{\exp(-\beta_- |x|)}{|x|^{1+\alpha}} & \text{for } x < 0, \\ \lambda \frac{\exp(-\beta_+ x)}{x^{1+\alpha}} & \text{for } x > 0, \end{cases} \quad (14)$$

$$\text{where } \beta_-, \lambda, \beta_+ \in \mathbb{R}^+. \quad (15)$$

The λ parameter is a measure of the arrival rate of jumps, both positive and negative. β_+ and β_- control the rate of exponential decay of the probability of up- and down-jumps of different sizes. α allows distinction between different classes of processes: depending on the value of α , the process may or may not be completely monotone, and may exhibit finite or infinite activity. The CGMY characteristic function is

$$\varphi_{X_t}[u] := \mathbb{E}[e^{iuX_t}] = \exp(iu\mu t + t\lambda\Gamma[-\alpha]\{(\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha\}). \quad (16)$$

Here $\Gamma[\cdot]$ is the mathematical Gamma function and $\mu = -\lambda\Gamma[-\alpha]\{(\beta_+ - 1)^\alpha - \beta_+^\alpha + (\beta_- + 1)^\alpha - \beta_-^\alpha\}$ ensures zero expected return.⁵

The PIDE for the first-passage probability for the CGMY model is

$$0 = -g_t + \lambda \int_{-\infty}^0 (g[s+x, t] - g[s, t] - g_s[s, t](e^x - 1)) \times \frac{e^{-\beta_- |x|}}{|x|^{1+\alpha}} dx + \lambda \int_0^{\infty} (g[s+x, t] - g[s, t] - g_s[s, t] \times (e^x - 1)) \frac{e^{-\beta_+ x}}{x^{1+\alpha}} dx. \quad (17)$$

For $\alpha > 0$, jumps have infinite activity and it is not possible to further simplify the PIDE (8), as in the case of Merton's model. To solve numerically the PIDE we follow Hirta and Madan (2003) and approximate the integrals to first or second order around the singularity of the jump measure at zero. In particular, we use finite differences on the mesh $[\ln(H), U] \times [0, T]$, where H is the first-passage level and U is the upper limit of numerical integration. Let $g[x, t]$ denote $g[x, t]$ for brevity, and Δ and h denote the steps in log-direction and time direction, respectively. For each s (where s takes values $s_j = \ln(H) + j\Delta$, for $j = 0, 1, \dots, N$), we write the sum of the two integrals in (17) as

$$I \approx I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \quad (18)$$

⁵ The appeal of the CGMY specification is as follows. First, keeping $\alpha < 0$ in (14) generates finite-activity compound Poisson jumps, similar to those in Merton's model. Even within the compound Poisson jump class, the distinct parametrization of upside and downside jumps with different scaling coefficients (β_+, β_-) allows us to differentiate the impact of upside vs. downside risks. Finally, allowing the exponent α to take on positive values can generate infinite-activity jumps with finite variation ($0 \leq \alpha < 1$), or with infinite variation ($1 \leq \alpha < 2$). Carr, Geman, Madan, and Yor (2002), Wu (2006), and Bakshi, Carr, and Wu (2008) show that CGMY performs well in fitting both time-series of returns and option prices, with typical point estimates of $\alpha \in (0, 1)$.

where

$$I_1 = \lambda \int_{-\infty}^{\ln(H)-s} (1 - g[s] + g_s[s]) \frac{e^{-\beta_-|x|}}{|x|^{1+\alpha}} dx, \tag{19}$$

$$I_2 = \lambda \int_{\ln(H)-s}^{-d} (g[s+x] - g[s] + g_s[s]) \frac{e^{-\beta_-|x|}}{|x|^{1+\alpha}} dx, \tag{20}$$

$$I_3 = -\lambda \int_{-\infty}^{-d} g_s[s] \frac{e^x e^{-\beta_-|x|}}{|x|^{1+\alpha}} dx, \tag{21}$$

$$I_4 = \lambda \int_{-d}^0 \left(g[s] + g_s[s]x + \frac{1}{2} g_{ss}x^2 - g[s] - g_s[s] \left(1 + x + \frac{1}{2}x^2 - 1 \right) \right) \frac{e^{-\beta_-|x|}}{|x|^{1+\alpha}} dx, \tag{22}$$

$$I_5 = \lambda \int_0^d \left(g[s] + g_s[s]x + \frac{1}{2} g_{ss}x^2 - g[s] - g_s[s] \left(1 + x + \frac{1}{2}x^2 - 1 \right) \right) \frac{e^{-\beta_+x}}{|x|^{1+\alpha}} dx, \tag{23}$$

$$I_6 = \int_d^U (g[s+x] - g[s] - g_s[s](e^x - 1)) \frac{e^{-\beta_+x}}{x^{1+\alpha}} dx. \tag{24}$$

Note that in I_1 the first term in the bracket is one instead of $g[s+x]$: For any log-price s , negative jumps x with absolute magnitudes larger than $s - \ln(H)$ take the price below the first-passage level, hence, $g[s+x] = 1$ for all such jumps. The discretization for each of the six integrals above is given in Appendix B. We solve the PIDE using an explicit finite-difference scheme.

4.3. Finite-moment log-stable model (FMLS)

The FMLS process, introduced in Carr and Wu (2003), can be considered as a special case of the CGMY process. It is pure-jump, has only down-sided jumps, and its Lévy measure is

$$k[x] = \frac{\lambda}{|x|^{1+\alpha}} \quad \text{where } x < 0, \quad \lambda \in \mathbb{R}^+, \quad \alpha \in (1, 2). \tag{25}$$

Alternatives with only downside jumps are potentially attractive as they focus on the phenomenon of primary importance in the VaR and risk management context. Since the numerator in (25) lacks the exponential damping factor of the CGMY Lévy measure, the left tail is fatter. Actually, the left tail is so fat that FMLS is only made a feasible model for asset returns by disallowing any up-jumps. Only this restriction ensures that all moments of the asset price are finite (unlike the stable processes with two-sided jumps). With $\mu = v^\alpha \sec(\pi\alpha/2)$, the FMLS characteristic function is

$$\varphi_{X_t}[u] := \mathbb{E}[e^{iuX_t}] = \exp\left(iu\mu t - t(iuv)^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right),$$

where $v = \left(\frac{\lambda\Gamma[\alpha/2]\Gamma[1-\alpha/2]}{2\Gamma[1+\alpha]}\right)^{1/\alpha}$. (26)

The PIDE for the first-passage probability in the FMLS case is analogous to (17), but only has an integral

corresponding to down-jumps:

$$0 = -g_t + \lambda \int_{-\infty}^0 (g[s+x, t] - g[s, t] - g_s[s, t](e^x - 1)) \frac{1}{|x|^{1+\alpha}} dx. \tag{27}$$

The discretization of the integral is given in Appendix B and the model is amenable to first-passage probability calculation.

5. Maximum likelihood estimation

Density functions for Lévy models are often not available in analytical closed form, preventing direct maximum likelihood estimation (e.g., Ait-Sahalia, 2002; Ait-Sahalia and Jacod, 2007; Singleton, 2001; Yu, 2007). However, since Lévy processes are completely described by their characteristic function through the Lévy–Khinchine theorem (i.e., Bertoin, 1996; Sato, 1999), there is a long-standing tradition to employ characteristic functions in estimation procedures. One approach is based on the observation that (i) the empirical characteristic function contains the same information as the likelihood function, and (ii) any set of moment conditions can be calculated using the empirical characteristic function (see Feuerverger and McDunnough, 1981; Carrasco and Florens, 2000; Singleton, 2001). Hence, GMM estimation techniques have been applied, and conditions under which empirical characteristic function based estimators attain the efficiency of maximum likelihood estimators have been derived (e.g., Sueishi and Nishiyama, 2006). Furthermore, empirical characteristic function based procedures have been generalized to handle latent variables such as stochastic volatility (e.g., Singleton, 2001; Knight and Yu, 2002), or to estimate multivariate models (e.g., Carrasco, Chernov, Florens, and Ghysels, 2002).

A related approach has been explored by Carr, Geman, Madan, and Yor (2002), Das (2002), Wu (2006, 2007), and Bakshi, Carr, and Wu (2008). They note that, given the fast Fourier transform procedure (e.g., Carr and Madan, 1999) to recover numerically the return density from the analytical characteristic function, Lévy models can be estimated through maximum likelihood. While the maximum likelihood estimation approach is particularly suited for the class of Lévy jump models with i.i.d. increments, it can also be modified to handle more intricate jump models with stochastic volatility through state space techniques (Bates, 2006; Bakshi, Carr, and Wu, 2008; Wu, 2007). Since we estimate our risk measures over a short, 10-day horizon, where jumps are known to be more relevant than stochastic volatility, we have not considered models with stochastic volatility to maintain parsimony.

Mostly for computational efficiency and the ease of obtaining jump parameters, we adopt the maximum likelihood approach based on the analytical characteristic function. Corresponding to a Lévy model with parameter vector θ , the log-likelihood function of the returns time series $\{X_t : t = 1, 2, \dots, T\}$ is readily computed with interpolation from the returns densities $\{\Phi[X_k; \theta] : k = 1, 2, \dots, \aleph\}$. Here \aleph is the number of grid points in the

Table 1

Analytical form of the characteristic functions used in the maximum likelihood estimation.

Entries display the characteristic function $\varphi_{X_t}[u] := \mathbb{E}[e^{iuX_t}]$ of log-returns X_t at horizon t for each model. We estimate each model on time-series of returns via maximum likelihood using the fast Fourier inversion of the characteristic function. The three models are: (i) the Merton (1976) jump-diffusion model (the JD), (ii) the exponentially dampened power law model of Carr, Geman, Madan, and Yor (2002) (the CGMY), and (iii) the finite-moment log-stable model of Carr and Wu (2003) (the FMLS). Fixing α in the CGMY model has negligible effect on the maximized log-likelihood, and we set $\alpha = 0.5$ to enhance the identification of $(\beta_-, \lambda, \beta_+)$. Setting $\alpha = 0.5$ is compatible with models exhibiting infinite-activity jumps with finite variation. In each model, μ is the convexity correction which ensures that expected returns over a 10-day horizon are zero, as we assume throughout.

Model	Characteristic function	Parameter vector, θ
JD	$\exp\left(iu\mu t - \frac{u^2\sigma^2 t}{2} + \lambda t \left(\exp(iu\mu_j - \frac{u^2\sigma_j^2}{2}) - 1\right)\right)$ <p>where $\mu = -\frac{\sigma^2}{2} - \lambda \exp\left(\mu_j + \frac{\sigma_j^2}{2}\right)$</p>	$\theta := (\sigma_j, \mu_j, \lambda, \sigma)$
CGMY	$\exp(iu\mu t + t\lambda\Gamma[-\alpha](\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha)$ <p>where $\mu = -\lambda\Gamma[-\alpha](\beta_+ - 1)^\alpha - \beta_+^\alpha + (\beta_- + 1)^\alpha - \beta_-^\alpha$</p>	$\theta := (\beta_-, \lambda, \beta_+, \alpha)$ $\alpha = 0.50$
FMLS	$\exp\left(iu\mu t - t(iu)^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right)$ <p>where $v = \left(\frac{\lambda\Gamma[\alpha/2]\Gamma[1-\alpha/2]}{2\Gamma[1+\alpha]}\right)^{1/\alpha}$ $\mu = v^\alpha \sec\left(\frac{\pi\alpha}{2}\right)$</p>	$\theta := (\alpha, \lambda)$ $\lambda \in \mathbb{R}^+, \alpha \in (1, 2)$

fast Fourier inversion of the characteristic function, and we employ a large $\aleph (= 2^{14})$ to ensure precise interpolation of the numerical density. To improve numerical stability, we standardize the return series as in Wu (2006). Following Aït-Sahalia (2002), the estimation is carried out by maximizing the likelihood function over the parameter space Θ :

$$\max_{\theta \in \Theta} \mathcal{L}[\theta] := \sum_{t=1}^T \log(\Phi[X_t; \theta]). \quad (28)$$

Numerical second derivatives of the likelihood function evaluated at the optimal parameter vector provide the Hessian matrix H . Standard errors can be computed from the Hessian as $\sqrt{\text{diag}(-H^{-1})}$.

If Δ is the time interval over which returns are calculated (e.g., $\Delta = \frac{1}{52}$ for weekly returns), the returns density $\Phi[X_k; \theta]$ at grid point k , for $k = 1, 2, \dots, \aleph$ derived via fast Fourier inversion of the characteristic function is

$$\Phi[X_k; \theta] = \frac{1}{\pi} \sum_{j=1}^{\aleph} \text{Re} \left[e^{-i(2\pi/\aleph)(j-1)(k-1)} e^{(1/2)i\pi\aleph(j-1)\ell} \right. \\ \left. \times \varphi_{X_{\Delta}}[(j-1)\ell; \theta] \frac{\ell}{3} (3 + (-1)^j - \delta_{j-1}) \right], \quad (29)$$

where $\varphi_{X_{\Delta}}[u; \theta]$ is the characteristic function of returns X_{Δ} , as presented in Table 1 for each of the three models; \hbar is the step in the grid at which the characteristic function is calculated; ℓ is the step in the grid at which the returns density is obtained; \aleph is the number of grid points in the fast Fourier inversion; δ_{j-1} is Kronecker's delta function, equal to one for $j = 1$ and 0 otherwise. The term $\frac{1}{3}(3 + (-1)^j - \delta_{j-1})$ in (29) incorporates Simpson's rule weights: it provides accurate integration even with a larger step \hbar , which allows for a finer grid of returns thereby ensuring precise interpolation of the returns density.

6. Empirical design and assessment of intra-horizon risk in jump models

For the empirical assessment exercises, we compute standard VaR and intra-horizon VaR-I for each of the three models with jumps. Bear in mind that the standard 99% VaR is the absolute value of the loss level *at the end of* the regulatory 10-day horizon, which is exceeded with probability 1%. On the other hand, 99% VaR-I is the absolute value of the loss level exceeded *on any day during* the regulatory 10-day horizon with probability 1%.

To compute standard VaR, we obtain numerically the 1% quantile of the 10-day return distribution. Implementation relies on the estimated model parameters, and we use the fast Fourier transform to invert the characteristic function for the 10-day returns which allows us to infer the probability density. We integrate numerically this density and calculate the 1% quantile of the obtained distribution, which provides the VaR measure. To compute VaR-I, we exploit model parameters and the first-passage density calculation procedures outlined in Section 4 to find the distribution of loss levels that are exceeded over a 10-day horizon. The 1% quantile of this distribution provides the VaR-I measure.

6.1. Universe of risk exposures

To consider a broad set of assets and a wide range of risks that may be impacting the portfolio of financial institutions (e.g., Green and Figlewski, 1999; Engle, 2004; Jorion, 2006), our empirical investigation employs return time series on (i) three equity indexes, (ii) two US Treasury bonds, (iii) two dollar-based exchange rates, and (iv) at-the-money put and call options on the S&P 500 index, with fixed maturity of 30 days. Together, these asset groups represent four types of risk exposures, which are

Table 2

Average and maximum VaR and VaR-I multiples for the jump models.

VaR is the 1% quantile of the 10-day returns distribution, and VaR-I is the 1% quantile of the first-passage distribution, obtained from each model estimation. Benchmark VaR is the 1% quantile of the Normal distribution $N(\hat{\mu}, \hat{\sigma})$, where $\hat{\mu} = 0$ and $\hat{\sigma}$ is the standard deviation of the returns time series, used in each VaR and VaR-I calculation. Reported are the values of the multiples $\text{VaR}/(2.32\hat{\sigma} - \hat{\mu})$ and $\text{VaR-I}/(2.32\hat{\sigma} - \hat{\mu})$. Each model is estimated on time series of 260 weekly (five trading day) returns and the estimations are performed once every month. The three models are: (i) the Merton jump-diffusion model (the JD), (ii) the exponentially dampened power law model of Carr, Geman, Madan, and Yor (2002) (the CGMY), and (iii) the finite-moment log-stable model of Carr and Wu (2003) (the FMLS). At-the-money call and put options on the S&P 500 index have a maturity of 30 days.

	Panel A: Average VaR multiples			Panel B: Maximum VaR multiples			Panel C: Average VaR-I multiples			Panel D: Maximum VaR-I multiples		
	JD	CGMY	FMLS	JD	CGMY	FMLS	JD	CGMY	FMLS	JD	CGMY	FMLS
S&P 500	1.21	1.33	1.44	1.33	1.37	1.69	1.60	1.39	1.94	1.77	1.50	2.08
FTSE	1.20	1.21	1.35	1.38	1.44	1.50	1.50	1.38	1.85	1.70	1.55	2.07
NIKKEI	1.14	1.11	1.39	1.19	1.25	1.75	1.28	1.27	1.80	1.40	1.34	2.12
1Y-TB	1.29	1.41	1.63	1.61	1.50	1.85	1.60	1.48	2.02	1.93	1.69	2.47
20Y-TB	1.21	1.23	1.59	1.27	1.27	1.87	1.43	1.32	2.30	1.51	1.39	2.45
\$-GBP	1.12	1.15	1.33	1.16	1.19	1.83	1.23	1.23	1.54	1.33	1.29	2.45
\$-JPY	1.24	1.22	1.47	1.37	1.45	1.56	1.49	1.38	1.83	1.72	1.52	2.04
ATM Put	1.53	1.53	1.67	1.62	1.58	1.86	1.89	1.69	2.15	1.97	1.72	2.35
ATM Call	1.37	1.29	1.68	1.39	1.31	1.85	1.61	1.45	2.55	1.67	1.47	2.64

equity, interest rate, currency risk, and equity volatility risk. We note that volatility risk is often surrogated with straddle returns which are correlated with changes in volatility. However, we show results with call and put returns separately, which still bear essential information about volatility risk, but may also reflect additional risks of concern to financial institutions.⁶

Specifically, the equity indexes employed in the study are S&P 500, FTSE, and NIKKEI. Based on constant maturity Treasury bond yields (from the Federal Reserve Board), we extract zero-coupon yield curves and calculate returns on zero-coupon Treasury bonds with maturities of one and 20 years. The exchange rates are \$-GBP and \$-JPY, with the foreign currency as the reference. Following Jackwerth and Rubinstein (1996) and Bakshi and Kapadia (2003), we construct time-series of returns for at-the-money (hereby ATM) calls and puts with maturity of 30 days (from OptionMetrics). The choice of assets is consistent with our general focus on intra-horizon risk in jump models. As shown in Ait-Sahalia (2004), Andersen, Bollerslev, Diebold, and Vega (2003, 2005), Bakshi, Carr, and Wu (2008), and Piazzesi (2005), incorporating the jump component is important in capturing large discontinuous movements in security prices.

To make our results comparable with other studies and with the post-Amendment evaluation period, the

1995–2005 sample period is adopted for our investigation. Consistent with the PIDE derived in Section 3, the return time-series are demeaned.

In trading the universe of assets, financial institutions have the flexibility of taking long or short positions. We assume that they take the riskier position, and hence, we choose the position (long or short) such that the skewness of the returns distribution faced by the financial institution is negative. While this procedure may potentially bias up estimated risk measures, it agrees with Green and Figlewski (1999) who discuss the concern of commercial banks for short positions and negatively skewed returns. It also agrees with Berkowitz and O'Brien (2005), who find that traders at large banks often switch positions from short to long and vice versa. This procedure is also appropriate in the case of options: banks are typically short volatility, hence, they have negatively skewed risk exposure.

6.2. Accounting for jumps alone can increase benchmark VaR up to 1.68 times

Table 2 reports 10-day 99% VaR and VaR-I for each Lévy jump model. We choose to estimate the models once every month with returns sampled at the weekly interval (five trading days) and relying on a window of 260 weekly observations (see also Manganelli and Engle, 2001). The appropriateness of these choices and robustness issues are detailed in Section 6.5.

Most importantly, Table 2 reports ratios of VaR and VaR-I to the benchmark VaR. Presenting the results in ratio format facilitates evaluation of the relevance of jumps and intra-horizon risk. One convenient benchmark is the 99% Normal VaR, which is consistent with the

⁶ We treat options and the associated volatility risk in the same way as other assets, which seemingly ignores the specifics of options as derivative instruments requiring specific option pricing models. Note, however, that all our estimations are performed on returns, not prices, and we use throughout statistical probability measures and there is no risk-neutral modeling. Hence, the consistent use of the same approach to both option returns and other asset returns is reasonable in our context.

current regulatory framework, that also uses a 99% VaR level and implicitly assumes a Brownian motion model for profits and losses via the “square root of t ” rule for transforming daily VaR into 10-day VaR.

We compute and report the following ratios (multiples) of VaR and VaR-I to the benchmark VaR:

$$\text{VaR}/(2.326\hat{\sigma} - \hat{\mu}) \quad \text{and} \quad \text{VaR-I}/(2.326\hat{\sigma} - \hat{\mu}), \quad (30)$$

where $\hat{\sigma}$ is the return standard deviation and $\hat{\mu}$ is the mean return. The denominator in (30) represents the 1% quantile of a Normal distribution $N(\hat{\mu}, \hat{\sigma})$. Since the return time-series are demeaned, $\hat{\mu} = 0$ throughout.

It is worth emphasizing that our findings about intra-horizon risk are qualitatively robust to the choice of a benchmark. We later report VaR and VaR-I multiples obtained with an alternative VaR methodology based on the filtered historical simulation of Barone-Adesi, Giannopoulos, and Vosper (1999) and Barone-Adesi, Engle, and Mancini (2008). Furthermore, to address the issue of potential misspecification of models, we perform in Section 6.6 a set of standard diagnostic tests.

Table 2 reports both average and maximum values for the ratio of jump model VaR and jump model VaR-I to the benchmark VaR. The maximum values are economically relevant as a stress test measure in risk management: even when large (adverse) return movements are not observed in each estimation period, akin to the classic peso problem, the extreme market situations are of significant concern to both financial institutions and regulators.

To go to the heart of the issues, consider the interpretation of the reported value of 1.21 for the S&P 500 index corresponding to the JD model in Panel A of Table 2: it indicates that the 1% quantile of the S&P 500 10-day return distribution, fitted with Merton's jump-diffusion (JD) is on average 1.21 times higher than the benchmark VaR over the sample period. Similarly, the reported 1.33 in Panel B indicates that the 99% VaR for S&P 500 10-day returns fitted with JD is at most 1.33 times higher than the benchmark VaR. A similar pattern exists across other jump models, implying that the benchmark VaR tends to substantially underestimate market risk exposures.

When averaged across the three models and within an asset class, VaR is 1.26 times higher than benchmark VaR in the case of equities, 1.38 times higher for bonds, 1.25 times higher for currencies, and 1.51 times higher for options. The magnitude of risk underestimation with the benchmark VaR is accordingly the highest for volatility exposures and consistent with the regulator's explicit concern about pitfalls in measuring option-related risks.

Comparison of VaR multiples across jump models is also informative: for a given asset class, the average multiples over benchmark VaR are consistently more pronounced for the FMLS. For instance, multiples above 1.60 are observed for FMLS for both options and Treasury bonds. The highest VaR multiples of 1.67 and 1.68 are obtained when the FMLS is applied to ATM put option returns and ATM call option returns, respectively. Additionally, as seen from Panel B of Table 2, fixing an exposure, the maximum 99% VaR multiples are also the

highest for the FMLS. In fact, the FMLS model generates the highest multiples irrespective of the type of exposure.

6.3. Accounting for intra-horizon risk in jump models can increase benchmark VaR up to 2.6 times

Turning our attention to what is gained by adopting the VaR-I risk measure, Panels C and D of Table 2 show that across models and within an asset class, average VaR-I is 1.56 times higher than benchmark VaR for equities, 1.67 times higher for bonds, 1.45 times higher for currencies, and 1.89 times higher for options. The magnitude of risk underestimation with the benchmark VaR is again most severe when measuring volatility exposures.

Returning to the theme of the relevance of intra-horizon risk, we observe further that the model with Poisson jumps, i.e., the JD, can produce average VaR-I multiples over benchmark VaR typically around 1.40–1.60, and up to 1.89 for puts. At the same time, CGMY typically yields lower multiples in the 1.30–1.50 range, while the multiples for FMLS are often about 2.0, and as high as 2.55 for calls. The maximum VaR-I multiples in Panel D show similar ordering across models. Note in particular that the maximum multiple of 2.64 is obtained for call options, again under the FMLS.

Fig. 1 plots the time-series variation in the estimated ratios of VaR-I to benchmark VaR corresponding to a sample of assets. The main point to note is the consistency with which the FMLS produces the highest multiples over the benchmark, suggesting that our conclusions based on average multiples are robust and not an artifact of outliers. There is a striking pattern that the FMLS generates the highest VaR-I multiples, and the CGMY the lowest, on a period-by-period basis, with the JD straddling in between the two models. We furthermore observe that the CGMY produces less period-to-period variation in the multiples over the benchmark, while the FMLS-based VaR-I exhibits the most variation.

The documented high VaR-I multiples for the FMLS can be explained by the fatter left tail of returns distribution under this model. This tail decays to zero so slowly that the returns distribution has infinite second and higher moments, in contrast to the other two models. The high VaR-I multiples for the FMLS can be further understood in terms of the Lévy densities of the different models. From Eqs. (11) and (14) the jump density decays exponentially or faster with the jump size in the JD and CGMY models. In contrast, for FMLS the jump density only decays as a power law, allowing for large jumps to occur with higher probability. This specific path property is imperative from the perspective of intra-horizon risk, since a sequence of large jumps can account for a large loss over a short interval, even if it is overall compensated by the drift of the process.

The goodness-of-fit diagnostic tests discussed later show no evidence that FMLS is any more misspecified relative to the other candidate models, and hence, the reported high multiples cannot be attributed to estimation error and implausible parameter inputs. The high VaR

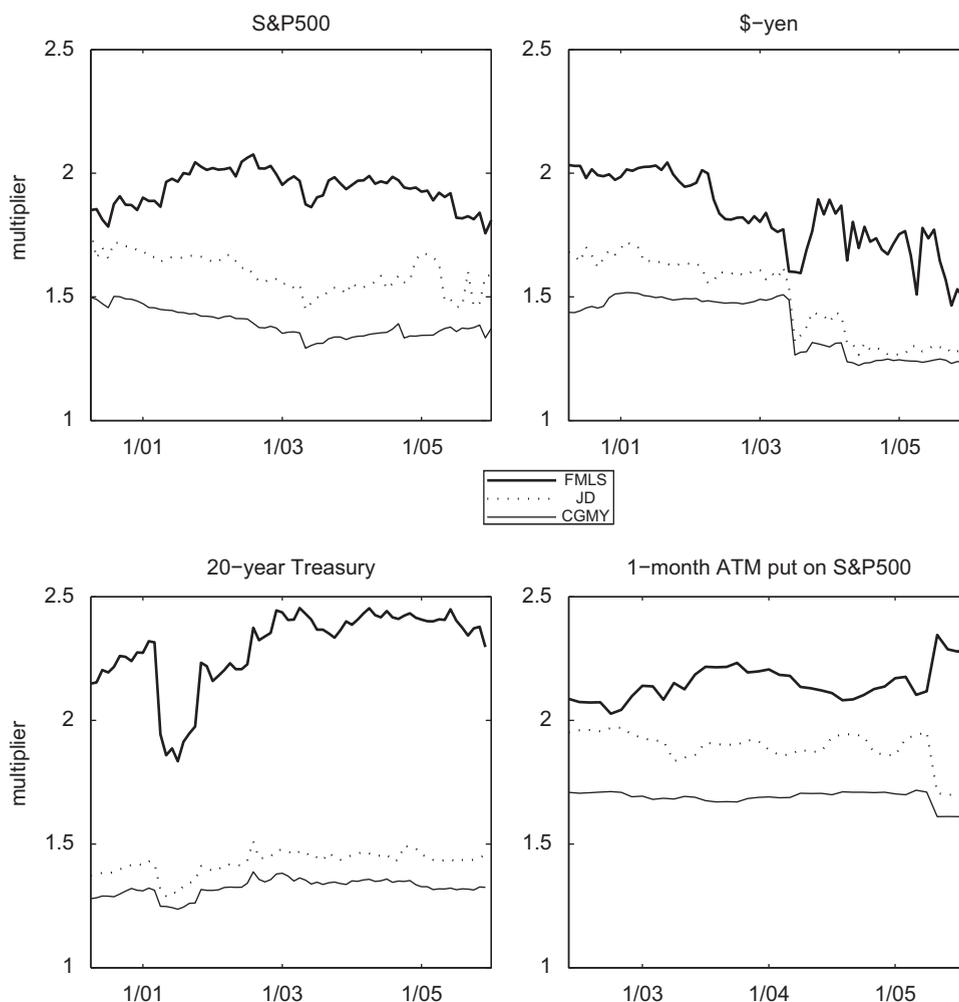


Fig. 1. VaR-I multiples. The graph plots VaR-I multiples over benchmark VaR. Each model is estimated on time-series of 260 weekly (five trading day) returns. VaR-I is the 1% quantile of the first-passage distribution obtained from each model estimation. Benchmark VaR is the 1% quantile of the Normal distribution $N(\hat{\mu}, \hat{\sigma})$, where $\hat{\mu} = 0$ and $\hat{\sigma}$ is the standard deviation of the returns time-series, used in each VaR calculation. Plotted in the graph are the values of VaR-I/ $(2.32\hat{\sigma} - \hat{\mu})$ for the Merton (1976) jump diffusion model (the JD), the exponentially dampened power law model of Carr, Geman, Madan, and Yor (2002) (the CGMY), and the finite-moment log-stable model of Carr and Wu (2003) (the FMLS).

and VaR-I multiples under the FMLS are a structural property of the model.

6.4. Jump risk increases the probability of large losses and amplifies intra-horizon risk

Can we distinguish the component of VaR-I, attributable to the jump model, from the component attributable to intra-horizon risk? We recognize that unambiguous separation may not be possible, if, for example, jump risk, while giving rise to an increased probability of large losses, also leads to an increased probability of intra-horizon losses and thereby induces higher VaR-I.

Table 2 can provide information on the connection between intra-horizon and jump risks embedded in the VaR-I risk measure in two ways. First, dividing the

numbers reported in Panel C of Table 2 by the corresponding entries in Panel A of Table 2 across each asset class, we observe that average VaR-I with jumps exceeds average VaR with jumps by 27.0% for equities, 21.4% for Treasury bonds, 19.6% for currencies, and 19.2% for options. Second, comparing Panels D and B shows that maximum VaR-I with jumps exceeds maximum VaR with jumps by about 20% across models in each asset class. Across assets, VaR-I exceeds VaR (both average and maximum) by about 20% for Merton's JD, 10% for CGMY, and 30% for FMLS.

To put the above magnitudes in perspective, note that, from an application of the reflection principle, the 1% quantile of the first-passage distribution for Brownian motion with no drift exceeds the 1% quantile of the respective Normal distribution by 10.7% (Feller, 1971). Therefore, VaR-I *without jumps* is 1.107 times higher than

our benchmark Normal VaR, irrespective of the underlying asset.

Observe that the percentage difference between VaR-I and VaR in jump models is not constant, and is typically higher than the respective difference in the Brownian motion case. Therefore, one may not be able to directly distinguish the intra-horizon risk component from the jump risk component in VaR-I. In particular, the respective percentage differences for the FMLS indicate that higher jump risk may give rise to an increased probability of intra-horizon losses, and hence, to higher VaR-I with jumps and to a stronger departure of VaR-I with jumps from VaR with jumps. We provisionally interpret these findings to mean that intra-horizon risk is dependent on the jump structure of a model, and that jump risk tends to amplify the intra-horizon risk component in VaR-I for the jump models.

From another angle, note that the multiples of jump risk measures or intra-horizon risk measures over benchmark VaR separately remain well below the Basel multipliers between three and four. In particular, the *average* multiples over the benchmark for VaR accounting only for jumps in Table 2 are typically about 1.2–1.4 and do not exceed 1.68. Even the *maximum* VaR multiple does not exceed 1.9. Furthermore, VaR-I without jumps only exceeds the benchmark VaR by 10.7%. On the other hand, the multiples for VaR-I with jumps in the range of 2.4–2.6 (both average and maximum) are revealing, and provide partial support for the Basel multipliers. Overall, our empirical investigation supports the view that intra-horizon risk and jump risk together can justify a significant part of the Basel multipliers. This conclusion, however, is subject to a few qualifications: First, it assumes that the benchmark for comparison is Normal VaR. As is now well-known, the Normal VaR as a risk measure is conceptually straightforward, but more sophisticated VaR measures are used in practice. Second, we have built in conservativeness in our risk estimates by considering individual assets instead of possibly diversified portfolios, and also by using the riskier of a short and long position. Third, we have limited our focus to traditional assets and vanilla options, which may not represent accurately all the risks in the trading portfolios of financial institutions. Nonetheless, our analysis corroborates the relevance of considering intra-horizon risk based on jump models, and its ability to narrow the gap with the Basel multipliers with certain models, especially the two-parameter FMLS.

6.5. Time scaling strongly impacts the calculation of risk measures in jump models

A critical design issue is the choice of frequency of the returns used in the calculation of 10-day risk measures. A common approach is to estimate VaR from daily returns and then scale the result with $\sqrt{10}$ to obtain the regulatory 10-day VaR. While this scaling does not introduce bias when returns follow Brownian motion, it may not be appropriate for the calculation of VaR in non-

Gaussian models for asset returns (see, among others, Basel Committee, 1996; Aït-Sahalia and Jacod, 2009).⁷

Departing for a moment from the Lévy processes considered in this paper, note that scaling by \sqrt{t} , and, more generally, by t^H , for $H > 0$, implies that the returns process is H-self-similar, i.e., the distribution of returns generated by this process over any horizon has the same shape, upon scaling. This property characterizes, among others, the α -stable processes, where $H = 1/\alpha$, and $\alpha \in (0, 2]$ is the stable index of the process. One special case in the stable class is Brownian motion, where $\alpha = 2$ and the scaling factor is \sqrt{t} , while, in general, α -stable processes scale as $t^{1/\alpha}$; see, in particular, Aït-Sahalia and Jacod (2009) for one such characterization. We note that FMLS can be considered an extreme-skewness version of a stable process, and hence, is self-similar and scales as $t^{1/\alpha}$. In contrast, Merton's JD and CGMY are not self-similar, hence, the shape of the returns distribution under these models changes with the horizon, and they do not exhibit $t^{1/\alpha}$ scaling.⁸

What are the implications of the above observations for calculating the risk measures? First, for Merton's JD and CGMY (and for any other Lévy process with finite variance), using parameters, obtained from model estimation on daily returns, to calculate 10-day risk measures, implicitly assumes that asset returns over longer horizons converge to the Normal distribution. If this assumption is not validated in the data, using daily parameters may result in understatement of risk. For example, in our 1995–2005 sample period, the skewness and excess kurtosis are -0.006 and 2.80 for daily S&P 500 returns, 0.24 and 3.40 for five-day returns and -0.31 and 2.52 for 10-day returns, clearly departing from the Lévy decay rates of $1/\sqrt{t}$ and $1/t$, respectively. A similar pattern is observed for the other assets, suggesting that it is inappropriate to use daily returns to obtain 10-day VaR or VaR-I measures in Merton's JD and CGMY models.

Second, an alternative approach would be to estimate Merton's JD and CGMY on daily returns, then calculate *daily* VaR or VaR-I, and finally scale the result by $\sqrt{10}$, consistent with the regulator's suggestion. While feasible, this approach is ad hoc, it is not consistent with the

⁷ Danielsson and Zigrand (2006) emphasize that when returns follow a jump-diffusion process, the square-root-of-time rule leads to a systematic underestimation of risk. In a GARCH framework, McNeil and Frey (2000) find evidence for a power law scaling for quantiles, where the power depends on the initial volatility level. Finally, Guidolin and Timmerman (2006) discuss quantile scaling for a number of GARCH models and find a widely divergent scaling behavior across these models. The issue of VaR scaling is far from being resolved and warrants further research.

⁸ This property follows on noting that both processes have independent increments with finite variance, hence, the Central Limit Theorem applies and model-generated returns at longer horizons are close to Normal, even if Merton's JD and CGMY may generate highly non-Normal returns over a short horizon. This intuition can be quantified by an explicit calculation of the time dependence of the higher moments of returns under the two models: it has been shown, for example, in Konikov and Madan (2002), that for a Lévy process with finite variance, skewness decreases in absolute value proportional to $1/\sqrt{t}$ and excess kurtosis decreases proportional to $1/t$. The rate of convergence to Normality for Lévy processes has also been illustrated in Wu (2006).

Table 3

VaR and VaR-I multiples at different horizons and return frequencies (single estimation, 2001–2005).

Panel A shows VaR and VaR-I multiples over benchmark VaR, obtained with parameters from a single estimation on *daily returns* over 2001–2005. VaR is the 1% quantile of the returns distribution at one-day, five-day, and 10-day horizons, obtained from each model estimation, and VaR-I is the 1% quantile of the first-passage distribution over a one-day, five-day, and 10-day horizons, obtained from each model estimation. Benchmark VaR is the 1% quantile of the Normal distribution $N(\bar{\mu}, \hat{\sigma})$, where $\bar{\mu} = 0$ and $\hat{\sigma}$ is the standard deviation of daily returns, scaled by 1, $\sqrt{5}$ and $\sqrt{10}$, respectively. Reported are the values of the multiples $\text{VaR}/(2.32\hat{\sigma} - \bar{\mu})$ and $\text{VaR-I}/(2.32\hat{\sigma} - \bar{\mu})$. Panel B shows VaR and VaR-I multiples, obtained with parameters from a single model estimation using *five-day returns* over 2001–2005. VaR and VaR-I are 1% quantiles of the respective distributions at five-day and 10-day horizons, and benchmark VaR is the 1% quantile of the Normal distribution $N(\bar{\mu}, \hat{\sigma})$, where $\bar{\mu} = 0$ and $\hat{\sigma}$ is the standard deviation of five-day returns scaled by 1 and $\sqrt{2}$, respectively.

		Panel A: 1-, 5-, and 10-day VaR and VaR-I multiples, estimated from daily returns								
		JD			CGMY			FMLS		
		1-day	5-day	10-day	1-day	5-day	10-day	1-day	5-day	10-day
S&P 500	VaR	1.21	1.06	1.04	1.17	1.05	1.03	1.11	1.13	1.13
	VaR-I	1.61	1.24	1.15	1.32	1.16	1.13	1.71	1.77	1.78
20Y-TB	VaR	1.19	1.07	1.05	1.22	1.10	1.08	1.21	1.24	1.24
	VaR-I	1.95	1.42	1.25	1.38	1.22	1.19	1.85	1.99	2.00
\$-JPY	VaR	1.20	1.07	1.05	1.16	1.06	1.05	1.10	1.11	1.12
	VaR-I	1.93	1.41	1.28	1.32	1.18	1.15	1.68	1.71	1.71
ATM Put	VaR	1.32	1.16	1.12	1.29	1.14	1.11	1.46	1.64	1.63
	VaR-I	1.78	1.40	1.28	1.49	1.28	1.23	2.13	2.17	2.18
		Panel B: 5- and 10-day VaR and VaR-I multiples, estimated from 5-day returns								
		JD		CGMY		FMLS				
		5-day	10-day	5-day	10-day	5-day	10-day	5-day	10-day	
S&P 500	VaR	1.10	1.07	1.22	1.15	1.18	1.19			
	VaR-I	1.71	1.59	1.37	1.28	1.79	1.81			
20Y-TB	VaR	1.19	1.15	1.12	1.07	1.24	1.26			
	VaR-I	1.74	1.43	1.39	1.35	2.26	2.35			
\$-JPY	VaR	1.08	1.06	1.07	1.05	1.04	1.04			
	VaR-I	1.30	1.23	1.30	1.24	1.55	1.55			
ATM Put	VaR	1.29	1.22	1.50	1.38	1.65	1.69			
	VaR-I	1.76	1.59	1.72	1.58	2.06	2.31			

structure of the two models or the nature of the risk measures (VaR-I in particular), and may require additional justification. We note that modifications of Lévy processes, which explicitly scale as $t^{1/\alpha}$, have been developed in Carr, Geman, Madan, and Yor (2007). Third, FMLS differs in an important way from the other Lévy models: it has infinite second and higher moments, is self-similar, and scales as $t^{1/\alpha}$. Therefore, using daily returns to obtain 10-day risk measures is consistent with the model structure. This approach would be further validated, if actual returns exhibit the self-similarity, inherent in the FMLS model. However, the evidence is mixed, in particular for daily and lower frequencies (see, e.g., Wu, 2006). In sum, Lévy models may not be able to capture properly the scaling behavior of asset returns, hence, for these models, the frequency of returns used in calculating risk measures is of critical importance.

The most appropriate approach to calculate the risk measures is, unfortunately, not feasible: estimating the distribution of 10-day returns using 10-day sampling requires data from the distant past, which are likely not relevant for risk measurement. Based on the above reasoning, Table 2 reported estimates for 10-day 99% VaR and VaR-I, taking parameters from estimations on weekly returns (i.e., five trading days). In this way the scaling issue is partly mitigated as we avoid the use of model parameters across two widely different frequencies (e.g., one day vs. 10 days), and at the same time avoid using data from the distant past that may be irrelevant to measuring contemporary exposures.

How severe empirically is the scaling problem for the particular horizon over which our risk measures are calculated? To explore this question, Table 3 compares results for VaR and VaR-I multiples at three horizons,

obtained with model parameters from estimations on returns at two different frequencies over a single time frame. Panel A of Table 3 shows one-day, five-day, and 10-day VaR and VaR-I multiples, obtained from *daily* returns over the single five-year sample period over 2001–2005. When S&P 500, the 20-year Treasury bond, the \$-JPY exchange rate, and put options are taken as the basis, it is clear that for the JD and CGMY models both VaR and VaR-I multiples sharply decrease as the horizon increases from one to five and to 10 days. For instance, the VaR-I multiple falls from 1.61 at the one-day horizon to 1.15 at the 10-day horizon for the JD on S&P 500 daily returns. This behavior is consistent with our view about the convergence of returns in these two models to the Normal distribution.

At the same time, the FMLS multiples tend to slightly increase as the horizon increases. A similar pattern is observed in Panel B of Table 3, where the multiples are calculated from five-day returns over the same five-year period. As in Panel A of Table 3, estimation is performed once, and the horizons are five and 10 days. The differences, however, are much smaller, given the smaller difference between the horizons. Comparing the three models, it appears that the FMLS risk measures change less with changes in the horizon, but are still empirically sensitive to changes in the frequency of returns used in estimations.

Overall, this part of the investigation provides the central insight that time scaling affects the estimated risk measures in an important way, and hence, substantial care must be taken in the estimation of internal models aimed at computing VaR at different horizons. Specifically, the choice of returns frequency at which the risk measures are estimated is not innocuous, and is likely to be a source of estimation uncertainty when lower frequency data cannot be employed. In spite of this complication, our conclusions regarding the relatively high VaR-I multiples over the benchmark remain intact for the FMLS.

6.6. Jump models yield similar performance yet they produce vastly different VaR-I multiples

One key observation to be made from Tables 2 and 3, and Fig. 1 is that the multiples over the benchmark tend to vary across models. In some cases the largest multiple for an asset is almost twice as high for some model than for another. These differences may be due to model specifics, e.g., FMLS having only one-sided jumps, unlike JD and CGMY, or JD having a diffusion component and jumps with finite activity, unlike FMLS and CGMY. Such an explanation would be consistent with our themes that the structure of jumps is an important determinant of intra-horizon risk, driving the differences in multiples across the Lévy models (this conjecture is further pursued through our comparative statics analysis).

Yet another reason for the divergence between the multiples could be that the models differ in their ability to calibrate to asset returns. If this is the case, then we should pay less attention to *worse* models, and consequently the differences in the multiples we obtain may

not be relevant. In what follows, we address this concern in two ways. First, we demonstrate that the models are estimated with reasonable precision and calibrate well to different types of return time-series. We appeal here to the Kolmogorov-Smirnov test to assess whether the empirical distribution function deviates significantly from the model distribution, as done also in Li, Wells, and Yu (2008).

Second, we evaluate the statistical significance of differential performance between any pair of models, using the Vuong test for non-nested model selection. The test statistic is (Vuong, 1989)

$$\frac{\mathcal{L}_1[\theta_1] - \mathcal{L}_2[\theta_2] - \frac{\ln(T)}{2}(n_1 - n_2)}{\sqrt{T}\omega_{1,2}} \quad \text{where}$$

$$\omega_{1,2}^2 = \frac{1}{T} \sum_{t=1}^T \left(\ln \frac{\Phi_1[X_t; \theta_1]}{\Phi_2[X_t; \theta_2]} \right)^2 - \left(\frac{1}{T} \sum_{t=1}^T \ln \frac{\Phi_1[X_t; \theta_1]}{\Phi_2[X_t; \theta_2]} \right)^2, \quad (31)$$

and the numerator of the test statistic is corrected for difference in the parameter vector dimensions following Schwarz's Bayesian information criterion with $n_1 := \dim[\theta_1]$ and $n_2 := \dim[\theta_2]$. The test statistic is asymptotically standard Normal distributed and hence, model 1 dominates model 2 at confidence level 5% if the statistic is greater than 1.65, or at 1% if it is greater than 2.32.

6.6.1. Model fits are similar and do not account for the consistent differences in VaR-I multiples

To stay focused we report parameter estimates for a subset of asset returns for each of the models in Table 4. The models are estimated on weekly returns over the entire sample period 1995–2005. Our estimation shows that model parameters are reasonable, and with only few exceptions the *t*-statistics are well in excess of 2.0 (reported in square brackets). For the FMLS model, generating the highest VaR and VaR-I multiples, the estimated exponent parameter α is in the range of 1.57–1.92, in line with those in Carr and Wu (2003) and Li, Wells, and Yu (2008), and fairly representative of values obtained over our subsamples. The value of the jump intensity parameter λ depends strongly on the frequency of our returns and the normalization we use, hence, it is not strictly comparable to the values reported in other studies. For the CGMY model, we uniformly estimate $\beta_+ > \beta_-$ implying more prominent left-tails.

Table 4 also shows *p*-values for the Kolmogorov-Smirnov test for models. In all cases *p*-values exceed 15%, showing that all models perform well for each of the four assets under this goodness-of-fit metric.

To draw comparisons with Tables 2 and 3, we also report VaR and VaR-I multiples that are now estimated from the entire 1995–2005 sample. In particular, the FMLS VaR-I multiples are the highest again, in the range of 2.00. Our findings on the relevance of incorporating intra-horizon risk remain valid even when the models and parameters are estimated over a longer sample.

Table 5 presents the Vuong (1989) test for the pairwise equivalence of the models, and generally implies lack of difference in model performance. Out of three pairwise comparisons for each of four assets, we only find

Table 4

Parameter estimates, VaR and VaR-I multiples over the entire sample.

Maximum likelihood estimation results in this table are based on weekly returns over the entire 1995–2005 sample period. Parameter estimates are shown with t -statistics in square brackets. \mathcal{L}/T is the maximized log-likelihood divided by T . For T observations X_t , $t = 1, \dots, T$, the Kolmogorov-Smirnov test statistic is $D = \max_{1 \leq t \leq T} (F(X_t) - t/T, t/T - F(X_t))$, where $F(X_t)$ is the cumulative distribution at X_t under the tested model. The p -values for the Kolmogorov-Smirnov goodness-of-fit test are reported in the row marked KS . The null hypothesis that the returns data come from the estimated model distribution cannot be rejected at the confidence level specified by the reported p -value. Experimental estimation shows that α has negligible effect on the likelihood function and we set $\alpha = 0.50$ in the CGMY model.

	S&P 500		20Y-TB		\$-JPY		ATM Put		
Panel A: JD									
λ	7.36	[1.31]	6.46	[1.57]	1.93	[1.25]	36.29	[1.72]	
μ_j	-0.01	[-1.25]	-0.01	[-1.58]	-0.01	[-1.69]	-0.01	[-2.13]	
σ_j	0.02	[3.86]	0.02	[4.19]	0.02	[2.53]	0.00	[2.17]	
σ	0.06	[11.81]	0.06	[15.67]	0.06	[21.11]	0.03	[6.45]	
\mathcal{L}/T	3.22		3.20		3.25		3.14		
KS	0.21		0.47		0.25		0.33		
VaR {VaR-I}	1.24	{1.57}	1.23	{1.55}	1.33	{1.69}	1.21	{1.63}	
Panel B: CGMY									
λ	4.70	[5.07]	6.24	[4.54]	5.37	[4.49]	4.36	[5.54]	
β_-	120.58	[5.96]	150.86	[6.19]	138.93	[5.82]	86.48	[6.12]	
β_+	174.47	[5.64]	229.20	[4.48]	179.24	[4.90]	651.95	[2.94]	
α	0.50		0.50		0.50		0.50		
\mathcal{L}/T	3.22		3.18		3.25		3.15		
KS	0.43		0.16		0.23		0.27		
VaR {VaR-I}	1.28	{1.41}	1.23	{1.35}	1.21	{1.33}	1.39	{1.49}	
Panel C: FMLS									
λ	0.05	[23.51]	0.05	[24.07]	0.05	[23.61]	0.06	[17.46]	
α	1.91	[6.30]	1.92	[6.23]	1.92	[5.77]	1.57	[1.80]	
\mathcal{L}/T	3.21		3.18		3.24		3.13		
KS	0.41		0.27		0.41		0.28		
VaR {VaR-I}	1.17	{1.98}	1.16	{1.95}	1.12	{1.88}	2.04	{2.10}	

Table 5

Pairwise comparison of model performance over the entire sample using the Vuong (1989) test.

The table reports test statistics for equivalence of each pair of non-nested models, where estimations are performed on weekly returns over the entire 1995–2005 sample period. The test statistic is (Vuong, 1989)

$$\frac{\mathcal{L}_1[\theta_1] - \mathcal{L}_2[\theta_2] - \frac{\ln(T)}{2}(n_1 - n_2)}{\sqrt{T}\omega_{1,2}} \quad \text{where}$$

$$\omega_{1,2}^2 = \frac{1}{T} \sum_{t=1}^T \left(\ln \frac{\Phi_1[X_t; \theta_1]}{\Phi_2[X_t; \theta_2]} \right)^2 - \left(\frac{1}{T} \sum_{t=1}^T \ln \frac{\Phi_1[X_t; \theta_1]}{\Phi_2[X_t; \theta_2]} \right)^2,$$

and the numerator of the test statistics includes a correction for the difference in the dimensions of the two parameter vectors following Schwarz's Bayesian information criterion with $n_1 = \dim[\theta_1]$ and $n_2 = \dim[\theta_2]$. Positive entries above 1.65 (2.32) indicate that the first model in the pair dominates the second rival model at the 5% (1%) confidence level. In the reverse, negative entries smaller than -1.65 (-2.32) indicate that the second model in the pair dominates at the 5% (1%) confidence level.

	S&P 500	20Y-TB	\$-JPY	ATM Put
JD vs. CGMY	0.01	0.13	0.08	-1.44
JD vs. FMLS	-0.17	-3.03	-3.54	0.11
CGMY vs. FMLS	0.13	-0.49	-0.67	0.37

statistically significant domination of FMLS over Merton's jump-diffusion for the Treasury bond and the \$-JPY. The results provide support for our conclusion that models with similar performance in time-series estimation generate widely different risk measures, indicating model risk. Table 5 also reveals that high FMLS multiples over the benchmark do not reflect inferior model fit.

While the estimates in Tables 4 and 5 are based on the entire sample of returns, we have also calculated the Kolmogorov-Smirnov statistic for each model, asset, and subperiod, used in computing the entries in Table 2. Table 6 shows the proportion of estimations for each model where the Kolmogorov-Smirnov test cannot reject the Lévy model at 1%. For example, the reported 0.04 in the third row, second column of the table indicates that in 4% of the estimations on NIKKEI returns, the CGMY model can be rejected at the 1% confidence level. With few exceptions, all models perform fairly well according to this metric, with rarely more than 5% rejections. The jump-diffusion JD tends to be rejected more often than the other models, indicating the importance of modeling small jumps (see also the simulation study of Li, Wells, and Yu, 2008; Wu, 2007). The FMLS, which produces the

Table 6

Assessing model goodness-of-fit across subsample estimations.

Proportion of estimations for each model where the Kolmogorov-Smirnov goodness-of-fit test cannot reject the model at 1%. The models are estimated once every month using weekly returns from the prior five years over the 1995–2005 sample period. For T observations X_t , $t = 1, \dots, T$, the test statistic is $D = \max_{1 \leq t \leq T} (F(X_t) - t/T, t/T - F(X_t))$, where $F(X_t)$ is the cumulative distribution at X_t under the tested model. With 260 observations, the critical value of the test statistics for a 1% confidence level is 0.1002.

	JD	CGMY	FMLS
S&P 500	0.00	0.00	0.00
FTSE	0.23	0.01	0.09
NIKKEI	0.00	0.04	0.00
1Y-TB	0.06	0.00	0.00
20Y-TB	0.00	0.00	0.00
\$-GBP	0.00	0.00	0.01
\$-JPY	0.01	0.00	0.00
ATM Put	0.10	0.00	0.00
ATM Call	0.05	0.00	0.00

highest VaR-I multiples, performs at par with CGMY. Overall, Table 6 shows that the ability of the models to fit the return time-series in the subsample estimations is similar and does not account for the consistent differences in the VaR and VaR-I multiples over the benchmark.

6.6.2. Sensitivity analysis shows that risk measures reflect model attributes

Two questions deserve further comment, given the empirically documented variation in risk measures across models. In what way does each jump parameter impact intra-horizon risk? What parameter combinations elevate first-passage probability? To address these issues, we conduct sensitivity analysis and report the findings in Table 7. Recall that the jump density decays exponentially (or faster) with the jump size in the JD and CGMY models, while it only decays as a power law for the FMLS. There are fundamental differences in the probability with which large jumps are allowed to occur across the jump models.

To start, consider the results for the Merton model presented in Panel A in Table 7. Here, to maintain parsimony, we assume three distinct values for each parameter σ_j , μ_j , and λ , denoted L (for low), M (for medium), and H (for high). The diffusion coefficient is set to 0.05. Appreciate that the highest (lowest) 10-day 99% VaR-I, per \$1 invested, is associated with the highest (lowest) σ_j , the lowest (highest) μ_j , and the highest (lowest) λ . In particular, the largest VaR-I difference is observed between HLH and LHL (i.e., 0.116 vs. 0.034). Our results confirm that parameter combinations that reflect a fatter left-tail of the returns distribution under the Merton model tend to produce higher first-passage probability.

Consistent with intuition, Panel B of Table 7 shows that VaR-I is sensitive to the dampening coefficient of downside jumps β_- in the CGMY model (with α fixed at 0.50

throughout): the weaker the dampening on β_- , the higher is VaR-I. Furthermore, VaR-I strongly rises as the arrival rate of jumps λ increases. Clearly, the effect of β_+ on VaR-I is negligible, when controlling for β_- and λ .

There are two final points to make based on Panel C of Table 7 for the FMLS. First, controlling for the FMLS α , a higher λ magnifies VaR-I. Second, controlling for λ , VaR-I is hump-shaped as the FMLS α varies. Observe that FMLS degenerates into Brownian motion when α equals 2.0 (Carr and Wu, 2003). However, when $\alpha < 2$, then FMLS is a pure-jump process with only downside jumps. The hump-shaped relation between the FMLS α and VaR-I likely reflects the working of the two distinct effects: First, as α increases, while staying far from 2.0, small jumps dominate (e.g., Ait-Sahalia and Jacod, 2009) and VaR-I increases. Second, as α increases further and tends to 2.0, the process becomes closer to continuous and VaR-I tends to decrease due to the lack of large jumps, thus reconciling the hump-shaped relation. In summary, under plausible parameter values, the FMLS produces the highest VaR-I, while CGMY produces the lowest VaR-I, consistent with Fig. 1. The jump model attributes map back into documented empirical behavior of risk measures.

6.7. Imposing more stringent VaR levels accentuates the impact of intra-horizon risk

The reported jump and intra-horizon risk measures are all based on the 99% regulatory VaR level and are compared to the benchmark 99% Normal VaR. However, this confidence level may not be the only one of relevance to market participants and regulators. For example, the Basel multipliers of three to four applied to internally estimated VaRs may be intended to enforce more stringent capital standards, corresponding to higher confidence levels. Such a view is plausible, given, for example, that VaR estimates used for a bank's credit book typically apply confidence levels of 99.95% or higher (together with longer investment horizons). Hence, a 99% trading VaR may result in a drastic mismatch in the levels of risk across the trading and credit books.

To illustrate the impact of intra-horizon risk at higher confidence levels, we next consider 99.9% VaR-I. Table 8 presents maximum multiples of 99.9% VaR-I over 99.9% Normal VaR for each of the jump models. The multiples are calculated as $\text{VaR-I}/(3.09\hat{\sigma} - \hat{\mu})$, where VaR-I is the 0.1% quantile of the first-passage distribution obtained from each model estimation.

Observe that the multiples are now uniformly higher than the respective ones obtained at the 99% level in Panel D of Table 2, and exceed them on average by 30%. This result reflects (i) the relatively heavier tails of the return distributions under jump models compared to the Normal distribution, and (ii) the possible impact of large jump activity.

In sum, imposing more stringent VaR standards only sharpens the deviation of the benchmark VaR from VaR-I. It furthermore emphasizes the importance of intra-horizon risk in calculating risk measures applicable to financial market data.

Table 7

Sensitivity analysis of VaR-I to model parameters.

The table shows 10-day, 99% VaR-I per \$1 invested for the Merton (1976) jump-diffusion model (the JD), the exponentially dampened power law model of Carr, Geman, Madan, and Yor (2002) (the CGMY), and the finite-moment log-stable model of Carr and Wu (2003) (the FMLS). VaR-I is the 1% quantile of the first-passage distribution over a 10-day horizon. To gauge sensitivity, each model parameter is allowed to take three values: low (denoted L), medium (denoted M), and high (denoted H). The parameter values used to calculate VaR-I are displayed in each row. For the Merton model, the diffusion parameter σ is held fixed at 0.05, and α in the CGMY model is held fixed at 0.50.

	JD				CGMY				FMLS			
	σ_j	μ_j	λ	VaR-I	β_-	λ	β_+	VaR-I	α	λ	VaR-I	
L/L/L	0.01	-0.030	5	0.068	100	3	155	0.043	L/L	1.55	0.02	0.021
L/L/M	0.01	-0.030	8	0.080	100	3	175	0.043	L/M	1.55	0.05	0.055
L/L/H	0.01	-0.030	10	0.085	100	3	195	0.043	L/H	1.55	0.08	0.088
L/M/L	0.01	-0.015	5	0.049	100	5	155	0.053	M/L	1.75	0.02	0.026
L/M/M	0.01	-0.015	8	0.055	100	5	175	0.052	M/ M	1.75	0.05	0.071
L/M/H	0.01	-0.015	10	0.058	100	5	195	0.051	M/H	1.75	0.08	0.113
L/H/L	0.01	0.000	5	0.034	100	7	155	0.060	H/L	1.95	0.02	0.023
L/H/M	0.01	0.000	8	0.037	100	7	175	0.059	H/M	1.95	0.05	0.058
L/H/H	0.01	0.000	10	0.038	100	7	195	0.058	H/H	1.95	0.08	0.093
M/L/L	0.02	-0.030	5	0.081	120	3	155	0.039				
M/L/M	0.02	-0.030	8	0.092	120	3	175	0.038				
M/L/H	0.02	-0.030	10	0.099	120	3	195	0.038				
M/M/L	0.02	-0.015	5	0.064	120	5	155	0.047				
M/M/ M	0.02	-0.015	8	0.071	120	5	175	0.046				
M/M/ H	0.02	-0.015	10	0.075	120	5	195	0.046				
M/H/L	0.02	0.000	5	0.048	120	7	155	0.054				
M/H/ M	0.02	0.000	8	0.054	120	7	175	0.053				
M/H/H	0.02	0.000	10	0.057	120	7	195	0.052				
H/L/L	0.03	-0.030	5	0.096	140	3	155	0.035				
H/L/M	0.03	-0.030	8	0.108	140	3	175	0.035				
H/L/H	0.03	-0.030	10	0.116	140	3	195	0.035				
H/M/L	0.03	-0.015	5	0.080	140	5	155	0.043				
H/M/ M	0.03	-0.015	8	0.089	140	5	175	0.042				
H/M/H	0.03	-0.015	10	0.095	140	5	195	0.042				
H/H/L	0.03	0.000	5	0.065	140	7	155	0.049				
H/H/M	0.03	0.000	8	0.073	140	7	175	0.049				
H/H/H	0.03	0.000	10	0.077	140	7	195	0.048				

6.8. Filtered historical simulation-based benchmark VaR does not dilute intra-horizon risk

Motivated by an extensive literature that has developed more realistic VaR estimation methods than the standard Normal VaR, we calculate VaR-I multiples over an alternative benchmark VaR for further confirmation of our findings. In particular, we use the filtered historical simulation method advocated by Barone-Adesi, Giannopoulos, and Vosper (1999) and Barone-Adesi, Engle, and Mancini (2008). This method compares favorably with other approaches, as demonstrated, e.g., in Kuester, Mittnik, and Paolella (2006) and Pritsker (2006).

The thrust of the filtered historical simulation (hereby FHS) method is to obtain an i.i.d. series of standardized returns to be used as an input in a bootstrap procedure for estimating VaR. The steps in applying the FHS are: (i) fit a first-order autoregressive model to the conditional mean of returns and get the residuals; (ii) rescale (or filter) these

residuals with contemporaneous volatility estimates obtained, e.g., in a GARCH model, thus removing serial correlation and volatility clustering; (iii) draw random samples from the standardized residuals, and (iv) use them recursively as innovations in a conditional variance equation to simulate future returns.

Rolling five-year prior daily returns are used to fit a GARCH(1,1) model, consistent with the estimation period for the jump models. Then we estimate an FHS-based VaR on each date in the sample. Table 9 reports maximum VaR-I multiples at the 99% level. Corresponding multiples over Normal VaR from Panel D of Table 2 are also shown for comparison.⁹

⁹ We present results for equity, interest rate, and currency risk but not for volatility risk. The reason is that non-overlapping 30-day option returns time-series are sampled at the monthly frequency, while the GARCH calibration requires data at a higher frequency to maintain precision.

Table 8

Impact of more stringent VaR levels: maximum multiples for 99.9% VaR-I.

This table presents maximum multiples of 99.9% VaR-I over 99.9% benchmark VaR and compares to Panel D of Table 2, which was based on 99% VaR-levels. 99.9% VaR-I is the 0.1% quantile of the first-passage distribution obtained from each model estimation. 99.9% benchmark VaR is the 0.1% quantile of the Normal distribution $N(\hat{\mu}, \hat{\sigma})$, where $\hat{\mu} = 0$ and $\hat{\sigma}$ is the standard deviation of the returns time-series, used in each VaR calculation. Reported are the values of the multiples $\text{VaR-I}/(3.09\hat{\sigma} - \hat{\mu})$. Each model is estimated on time-series of 260 weekly (five trading day) returns and the estimations are performed once every month.

	JD	CGMY	FMLS
S&P 500	2.10	1.96	3.30
FTSE	2.56	1.96	2.48
NIKKEI	1.73	1.72	2.70
1Y-TB	2.30	3.27	3.08
20Y-TB	1.88	1.46	2.95
\$-GBP	1.72	1.91	3.34
\$-JPY	2.19	1.87	2.81
ATM Put	2.20	1.79	3.58
ATM Call	2.38	1.98	2.91

Table 9

VaR-I multiples based on the Filtered historical simulation (FHS) benchmark.

This table presents maximum multiples of VaR-I over benchmark VaR, where the benchmark VaR is now based on the Filtered historical simulation (FHS) methodology of Barone-Adesi, Giannopoulos, and Vosper (1999) and Barone-Adesi, Engle, and Mancini (2008). Multiples over Normal VaR are displayed alongside for comparison. Using rolling five years of preceding daily returns, the steps in applying FHS are: (i) fit a first-order autoregressive model to the conditional mean of returns and get the residuals; (ii) rescale (or filter) these residuals with contemporaneous volatility estimates obtained in a GARCH(1,1) model to remove serial correlation and volatility clustering; (iii) draw random samples from the standardized residuals, and (iv) use them recursively as innovations in a conditional variance equation to simulate future 10-day returns. VaR-I is estimated on time series of 260 non-overlapping weekly (five trading day) returns over the 1995–2005 sample period, and the estimations are performed once every month for each of the models.

	S&P 500		
	JD	CGMY	FMLS
VaR-I multiples over 99% FHS VaR	1.83	1.33	1.81
VaR-I multiples over 99% Normal VaR	1.94	1.50	2.08
	20Y-TB		
	JD	CGMY	FMLS
VaR-I multiples over 99% FHS VaR	1.90	1.42	2.52
VaR-I multiples over 99% Normal VaR	1.95	1.39	2.45
	\$-JPY		
	JD	CGMY	FMLS
VaR-I multiples over 99% FHS VaR	1.93	1.62	2.01
VaR-I multiples over 99% Normal VaR	1.81	1.52	2.04

The basic message from Table 9 is twofold. First, maximum VaR-I multiples over FHS-based VaR for the S&P 500 index are 10–15% lower than those over Normal VaR, implying that FHS-based VaR is higher than Normal VaR. At the same time, the maximum VaR-I multiples over the two benchmarks are similar for the 20-year Treasury bond and the \$-JPY return series. In all cases, however, the VaR-I multiples over the two benchmarks differ significantly on a period-by-period basis and in their averages (not reported), reflecting the different nature of the two benchmarks. Second, the maximum VaR-I multiples typically range between 1.3 and 2.0, and are again the highest for the FMLS model. Overall, the impact of intra-horizon risk in jump models is not diluted even when we appeal to the filtered historical simulation-based VaR as the benchmark.

7. Concluding remarks and summary

This paper studies a risk measure, based on first-passage probability for jump models, which accommodates intra-horizon risk. The proposed risk measure improves on standard VaR by relaxing the end-of-horizon feature and focuses instead on the profit and loss distribution over the trading horizon. We present methods to compute the risk measure for three Lévy jump models of asset returns, with either finite or infinite activity of jumps.

Our approach for computing first-passage probability for jump models is general and can be applied to problems beyond the risk management field, such as those arising in characterizing default boundaries (e.g., Longstaff and Schwartz, 1995). We address time scaling issues that arise when model parameters are estimated with high frequency return observations, but risk measures must be calculated over a longer horizon.

We find, first, that risk measures taking into account intra-horizon risk in jump models consistently exceed the benchmark VaR, and can be up to 2.64 times higher. Validating the uneasiness of regulators with respect to options, the departure from benchmark VaR is most pronounced for volatility exposures. Second, allowing for an intra-horizon risk component can magnify the counterpart jump model VaR by at least 20%. Third, large variation in risk measures is observed across jump models, indicative of model risk. Fourth, the finite-moment log-stable model of Carr and Wu (2003) consistently provides the most conservative risk measures. In addition, we consider a more stringent 99.9% VaR level and an alternative VaR benchmark based on the filtered historical simulation. We find that 99.9% VaR-I multiples are uniformly higher than the 99% counterpart, potentially reflecting the impact of large jumps. Furthermore, internalizing the effect of an alternative benchmark does not dilute the role of intra-horizon risk. Overall, we conclude that omitting intra-horizon risk can lead to underestimation of risk exposure in financial markets.

Appendix A

Following Artzner, Delbaen, Eber, and Heath (1997, 1999), let X and Y be real-valued random variables, defined on the set of states of nature at a future date T . Interpret X and Y as

possible dollar return (profit or loss) at T of two positions or portfolios currently held. Let \mathbb{R} denote the real line.

Definition. A map $\rho : X \rightarrow \mathbb{R}$ is a coherent risk measure, if it satisfies the following axioms:

1. *Positive homogeneity:* For $c > 0$, $\rho(cX) = c\rho(X)$.
2. *Monotonicity:* $\rho(Y) \leq \rho(X)$ if $X \leq Y$.
3. *Translation Invariance:* $\rho(X + k_0) = \rho(X) - k_0$ for any constant k_0 .
4. *Subadditivity:* $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Is VaR-I, as defined in Eq. (1), a coherent risk measure? Intuitively, VaR-I should satisfy the homogeneity, monotonicity, and translation invariance axioms, since the mapping $\{\rho : X \rightarrow \text{quantiles of } X\}$ satisfies these axioms, as in the case of VaR, and the mapping $\{\phi : X \rightarrow X_t^{\min}\}$ is linear and monotonic, and hence, the composition $\rho \circ \phi$, defining VaR-I also satisfies the axioms. At the same time, VaR-I, being a quantile measure like VaR, may not satisfy the subadditivity axiom. To verify this intuition, we consider in turn each property in the above definition, with the understanding that it now refers to the paths X_t and Y_t of the dollar returns of two positions or portfolios over $[0, T]$.

- *Positive homogeneity:* VaR-I can be interpreted as the absolute value of the lowest level that is hit by each path among 1% of the paths of the dollar returns process over a 10-day horizon. Scaling the position, represented by X_t , by a factor c results in an equivalent scaling of each path of the dollar returns process, therefore VaR-I grows linearly with the factor c and satisfies positive homogeneity.
- *Monotonicity:* If $X \leq Y$ pointwise on each path, then, following the above interpretation, the 1% paths defining VaR-I for X cannot exceed the VaR-I level for Y , hence, VaR-I for X is greater than VaR-I for Y , therefore, VaR-I satisfies the monotonicity axiom.
- *Translation invariance:* Adding a riskfree asset (long or short) to a position or a portfolio shifts all the paths by a constant amount (positive or negative), including the 1% paths defining VaR-I. Hence, VaR-I also shifts by this constant amount and therefore VaR-I is translation invariant.
- *Subadditivity:* VaR-I will not satisfy subadditivity in general, which can be illustrated through an example. Suppose a stock S has a current price of \$100. Let also A (B) be a short position in a derivative paying \$1 million if the price of S hits the level \$120 (\$85) in the next 10 days, and nothing otherwise. Let the probability of the price hitting \$120 (\$85) in the next 10 days be 0.9%, and hence, the 1% VaR-I for A or B separately is zero. However, a portfolio composed of A and B pays out \$1 million with probability of about 1.8% (ruling out extreme serial dependence in the returns process for S), hence, the 1% VaR-I for the portfolio dramatically exceeds the combined 1% VaR-I's for the two components. Therefore, VaR-I is not subadditive and consequently is not a coherent risk measure.

It is known that standard VaR is not subadditive, implying that a diversified portfolio may in principle have

a higher VaR than a non-diversified portfolio. However, it has been argued that for realistic (joint) distributions of the components, VaR is subadditive (Ebrechts, McNeil, and Straumann, 2002; Danielsson, Jorgensen, Samorodnitsky, Sarma, and de Vries, 2005). Due to the general lack of analyticity of first-passage probability of Lévy jump models (see, e.g., Kyprianou, 2006), it is arduous to derive analogous conditions under which VaR-I with jumps can be subadditive. For an additional perspective on subadditivity, see Dhaene, Goovaerts, and Kaas (2003) and Heyde, Kou, and Peng (2006).

Appendix B

B.1. Discretization of the PIDE for the CGMY first-passage density

We solve (17) using finite differences on the mesh $[s_{\min}, s_{\max}] \times [0, T]$, with $s_j = s_{\min} + j\Delta$, $j = 0, 1, \dots, N$ and $t_i = ih$, $i = 0, 1, \dots, M$. Here s_j denote log-prices, s_{\min} is the log of the first-passage level $H < S_0$, Δ is the step in log-price direction, and h is the step in time direction, and let g_j denote $g[s_j, t_j]$, for $j = 2 : N - 1$. We need approximations to the integrals in (18).

Case 1: $0 < \alpha < 1$. The following integral is essential in what follows:

$$\lambda \int_{\kappa}^{\zeta} \frac{e^{-\gamma x}}{x^{1+\alpha}} dx = -\frac{\lambda \gamma^\alpha}{\alpha} \left(\frac{e^{-\gamma \zeta}}{(\gamma \zeta)^\alpha} - \frac{e^{-\gamma \kappa}}{(\gamma \kappa)^\alpha} + \Gamma[1 - \alpha] \right) \times (\Gamma_{inc}[\gamma \zeta, 1 - \alpha] - \Gamma_{inc}[\gamma \kappa, 1 - \alpha]), \quad (32)$$

where $0 \leq \kappa < \zeta$, and $\Gamma_{inc}[\cdot, \cdot]$ is the lower incomplete gamma function. With (32), for any t_i and $j = 2 : N - 1$:

$$I_1 = -\left(1 - g_j + \frac{g_{j+1} - g_j}{\Delta} \right) \frac{\lambda \beta^\alpha}{\alpha} \times \left(-\frac{e^{-\beta_- \Delta(j-1)}}{(\beta_- \Delta(j-1))^\alpha} + \Gamma[1 - \alpha] \right) \times (1 - \Gamma_{inc}[\beta_- \Delta(j-1), 1 - \alpha]), \quad (33)$$

and,

$$I_3 = \frac{g_{j+1} - g_j}{\Delta} \lambda \int_{-\infty}^{-\Delta} \frac{e^{-(\beta_- + 1)|x|}}{|x|^{1+\alpha}} dx = \frac{g_{j+1} - g_j}{\Delta} \lambda \int_{\Delta}^{\infty} \frac{e^{-(\beta_- + 1)x}}{x^{1+\alpha}} dx, = -\frac{g_{j+1} - g_j}{\Delta} \frac{\lambda (\beta_- + 1)^\alpha}{\alpha} \left(-\frac{e^{-(\beta_- + 1)\Delta}}{[(\beta_- + 1)\Delta]^\alpha} + \Gamma[1 - \alpha] \right) \times (1 - \Gamma_{inc}[(\beta_- + 1)\Delta, 1 - \alpha]), \quad (34)$$

$$I_4 = \frac{\lambda}{2} \left(\frac{g_{j+1} - 2g_j + g_{j-1}}{\Delta^2} - \frac{g_{j+1} - g_j}{\Delta} \right) \int_{-\Delta}^0 \frac{x^2 e^{-\beta_- |x|}}{|x|^{1+\alpha}} dx = \frac{\lambda \beta_-^{\alpha-2}}{2} \left(\frac{g_{j+1} - 2g_j + g_{j-1}}{\Delta^2} - \frac{g_{j+1} - g_j}{\Delta} \right) \Gamma[2 - \alpha] \times \Gamma_{inc}[\beta_- \Delta, 2 - \alpha], \quad (35)$$

$$\begin{aligned}
I_5 &= \frac{\lambda}{2} \left(\frac{g_{j+1} - 2g_j + g_{j-1}}{\Delta^2} - \frac{g_{j+1} - g_j}{\Delta} \right) \int_0^{\Delta} \frac{x^2 e^{-\beta_+ x}}{x^{1+\alpha}} dx \\
&= \frac{\lambda \beta_+^{\alpha-2}}{2} \left(\frac{g_{j+1} - 2g_j + g_{j-1}}{\Delta^2} - \frac{g_{j+1} - g_j}{\Delta} \right) \\
&\quad \times \Gamma[2 - \alpha] \Gamma_{inc}[\beta_+ \Delta, 2 - \alpha]. \tag{36}
\end{aligned}$$

For the remaining two integrals I_2 and I_6 note that there is an I_2 term only for $j > 2$, since for $j = 2$ down-jumps are small in absolute value and are given only by I_4 . Also, there is an I_6 term only for $j < N - 1$.

$$\begin{aligned}
I_2 &\approx \lambda \sum_{k=1}^{j-2} \int_{s_k - s_j}^{s_{k+1} - s_j} \left(g_{k+1} + \frac{g_{k+1} - g_k}{\Delta} (x - s_{k+1} + s_j) - g_j \right. \\
&\quad \left. + \frac{g_{j+1} - g_j}{\Delta} \right) \frac{e^{-\beta_- |x|}}{|x|^{1+\alpha}} dx \\
&= \lambda \sum_{k=1}^{j-2} \left(g_{k+1} + \frac{g_{k+1} - g_k}{\Delta} (s_j - s_{k+1}) - g_j + \frac{g_{j+1} - g_j}{\Delta} \right) \\
&\quad \times \int_{s_k - s_j}^{s_{k+1} - s_j} \frac{e^{-\beta_- |x|}}{|x|^{1+\alpha}} dx + \lambda \sum_{k=1}^{j-2} \frac{g_{k+1} - g_k}{\Delta} \int_{s_k - s_j}^{s_{k+1} - s_j} \frac{e^{-\beta_- |x|}}{|x|^\alpha} dx. \tag{37}
\end{aligned}$$

The integral in the first summand in (37) is equal to

$$\begin{aligned}
I_2^1 &= \frac{\beta_-^\alpha}{\alpha} \left(\frac{e^{-\beta_- \Delta(j-k)}}{(\beta_- \Delta(j-k))^\alpha} - \frac{e^{-\beta_- \Delta(j-k-1)}}{(\beta_- \Delta(j-k-1))^\alpha} \right) \\
&\quad + \frac{\beta_-^\alpha}{\alpha} \Gamma(1 - \alpha) (\Gamma_{inc}[\beta_- \Delta(j-k), 1 - \alpha] \\
&\quad - \Gamma_{inc}[\beta_- \Delta(j-k-1), 1 - \alpha]), \tag{38}
\end{aligned}$$

and the integral in the second summand in (37) is equal to

$$I_2^2 = \beta_-^{1-\alpha} (\Gamma_{inc}[\beta_- \Delta(j-k), 1 - \alpha] - \Gamma_{inc}[\beta_- \Delta(j-k-1), 1 - \alpha]). \tag{39}$$

In a similar way:

$$\begin{aligned}
I_6 &\approx \lambda \sum_{k=j+1}^{N-1} \int_{s_k - s_j}^{s_{k+1} - s_j} \left(g_k + \frac{g_{k+1} - g_k}{\Delta} (x - s_k + s_j) \right. \\
&\quad \left. - g_j - \frac{g_{j+1} - g_j}{\Delta} (e^x - 1) \right) \frac{e^{-\beta_+ x}}{x^{1+\alpha}} dx, \\
&= \lambda \sum_{k=j+1}^{N-1} \left(g_k + \frac{g_{k+1} - g_k}{\Delta} (s_j - s_k) - g_j + \frac{g_{j+1} - g_j}{\Delta} \right) \\
&\quad \times \int_{s_k - s_j}^{s_{k+1} - s_j} \frac{e^{-\beta_+ x}}{x^{1+\alpha}} dx \\
&\quad - \lambda \sum_{k=j+1}^{N-1} \frac{g_{j+1} - g_j}{\Delta} \int_{s_k - s_j}^{s_{k+1} - s_j} \frac{e^{-(\beta_+ - 1)x}}{x^{1+\alpha}} dx \\
&\quad + \lambda \sum_{k=j+1}^{N-1} \frac{g_{k+1} - g_k}{\Delta} \int_{s_k - s_j}^{s_{k+1} - s_j} \frac{e^{-\beta_+ x}}{x^\alpha} dx. \tag{40}
\end{aligned}$$

The integral in the first summand in (40) is equal to

$$\begin{aligned}
I_6^1 &= \frac{e^{-\beta_+ \Delta(k+1-j)}}{(\beta_+ \Delta(k+1-j))^\alpha} - \frac{e^{-\beta_+ \Delta(k-j)}}{(\beta_+ \Delta(k-j))^\alpha} \\
&\quad + \Gamma[1 - \alpha] (\Gamma_{inc}[\beta_+ \Delta(k+1-j), 1 - \alpha] \\
&\quad - \Gamma_{inc}[\beta_+ \Delta(k-j), 1 - \alpha]), \tag{41}
\end{aligned}$$

and the integral in the second and third summands, respectively, in (40) are equal to

$$\begin{aligned}
I_6^2 &= \frac{e^{-(\beta_+ - 1)\Delta(k+1-j)}}{((\beta_+ - 1)\Delta(k+1-j))^\alpha} - \frac{e^{-(\beta_+ - 1)\Delta(k-j)}}{((\beta_+ - 1)\Delta(k-j))^\alpha} \\
&\quad + \Gamma[1 - \alpha] (\Gamma_{inc}[(\beta_+ - 1)\Delta(k+1-j), 1 - \alpha] \\
&\quad - \Gamma_{inc}[(\beta_+ - 1)\Delta(k-j), 1 - \alpha]), \tag{42}
\end{aligned}$$

$$\begin{aligned}
I_6^3 &= \beta_+^{1-\alpha} (\Gamma_{inc}[\beta_+ \Delta(k-j), 1 - \alpha] \\
&\quad - \Gamma_{inc}[\beta_+ \Delta(k+1-j), 1 - \alpha]). \tag{43}
\end{aligned}$$

Thus, we have a complete characterization under this case.

Case 2: $1 < \alpha < 2$. The only difference in this case is that, for $0 < \kappa < \zeta$, the integral of the Lévy density in (32) becomes

$$\begin{aligned}
\lambda \int_{\kappa}^{\zeta} \frac{e^{-\gamma x}}{x^{1+\alpha}} dx &= \frac{\lambda \gamma^\alpha}{\alpha(1-\alpha)} \left(\frac{e^{-\gamma \zeta}}{(\gamma \zeta)^\alpha} (1 - \alpha + \gamma \zeta) - \frac{e^{-\gamma \kappa}}{(\gamma \kappa)^\alpha} \right. \\
&\quad \left. \times (1 - \alpha + \gamma \kappa) \right) + \frac{\lambda \gamma^\alpha}{\alpha(1-\alpha)} \Gamma[2 - \alpha] \\
&\quad \times (\Gamma_{inc}[\gamma \zeta, 2 - \alpha] - \Gamma_{inc}[\gamma \kappa, 2 - \alpha]). \tag{44}
\end{aligned}$$

All other approximations and calculations follow exactly the previous case and are omitted.

B.2. Discretization of the PIDE for the FMLS first-passage density

We approximate the integral in (27) as

$$I \approx I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \lambda \int_{-\infty}^{\ln(H)-s} (1 - g[s] + g_s[s]) \frac{1}{|x|^{1+\alpha}} dx, \tag{45}$$

$$I_2 = \lambda \int_{\ln(H)-s}^{-\Delta} (g[s+x] - g[s] + g_s[s]) \frac{1}{|x|^{1+\alpha}} dx, \tag{46}$$

$$I_3 = -\lambda \int_{-\infty}^{-\Delta} g_s[s] \frac{e^x}{|x|^{1+\alpha}} dx, \tag{47}$$

$$\begin{aligned}
I_4 &= \lambda \int_{-\Delta}^0 \left(g[s] + g_s[s]x + \frac{1}{2} g_{ss}x^2 - g[s] - g_s[s] \right. \\
&\quad \left. \times \left(1 + x + \frac{1}{2} x^2 - 1 \right) \right) \frac{1}{|x|^{1+\alpha}} dx. \tag{48}
\end{aligned}$$

The integral of the Lévy measure is

$$\lambda \int_{-\zeta}^{-\kappa} \frac{1}{|x|^{1+\alpha}} dx = \frac{\lambda}{\alpha} \left(\frac{1}{\kappa^\alpha} - \frac{1}{\zeta^\alpha} \right), \tag{49}$$

where $0 < \kappa < \zeta$. Using (49), for any t_i and any $j = 2 : N - 1$:

$$I_1 = \left(1 - g_j + \frac{g_{j+1} - g_j}{\Delta} \right) \frac{\lambda}{\alpha(\Delta(j-1))^\alpha}, \tag{50}$$

and

$$\begin{aligned}
I_3 &= -\frac{g_{j+1} - g_j}{\Delta} \lambda \int_{-\infty}^{-\Delta} \frac{e^{-|x|}}{|x|^{1+\alpha}} dx = -\frac{g_{j+1} - g_j}{\Delta} \lambda \int_{\Delta}^{\infty} \frac{e^{-x}}{x^{1+\alpha}} dx \\
&= \frac{g_{j+1} - g_j}{\Delta} \frac{\lambda}{\alpha(\alpha-1)} \left(-\frac{e^{-\Delta}}{\Delta^\alpha} (1 - \alpha + \Delta) \right. \\
&\quad \left. + \Gamma[2 - \alpha] (1 - \Gamma_{inc}[\Delta, 2 - \alpha]) \right), \tag{51}
\end{aligned}$$

$$I_4 = \frac{\lambda}{2} \left(\frac{g_{j+1} - 2g_j + g_{j-1}}{\Delta^2} - \frac{g_{j+1} - g_j}{\Delta} \right) \int_{-\Delta}^0 \frac{x^2}{|x|^{1+\alpha}} dx$$

$$= \frac{\lambda}{2} \left(\frac{g_{j+1} - 2g_j + g_{j-1}}{\Delta^2} - \frac{g_{j+1} - g_j}{\Delta} \right) \frac{\Delta^{2-\alpha}}{2-\alpha}. \quad (52)$$

For $j = 3 : N - 1$:

$$I_2 \approx \lambda \sum_{k=1}^{j-2} \int_{s_k - s_j}^{s_{k+1} - s_j} \left(g_{k+1} + \frac{g_{k+1} - g_k}{\Delta} (x - s_{k+1} + s_j) \right.$$

$$\left. - g_j + \frac{g_{j+1} - g_j}{\Delta} \right) \frac{1}{|x|^{1+\alpha}} dx,$$

$$= \lambda \sum_{k=1}^{j-2} \left(g_{k+1} + \frac{g_{k+1} - g_k}{\Delta} (s_j - s_{k+1}) - g_j + \frac{g_{j+1} - g_j}{\Delta} \right)$$

$$\times \int_{s_k - s_j}^{s_{k+1} - s_j} \frac{1}{|x|^{1+\alpha}} dx + \lambda \sum_{k=1}^{j-2} \frac{g_{k+1} - g_k}{\Delta} \int_{s_k - s_j}^{s_{k+1} - s_j} \frac{1}{|x|^\alpha} dx. \quad (53)$$

The integral in the first summand is equal to

$$I_2^1 = \frac{1}{\alpha} \left(\frac{1}{(\Delta(j-k-1))^\alpha} - \frac{1}{(\Delta(j-k))^\alpha} \right), \quad (54)$$

and the integral in the second summand in (44) is equal to

$$I_2^2 = \frac{1}{\alpha-1} \left(\frac{1}{(\Delta(j-k-1))^{\alpha-1}} - \frac{1}{(\Delta(j-k))^{\alpha-1}} \right). \quad (55)$$

This completes the description of the first-passage density problem.

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