

An Inquiry into the Nature and Sources of Variation in the Expected Excess Return of a Long-Term Bond*

Gurdip Bakshi^{a†} Fousseni Chabi-Yo^{b‡} Xiaohui Gao^{c§}

^a*Smith School of Business, University of Maryland, College Park, MD 20742, USA*

^b*Fisher College of Business, Ohio State University, Columbus, OH 43210, USA*

^c*Smith School of Business, University of Maryland, College Park, MD 20742, USA*

First draft, January 2015

October 20, 2015

Abstract

We use a notion of positive dependence between the permanent and transitory components of the stochastic discount factor to develop a lower bound on the expected excess return of a long-term bond. This lower bound is a crucial number, as it represents the minimum expected excess return demanded by investors and can be extracted from options on the 30-year Treasury bond futures. Our implementation reveals that the annualized lower bound ranges from 0.22% to 6.07%, with an unconditional average of 1.18%. The developed results are useful for thinking about cost of debt and measuring investor reaction to monetary policy shocks.

KEY WORDS: Long-term Treasury bond, expected excess return, lower bound, options on bond futures

JEL CLASSIFICATION CODES: E43, E52, E58, G12, G14, G21

*We are especially grateful for discussions and feedback of Craig Burnside, Reena Aggarwal, Jennie Bai, Turan Bali, Svetlana Bryzgalova, John Campbell, John Cochrane, John Crosby, Steve Heston, Nikunj Kapadia, Mark Loewenstein, Dilip Madan, Ian Martin (indirectly through John Campbell), Pete Kyle, Shrihari Santosh, Rohan Williamson, and Lai Xu. The seminar participants at Georgetown University, SIRE asset pricing workshop (University of Glasgow), and the University of Maryland provided useful comments and questions. We welcome comments, including references to related papers we have inadvertently overlooked. All errors are our responsibility.

[†]Tel.: +1-301-405-2261. *E-mail address:* gbakshi@rhsmith.umd.edu

[‡]Tel.: +1-614-292-8477. *E-mail address:* chabi-yo_1@fisher.osu.edu

[§]Tel.: +1-301-405-8644. *E-mail address:* xiaohui@rhsmith.umd.edu

1. Introduction

A fundamental question in financial economics is how to estimate the expected excess return of an asset. The challenging nature of this problem is eloquently stated by Black (1993, page 36):

The key issue in investments is estimating expected return. It is neither explaining return nor, as Fama and French suggest, explaining average return. ... Estimating expected return is hard.

Merton (1980) more generally recognized early on the importance of estimating expected return.

The focus of this paper is on the conditional expected excess return of a long-term Treasury bond, a topic that sits at the intersection of investments, asset pricing, and corporate finance.¹ Our approach to studying the nature and sources of variation in the expected excess return is to propose a lower bound, which we extract from a portfolio of options on the 30-year Treasury bond futures, accounting for the fact that options are not traded on Treasury bonds. The knowledge of the lower bound is conceptually important, as it quantifies the minimum expected compensation for investing in the long-term bond.

The core idea is linked to a theory that postulates that the stochastic discount factor can be uniquely decomposed into its transitory and permanent components and that the return of a long-term discount bond is the inverse of the transitory component (Alvarez and Jermann (2005) and Hansen and Scheinkman (2009)). We show that the lower bound on the expected excess return is equal to the risk-neutral return variance of the Treasury bond futures, multiplied by the gross interest rate. The intuition is that suppliers of capital dislike shocks that make the risk-neutral return distributions more volatile, thereby raising their minimum expected excess return.

Each period, our estimates of the lower bound are forward-looking, incorporate the risk and pricing of interest rate movements, and respond to market conditions (e.g., monetary and fiscal policy). Moreover, our characterizations of the lower bound do not rely on shape restrictions on the stochastic discount factor.

While many economists have proposed conceptually innovative mechanisms to generate the expected excess return of the long-term bond (see the survey papers by Piazzesi (2010) and Gurkaynak and Wright (2011)), we offer the distinction of exploiting the information content of traded option prices for the expected excess return of the bond. Implementation shows that the lower bound has an unconditional average

¹The estimate of the expected excess return of the long-term Treasury bond is central to numerous calculations involving net present values, capital budgeting, and investment planning. Shocks to the expected excess return could impact the borrowing cost of the Treasury, which could percolate to the timing and size of Treasury issuances and to the public ownership of Treasury bonds.

of 1.18% over the sample period of 1982:10 to 2013:12. The minimum required compensation is importantly never negative, ranging between 0.22% and 6.07%.

What are the channels that underlie the variations in the minimum expected excess return of a long-term bond? We link the patterns of expected excess returns to three economically motivated variables. First, we show that a one-standard-deviation increase in the slope of the Treasury yield curve over the previous month increases the lower bound by an average of 0.07% (all units are annualized), and the effect is statistically significant. Next, we construct the return of a security with the payoff equal to the stochastic discount factor (SDF). This channel operates through the effect that the return of the SDF security is high during bad times, and we find that a one-standard-deviation movement in this return raises the lower bound by 0.18%. We also explore an asset that is often considered to be a hedge against economic tail risks. Specifically, we take the returns of a fully collateralized futures position in gold, and find that it exerts a positive and statistically significant effect on the expected excess returns of a bond over certain periods. A one-standard-deviation increase in the return of gold lifts the lower bound by 0.10%. Moreover, the effects remain robust when we control for a set of variables that are thought to impact the expected excess return of a bond – specifically, aggregate output growth, inflation, and the supply of Treasury bonds.

Relation to the Literature. The line of inquiry pursued here traverses other strands of research. First, our theoretical approach to the determination of expected excess return is related to the status of Treasury bonds as a global safe asset (e.g., Gorton and Ordonez (2013) and Gourinchas and Jeanne (2013)), so the time-varying nature of the lower bound could help to benchmark the expected return of other assets belonging to this class, and risky assets in general.

Second, the focus on option-inferred expected excess returns departs from studies that exploit a parametric model to understand model-implied expected excess returns of a long-term bond (e.g., see, among others, Alvarez and Jermann (2005, Sections 3.1 and 4.3), Bakshi and Chabi-Yo (2012, Section 5.2, Table 2), and Campbell, Sunderam, and Viceira (2013, Table 2)). We differ from Kazemi (1992, Proposition 3), who develops the return generating process of a long-term bond under a stationary environment, and also from Dybvig, Ingersoll, and Ross (1996, Theorem 2), who formalize the properties of the yield (but not the expected return) of an infinite-maturity discount bond. Central to our theoretical analysis is the dependence between the permanent and transitory components of the SDF, which we show has implications for the conditional expected excess return of the long-term bond. Despite the fact that much has been written on the Treasury bond market, we are far from finding a consensus on what could be point estimates of the

conditional and unconditional expected excess return of the long-term bond.

Our study is related to the interesting papers by Martin (2013, 2015), and we differ in four ways. First, Martin (2013, 2015) derives a lower bound on the expected excess return of the equity market. Second, the two papers adopt different approaches to obtain a theoretical lower bound on the expected excess return. In particular, Martin (2013, Section 4; 2015, Section 2.1) proposes a shape restriction on the SDF, which entails that the SDF is declining in equity market returns. Instead, our characterizations rely on the property that the permanent and the transitory components of the SDF exhibit positive quadrant dependence under a class of bivariate distributions, which implies a positive correlation. Going beyond the extant literature, our analysis also exposes the consequences of the dependence between the permanent and the transitory components of the SDF. Third, our construction of the options-based computation of the theoretical lower bound is framed in a stochastic interest rate setting, leading to a slight refinement of the lower bound formulation. Lastly, we showcase the estimates of the lower bound on the expected excess return of a 30-year Treasury bond.

Finally, the theoretical paper by Martin and Ross (2013) assumes a finite number of payoff states, unique transition-independent pricing kernel, and a time-invariant matrix of Arrow-Debreu prices, and derives the expected bond return (conditional on the current state; their Result 7). A distinguishing element of our paper is that it is not predicated on the absence of the permanent (martingale) component of the SDF, a feature that is important in light of Borovicka, Hansen, and Scheinkman (2014, Section 5).

2. Expected excess return of a long-term discount bond

The economic setup reflects the understanding that options are only available on Treasury bond futures. The key result is a lower bound on the expected excess return of the long-term discount bond. We show additionally how to operationalize the lower bound using options on the long-term Treasury bond futures. Our study has implications for the modeling of the stochastic discount factor, and, particularly, its transitory component, which determines risk and pricing in the long-term bond market.

2.1. Return of the long-term bond and the transitory components of the stochastic discount factor

Consider an economy outlined in Hansen and Scheinkman (2009), in which the absence of arbitrage guarantees the existence of a pricing kernel at time t , denoted M_t . Let z_t represent the set of state variables.

The work of Hansen and Scheinkman (2009, page 179) shows that M_t can be expressed as:

$$M_t = \exp(\rho t) \widehat{M}_t \frac{\phi[z_0]}{\phi[z_t]}. \quad \text{Hence, over two dates,} \quad \frac{M_{t+1}}{M_t} = \exp(\rho) \frac{\widehat{M}_{t+1}}{\widehat{M}_t} \frac{\phi[z_t]}{\phi[z_{t+1}]}, \quad (1)$$

where ρ is the deterministic growth rate component (the eigenvalue), \widehat{M}_t is a martingale, and $\log(\widehat{M}_t)$ has stationary increments. Moreover, $\phi[z_t]$ is a positive function and $\frac{\phi[z_t]}{\phi[z_{t+1}]}$ represents the transitory contribution to the stochastic discount factor, that is, $m_{t+1} \equiv \frac{M_{t+1}}{M_t} > 0$. The analysis maintains that m_{t+1} is stationary.

Hansen and Scheinkman (2009, Proposition 2, page 201) formalize how an appropriately solved eigenfunction problem could ensure the *uniqueness* of the decomposition in equation (1), specifically,

$$\phi[z_t] \text{ and } \rho \text{ are solutions to } E_t(M_{t+1} \phi[z_{t+1}]) = e^\rho M_t \phi[z_t], \quad (2)$$

where $E_t(\cdot)$ indicates conditional expectation under the physical probability measure. The eigenfunction problem may yield distinct eigenvalues ρ leading to different martingales (different permanent components). However, Hansen (2012, Sections 6.2–6.3), Hansen (2013, Sections 2.2–2.3), and Christensen (2014) show that there is at most one such decomposition for which the martingale induces stochastically stable dynamics. In such case, the permanent component of the SDF is $\frac{M_{t+1}^P}{M_t^P} = \frac{\widehat{M}_{t+1}}{\widehat{M}_t}$, together with:

$$\frac{M_{t+1}^P}{M_t^P} = \frac{M_{t+1}}{M_t} / \frac{M_{t+1}^T}{M_t^T} > 0, \quad \text{where the transitory component is } \frac{M_{t+1}^T}{M_t^T} = e^\rho \frac{\phi[z_t]}{\phi[z_{t+1}]} > 0. \quad (3)$$

$\phi[z_t]$ represents the unique eigenfunction that guarantees stable dynamics. The permanent and the transitory components of the SDF *are correlated*. In what follows, we write $m_{t+1}^P \equiv \frac{M_{t+1}^P}{M_t^P}$ and $m_{t+1}^T \equiv \frac{M_{t+1}^T}{M_t^T}$ for brevity.

The discount bond price at date t , denoted by $V_t[1_{t+k}]$, represents a claim to \$1 at date $t+k$. The absence of arbitrage implies the pricing relations (e.g., Alvarez and Jermann (2005, pages 1980–1981)):

$$V_t[1_{t+k}] = E_t \left(\frac{M_{t+k}}{M_t} \times 1 \right). \quad \text{Hence,} \quad R_{t+1,k} \equiv \frac{V_{t+1}[1_{t+k}]}{V_t[1_{t+k}]} = \frac{E_{t+1} \left(\frac{M_{t+k}}{M_{t+1}} \times 1 \right)}{E_t \left(\frac{M_{t+k}}{M_t} \times 1 \right)}. \quad (4)$$

The gross return of a k -period bond is $R_{t+1,k}$ and that of an infinite-maturity bond is $R_{t+1,\infty} \equiv \lim_{k \rightarrow \infty} R_{t+1,k}$. The stationarity of m_{t+1} implies that $R_{t+1,\infty}$ is stationary. Under the setup in Alvarez and Jermann (2005, Assumptions 1 and 2) and the stationarity of bond prices, $-\frac{1}{k} \log(V_t[1_{t+k}])$ and $V_t[1_{t+k}]$ are also well-defined for large k . The gross return of a risk-free bond is $R_{t+1,f} \equiv \frac{1}{E_t(m_{t+1})}$ (which is known at date t).

To link the return of the infinite-maturity discount bond and the transitory component of the stochastic discount factor, we invoke a result due to Alvarez and Jermann (2005, proof of Proposition 2, page 2007):

$$(m_{t+1}^T)^{-1} = R_{t+1,\infty}. \quad (5)$$

When equation (5) is satisfied, the decomposition in equation (1) is unique (e.g., Alvarez and Jermann (2005, Proposition 1)). The gross return of the long-term discount bond satisfies $E_t(m_{t+1} R_{t+1,\infty}) = 1$.

2.2. Lower bound on the conditional expected excess return of the long-term bond

The economic question of interest is how to characterize the behavior of the expected excess return of the long-term bond (i.e., $E_t(R_{t+1,\infty}) - R_{t+1,f}$), in particular, the lower bound. We obtain this bound in a model-free manner by adopting an options-based approach, and by exploiting a concept of dependence between the permanent and the transitory components of m_{t+1} .

For our analysis, suppose $0 < \underline{a} \leq m_{t+1}^P \leq \bar{a} < +\infty$ and $0 < \underline{b} \leq m_{t+1}^T \leq \bar{b} < +\infty$. Moreover, the two random variables (m_{t+1}^P, m_{t+1}^T) have a cumulative distribution function $\mathbb{H}[a, b] = \text{Prob}[m_{t+1}^P \leq a, m_{t+1}^T \leq b]$, and marginal distribution functions $\mathbb{F}[a] = \text{Prob}[m_{t+1}^P \leq a]$ and $\mathbb{G}[b] = \text{Prob}[m_{t+1}^T \leq b]$, respectively.

We suppose $E_t(m_{t+1}^P m_{t+1}^T) < +\infty$, $E_t(m_{t+1}^T) < +\infty$, and $E_t(1/m_{t+1}^T) < +\infty$. Moreover, $E_t((m_{t+1}^P)^2) < +\infty$, $E_t((m_{t+1}^T)^2) < +\infty$, and $E_t((m_{t+1}^T)^{-2}) < +\infty$. Then, (i) the covariance $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T)$ exists, (ii) the covariance $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$ also exists by applying Cauchy-Schwarz to $E_t(m_{t+1}^P \times \frac{1}{m_{t+1}^T})$.

In what follows, we consider a notion of dependence between m_{t+1}^P and m_{t+1}^T under a class of bivariate distributions $\mathbb{H}[a, b]$. The degenerate case, $m_{t+1}^P = 1$, which is at the core of Martin and Ross (2013), does not interest us.

2.2.1. The link between the price of futures and the long-term bond

Options are traded on the futures of the long-term Treasury bond (for example, with a maturity of 30 years). In this light, we exploit the arbitrage-free link between the price of futures and the long-term bond and modify the pricing equations.

Let f_t represent the time- t price of a one-period futures contract on $V_{t+1}[1_{t+k}]$, where k is presumed to be large. Hence, at time $t+1$, $f_{t+1} = V_{t+1}[1_{t+k}]$. Then $f_t = E_t(\frac{m_{t+1}}{E_t(m_{t+1})} V_{t+1}[1_{t+k}]) = R_{t+1,f} V_t[1_{t+k}]$, akin to

a cost-of-carry relationship (e.g., Cox, Ingersoll, and Ross (1981, equation (46))). Therefore, we may write $R_{t+1,\infty} - R_{t+1,f}$ as:

$$R_{t+1,\infty} - R_{t+1,f} = R_{t+1,f} \left(\frac{f_{t+1}}{f_t} - 1 \right). \quad (6)$$

One can interpret $\frac{f_{t+1}}{f_t} - 1$ as the excess return of a fully collateralized long futures position, since the gross return is $\frac{1}{f_t} (f_{t+1} + (R_{t+1,f} - 1)f_t)$. Equation (6) maps the quantity $R_{t+1,\infty} - R_{t+1,f}$ into $R_{t+1,f} \left(\frac{f_{t+1}}{f_t} - 1 \right)$.

2.2.2. Arbitrage-free pricing of claims on the excess futures return

To proceed with our developments, let $E_t^* (H[f_{t+1}])$ indicate the expectation of a generic payoff $H[f_{t+1}]$ under the risk-neutral (pricing) measure, defined as (e.g., Singleton (2006, pages 202–203)):

$$E_t^* (H[f_{t+1}]) = E_t \left(\frac{m_{t+1}}{E_t(m_{t+1})} H[f_{t+1}] \right), \quad (7)$$

where the normalization $E_t(m_{t+1})$ on the right-hand side of equation (7) ensures that the risk-neutral density integrates to unity.

In addition, if $H[f_{t+1}]$ is twice continuously differentiable with bounded expectation, then we can use the methods of Bakshi and Madan (2000, Appendix A.3) and Carr and Madan (2001, equation (1)) to express the arbitrage-free value of $H[f_{t+1}] = \left(\frac{f_{t+1}}{f_t} - 1 \right)^n$, up to any power $n \geq 2$ as (see Appendix B):

$$E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^n \right) = \frac{n(n-1)R_{t+1,f}}{f_t^2} \left(\int_{\{K > f_t\}} \left(\frac{K}{f_t} - 1 \right)^{n-2} C_t[K] dK + \int_{\{K < f_t\}} \left(\frac{K}{f_t} - 1 \right)^{n-2} P_t[K] dK \right), \quad (8)$$

where $C_t[K]$ ($P_t[K]$) is the time t price of the out-of-the-money call (put) written on the *futures* of the long-term bond with the strike price K .

Equation (8) captures the price of the “power contract,” aptly accounts for the effect of future interest rate uncertainty, and does not invoke assumptions about the return dynamics of the bond futures or the form of the stochastic discount factor.

We employ equation (6), together with equation (8), to show that the theoretical lower bound on the expected excess return of the long-term bond can be recovered from options (on bond futures), an approach also pursued by Martin (2013, 2015) in the context of the expected excess return of the equity market.

2.2.3. Conditional covariance between the permanent and the transitory components of the SDF

Using the relation $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) = E_t(m_{t+1}) - E_t(m_{t+1}^P)E_t(m_{t+1}^T) = \frac{1}{R_{t+1,f}} - E_t\left(\frac{1}{R_{t+1,\infty}}\right)$, we obtain

$$\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) \geq 0, \quad \text{provided} \quad \frac{1}{R_{t+1,f}} \geq E_t\left(\frac{1}{R_{t+1,\infty}}\right), \quad \text{or when} \quad E_t(m_{t+1}) \geq E_t(m_{t+1}^T). \quad (9)$$

Moreover, by the convexity of $1/R_{t+1,\infty}$ and Jensen's inequality, it then holds that $\frac{1}{R_{t+1,f}} \geq E_t\left(\frac{1}{R_{t+1,\infty}}\right) \geq \frac{1}{E_t(R_{t+1,\infty})}$. Thus, a sufficient condition for the conditional covariance between the permanent and transitory components of m_{t+1} to be positive is that the expected excess return of the long-term bond be positive.

The result $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) \geq 0$ does not rely on shape restrictions on m_{t+1} . Moreover, the inequality in equation (9) does not hinge on assumptions about the dynamics of the permanent and transitory components of m_{t+1} , when m_{t+1} can be decomposed into its permanent and transitory components.

Zero correlation between m_{t+1}^P and m_{t+1}^T yields $\frac{1}{1+r_{t+1,f}} = E_t\left(\frac{1}{1+r_{t+1,\infty}}\right)$, for $r_{t+1,f} \equiv R_{t+1,f} - 1$ and $r_{t+1,\infty} \equiv R_{t+1,\infty} - 1$, and, hence, to first-order, $E_t(r_{t+1,\infty}) = r_{t+1,f}$ (given the Taylor series $(1+x)^{-1} = 1-x+x^2-x^3 \dots \approx 1-x$, for small x). Likewise, $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) < 0$ implies, to first-order, $E_t(r_{t+1,\infty}) < r_{t+1,f}$. The intuition is that $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) < 0$ can produce a less volatile m_{t+1} , which can lower risk premiums.

A rationale for $E_t(m_{t+1}) \geq E_t(m_{t+1}^T)$ can also be garnered from Bakshi and Chabi-Yo (2012, Section 4, Table 5), where it is shown that the unconditional covariance $\text{Cov}(m_{t+1}^P, m_{t+1}^T)$ inferred from Treasuries is positive. The case $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) \geq 0$, namely, $E_t(m_{t+1}) \geq E_t(m_{t+1}^T)$, is of economic interest.

2.2.4. Characterizing the lower bound on the expected excess return

For the results to follow, recognize that $R_{t+1,\infty} = (m_{t+1}^T)^{-1}$, $m_{t+1}R_{t+1,\infty}^2 = m_{t+1}^P/m_{t+1}^T$, and $R_{t+1,\infty} - R_{t+1,f} = R_{t+1,f}\left(\frac{f_{t+1}}{f_t} - 1\right)$ from equation (6). Thus, we may express

$$\text{Cov}_t\left(m_{t+1}^P, \frac{1}{m_{t+1}^T}\right) = E_t\left(\frac{m_{t+1}^P}{m_{t+1}^T}\right) - E_t(R_{t+1,\infty}), \quad (10)$$

and

$$R_{t+1,f}E_t^*\left(\left(\frac{f_{t+1}}{f_t} - 1\right)^2\right) = E_t(m_{t+1}R_{t+1,\infty}^2) - R_{t+1,f} = E_t\left(\frac{m_{t+1}^P}{m_{t+1}^T}\right) - R_{t+1,f}. \quad (11)$$

Subtracting the terms in equations (10) from (11) on each side of the equality, and rearranging, we obtain the following representation:

$$E_t(R_{t+1,\infty}) - R_{t+1,f} = \underbrace{R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right)}_{\text{scaled risk-neutral futures return variance}} - \text{Cov}_t \left(m_{t+1}^P, \frac{1}{m_{t+1}^T} \right). \quad (12)$$

Equation (12) is the long-term bond analog of a relation in Martin (2013, equation (23)), presented in the context of the equity market portfolio. In our setting, equation (12) furnishes economic content by exposing the consequences of the dependence between m_{t+1}^P and m_{t+1}^T .

For the characterizations to follow, we consider the following definition of dependence.

Definition 1 (*Lehmann (1966, equation (2.1)) and Kimeldorf and Sampson (1989, Definition 2.1)*) *The pair $(m_{t+1}^P, m_{t+1}^T) \in \mathcal{P}$, the class of bivariate distributions that exhibit positive quadrant dependency, provided:*

$$\mathbb{H}[a, b] \geq \mathbb{F}[a] \mathbb{G}[b], \quad \text{for all } a, b, \quad (13)$$

where the joint distribution is $\mathbb{H}[a, b]$, and the marginal distributions are $\mathbb{F}[a]$ and $\mathbb{G}[b]$, respectively.

Equation (13) mathematically describes the tendency of two random variables to assume concordant (discordant) values, whereby positive dependence is captured by comparing the bivariate distribution with the distribution corresponding to the independence hypothesis. Economically, the positive quadrant dependence implies that the probability that both m_{t+1}^P and m_{t+1}^T are small (or large) is at least as great as it would be under independence. In other words, both variables display a tendency to move downwards or upwards.

Positive quadrant dependence is a weak condition of positive dependence (e.g., Mari and Kotz (2004, page 34)) and implies $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) \geq 0$ (Lehmann (1966, Lemma 2 and Lemma 3)). The class of distributions for which $(\log(m_{t+1}^P), \log(m_{t+1}^T)) \in \mathcal{P}$ includes the bivariate normal with a positive correlation coefficient (Lehmann (1966, Example 1, part (ii)); other examples are highlighted in Balakrishnan and Lai (2009)). We state the following result relevant to our analysis.

Lemma 1 *When the random variables (m_{t+1}^P, m_{t+1}^T) satisfy the positive quadrant dependency property, then*

$$\text{Cov}_t \left(m_{t+1}^P, \frac{1}{m_{t+1}^T} \right) \leq 0. \quad (14)$$

Proof. See Appendix A. ■

We can now formalize the lower bound on the expected excess return of the long-term discount bond, which is at the center of the empirical investigation.

Proposition 1 *Suppose (m_{t+1}^P, m_{t+1}^T) are positively quadrant dependent. The theoretical lower bound on the expected excess return of the long-term discount bond satisfies:*

$$E_t(R_{t+1,\infty}) - R_{t+1,f} \geq \underbrace{R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right)}_{\text{Lower bound}}. \quad (15)$$

Proof. See Appendix B. ■

The next result shows how to recover the theoretical lower bound from the options on the futures of the long-term bond.

Proposition 2 *The theoretical lower bound in equation (15) can be obtained as the value of the power contract in equation (8), evaluated at $n = 2$, under all martingale pricing measures:*

$$R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right) = \underbrace{\frac{2R_{t+1,f}^2}{f_t^2} \left(\int_{\{K > f_t\}} C_t[K] dK + \int_{\{K < f_t\}} P_t[K] dK \right)}_{\text{Can be recovered from prices of calls, puts, and futures}}, \quad (16)$$

where $C_t[K]$ ($P_t[K]$) is the time t price of the out-of-the-money call (put) written on the futures of the long-term bond with the strike price K .

Proof. See Appendix C. ■

A lower bound on the expected excess return is an important number, as it conveys the minimum expected compensation demanded by investors for committing their capital to the long-term bond at date t , as opposed to the upper bound, which conveys the maximum compensation. Accordingly, the *lower bound* in equation (15) must be distinguished from the *unconditional upper bound* in Cochrane (2005, equation 1.17, page 17) and must not be confused with the restriction $|E(R_{t+1,\infty}) - R_{t+1,f}| \leq \frac{1}{E(m_{t+1})} \sqrt{\text{Var}(m_{t+1}) \text{Var}(R_{t+1,\infty})}$, which holds when m_{t+1} prices the long-term bond. Such an upper bound requires knowledge of the unconditional variance of m_{t+1} (or its Hansen and Jagannathan (1991, equation (12)) lower bound).

Proposition 1 and Appendix B develop the view that the expected excess return is bounded by the vari-

ance of the excess futures return under the risk-neutral measure, scaled by $R_{t+1,f}$. The rationale follows from rewriting equation (10) as $E_t(R_{t+1,\infty}) = E_t(m_{t+1}R_{t+1,\infty}^2) - \text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$, which guides the intuition that return movements in the tails exert a positive influence on $E_t(R_{t+1,\infty})$. Ceteris paribus, the more negative is the variance risk premium implicit in $E_t(m_{t+1}R_{t+1,\infty}^2)$, the higher the minimum required return.

The result in equation (16) reveals that the variance of the excess futures return under the risk-neutral measure can be obtained through a positioning in out-of-the-money calls and puts on the bond futures, where each option is weighted by $\frac{2R_{t+1,f}}{f_t^2}$. Thus, the theoretical lower bound in equation (15) is a data-inferred quantity. Importantly, the changing nature of the expected excess return of the long-term bond is reflected in the changing nature of the risk-neutral variance of the excess futures return.

While our lower bound result in equation (15) relies on positive quadrant dependence between m_{t+1}^P and m_{t+1}^T , which implies $E_t(m_{t+1}) \geq E_t(m_{t+1}^T)$ (i.e., $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) \geq 0$), what is a possible economic interpretation? We develop the key intuition from several perspectives:

Case 1 Consider economies in which the pricing kernel is persistent (as in many equilibrium models):

$$\log(M_{t+1}) = \log(\beta_0) + \nu \log(M_t) + \tilde{\varepsilon}_{t+1}, \quad \text{where } \tilde{\varepsilon}_{t+1} \sim N(0, \sigma_{\tilde{\varepsilon}}^2), \quad |\nu| < 1, \quad \text{and } \beta_0 > 0. \quad (17)$$

Then $E_t\left(\log\left(\frac{R_{t+1,k}}{R_{t+1,f}}\right)\right) = \frac{\sigma_{\tilde{\varepsilon}}^2}{2}(1 - \nu^{2(k-1)}) \equiv h_t[k]$ (see Alvarez and Jermann (2005, equation (8))). Hence, $\lim_{k \rightarrow \infty} h_t[k] = E_t\left(\log\left(\frac{R_{t+1,\infty}}{R_{t+1,f}}\right)\right) = \frac{\sigma_{\tilde{\varepsilon}}^2}{2}$, implying that the term premium (the conditional expected log excess return) is linked to $\sigma_{\tilde{\varepsilon}}^2$, the volatility of the innovation of the pricing kernel M_t .

In economic terms, the positive term premium is compatible with $E_t(m_{t+1}) \geq E_t(m_{t+1}^T)$. To show this result, note, by Jensen's inequality, $\log\left(E_t\left(\frac{R_{t+1,\infty}}{R_{t+1,f}}\right)\right) \geq E_t\left(\log\left(\frac{R_{t+1,\infty}}{R_{t+1,f}}\right)\right)$ and, therefore, $E_t\left(\frac{R_{t+1,\infty}}{R_{t+1,f}}\right) \geq 1$. Next, we can write equation (9) as $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) = \frac{1}{R_{t+1,f}}\left(1 - E_t\left(\frac{R_{t+1,f}}{R_{t+1,\infty}}\right)\right)$. Invoking Jensen's inequality yields $E_t\left(\frac{R_{t+1,f}}{R_{t+1,\infty}}\right) \leq \frac{R_{t+1,f}}{E_t(R_{t+1,\infty})} \leq 1$, which then establishes $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) \geq 0$. ♣

Case 2 The inequality $E_t(m_{t+1}) \geq E_t(m_{t+1}^T)$ is consistent with a class of term structure models where the conditional expected excess return is positive at all maturities, for example, the translated version of Cox, Ingersoll, and Ross (1985) and Pearson and Sun (1994, equation (5)), and also Wachter (2013, equation (30)). The models of Constantinides (1992, equation (23)), Campbell and Cochrane (1999), Bansal and Yaron (2004), Cochrane and Piazzesi (2005), and Gabaix (2012) also admit a positive conditional expected excess return of a long-term discount bond under reasonable parameterizations.

Campbell, Sunderam, and Viceira (2013, equation (6)) construct a model in which the nominal bond yields are linear-quadratic functions of five state variables. Their model has two distinguishing attributes. First, the long-term (i.e., 10-year) bond could serve as a hedge against certain state variables. Second, the model accommodates positive expected excess return of a long-maturity discount bond. ♣

Case 3 Our thrust is a model that generates positive quadrant dependency of $(\log(m_{t+1}^P), \log(m_{t+1}^T))$ when the eigenfunction problem is solved to derive m_{t+1}^P and m_{t+1}^T . The realism of the economy depends critically on whether model parameterizations translate into a positive or a negative value for $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$.

Suppose the shocks $(\tilde{\boldsymbol{\epsilon}}_{t+1}, \tilde{\boldsymbol{\omega}}_{t+1})'$ to the economy are distributed bivariate standard normal with a constant correlation coefficient ψ . We specify a model for m_{t+1} as follows:

$$\log(m_{t+1}) = \varphi + \eta(\theta_z - z_t) - \frac{1}{2}\sigma_p^2 z_t - \sigma_p \sqrt{z_t} \tilde{\boldsymbol{\epsilon}}_{t+1} - \sigma_\infty \sqrt{z_t} \tilde{\boldsymbol{\omega}}_{t+1}, \quad (18)$$

$$z_{t+1} - z_t = \kappa_z(\theta_z - z_t) + \sigma_z \sqrt{z_t} \tilde{\boldsymbol{\omega}}_{t+1}, \quad (\text{with } 2\kappa_z \theta_z \geq \sigma_z^2) \quad (19)$$

where z_t reflects the state of the economy, and $\varphi, \eta, \sigma_p, \sigma_\infty, \kappa_z, \theta_z$ and σ_z are constants.

Section III of the Internet Appendix shows that the eigenfunction problem $E_t \left(m_{t+1} \frac{\phi[z_{t+1}]}{\phi[z_t]} \right) = e^\rho$ has solution of the form $\phi[z_t] = \exp(c z_t)$, where the constants c and ρ are presented in (A6) and (A7) of the Internet Appendix. Then, $m_{t+1}^T = e^\rho \frac{\phi[z_t]}{\phi[z_{t+1}]} = \exp(\rho - c \kappa_z \theta_z + c \kappa_z z_t - c \sigma_z \sqrt{z_t} \tilde{\boldsymbol{\omega}}_{t+1})$, and $m_{t+1}^P = m_{t+1} / m_{t+1}^T$.

The pair $(\log(m_{t+1}^P), \log(m_{t+1}^T))$ is bivariate normal with covariance $c \sigma_z (\sigma_p \psi + \sigma_\infty) z_t + c^2 \sigma_z^2 z_t$, which can attain positive or negative values (values of $\psi > 0$ can counteract positive dependence since c is typically negative). Our constructions offer tractability in modeling the dependence between the m_{t+1}^P and the m_{t+1}^T components of the SDF, keeps all expressions in closed-form, and enables us to study the properties of m_{t+1}^P and m_{t+1}^T in a unified framework.

To address the quantitative implications of the model, we derive the following quantities:

$$R_{t+1,f} = (E_t(m_{t+1}))^{-1} = \exp \left(-\varphi - \eta \theta_z + \left(\eta - \frac{1}{2} \sigma_\infty^2 - \psi \sigma_p \sigma_\infty \right) z_t \right), \quad (20)$$

$$E_t(R_{t+1,\infty}) = \exp \left(-\rho + c \kappa_z \theta_z + \left(\frac{1}{2} c^2 \sigma_z^2 - c \kappa_z \right) z_t \right), \quad (21)$$

$$E_t \left(\frac{m_{t+1}^P}{m_{t+1}^T} \right) = \exp \left(-\rho + c \kappa_z \theta_z - (\eta + 2c \kappa_z) z_t + \frac{1}{2} (2c \sigma_z - \sigma_\infty)^2 z_t - \psi \sigma_p (2c \sigma_z - \sigma_\infty) z_t \right). \quad (22)$$

We then obtain $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$ and $R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right)$ via equations (10) and (11), respectively.

We simulate the model under various parameterizations and focus on two sets of values in Table 1 that are consistent with $\text{Cov}_t(\log(m_{t+1}^P), \log(m_{t+1}^T)) > 0$. A number of insights stem from our analysis. First, the documented positive covariance between m_{t+1}^P and m_{t+1}^T connects with a negative covariance between m_{t+1}^P and $1/m_{t+1}^T$. Second, many features of the bond markets are broadly aligned with the data counterparts, including the risk-free return and the excess return of the long-term bond. However, a new data dimension matched in our model setting is the risk-neutral return volatility of the futures on the long-term bond.

The Internet Appendix isolates the implications of parameterizations that force $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) < 0$. Such economies can be distinguished by the feature that they yield the wrong order of magnitudes for the expected excess return of the long-term bond, and particularly the risk-neutral return volatility of the futures of the long-term bond. Overall, our exercises reinforce that a positive value for $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T)$, and a negative value for $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$, is needed to plausibly capture stylized elements of the data. ♣

Case 4 Entropy-based codependence, as explored in both Hansen (2012, Section 4.3) and Bakshi and Chabi-Yo (2013), is a stronger form of dependence than positive quadrant dependency. Using the definition of entropy, that is, $L_t[\tilde{u}] = \log(E_t[\tilde{u}]) - E_t(\log(\tilde{u}))$, one may note, in our context, that

$$\mathbb{C}_t \equiv \underbrace{L_t[m_{t+1}^P/m_{t+1}^T] - L_t[m_{t+1}^P] - L_t[1/m_{t+1}^T]}_{\text{entropy-based codependence}} = \log(E_t(m_{t+1}^P/m_{t+1}^T)) - \log(E_t(R_{t+1,\infty})). \quad (23)$$

Negative entropy-based codependence implies $\mathbb{C}_t < 0$ and $\exp(\mathbb{C}_t) < 1$ and, therefore, $\frac{E_t(m_{t+1}^P/m_{t+1}^T)}{E_t(R_{t+1,\infty})} < 1$. Hence, $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T}) < 0$ from equation (10). Thus, the theoretical lower bound in (15) of Proposition 1 can also emerge as a consequence of negative entropy-based codependence between m_{t+1}^P and $1/m_{t+1}^T$. ♣

Our characterizations in Propositions 1 and 2 may evoke references to Martin (2013, equations (24) and (12)) (or Martin (2015, equations (3) and (14))). There are several differences. First, Martin lower bounds the expected excess return of the equity market, while our analysis centers on the long-term bond. In fact, Martin features the restriction that the SDF is declining in equity market returns (his NCC condition) under which the equity premium is bounded by the discounted risk-neutral equity return variance (e.g., pages 22 and 23 of Martin (2013)). Instead, we transform the equality in (12) into an inequality, via a positive quadrant dependency condition (that implies $E_t(m_{t+1}) \geq E_t(m_{t+1}^T)$). Second, our formula for the lower bound in equation (16) is not the same as that stated in Martin (2015, equation (14)). While both papers obtain a model-free representation of the lower bound by valuing the squared contract, this treatment is not

unique to either paper (the power contract is considered, among others, by Bakshi, Kapadia, and Madan (2003, equations (4) and (7)) and Carr and Lee (2008, equation (5.2)); see also Singleton (2006, page 410)). We go a bit further given our setting of interest rate sensitive claims, accounting for the slight difference from Martin (2013, 2015). Finally, we pursue an empirical investigation that could increase our understanding of risk and pricing in the bond market.

Like the conditional expected return of the equity market (e.g., Merton (1980)), the expected return of a long-term bond is an important concept, prompting many authors, including Jarrow (1978), Fama and French (1989), Ilmanen (1995), Dybvig, Ingersoll, and Ross (1996), Bansal and Lehmann (1997), Campbell and Viceira (2001), Campbell, Sunderam, and Viceira (2013), and Martin and Ross (2013) to study the problem from theoretical and empirical perspectives.² Proposition 2 offers an alternative in that the theoretical lower bound on the expected excess return can be inferred from a portfolio of options on the futures of the long-term bond, which holds under all martingale pricing measures.³

3. Description of the options data on the 30-year Treasury bond futures

There are two sources of variation in the expected excess returns of a long-term bond, according to the derived equation (12): $E_t(R_{t+1,\infty}) - R_{t+1,f} = R_{t+1,f} E_t^*((\frac{f_{t+1}}{f_t} - 1)^2) - \text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}})$. First, shocks to expected excess returns can be traced to shocks to $R_{t+1,f} E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)$. The generality of our analysis comes from two aspects: (i) we do not specify the form of the stochastic discount factor, and (ii) $E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)$ can be directly recovered from options written on the long-term bond futures. Second, shocks to the expected excess returns can be traced to shocks to $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}})$. However, the conditional covariance term cannot be determined without specifying and estimating a model (as can also be seen from our Case 3).

Our approach motivates a lower bound on $E_t(R_{t+1,\infty}) - R_{t+1,f}$. Three inputs are required to compute the lower bound in equation (16) of Proposition 2. Specifically, the implementation requires the futures price, the prices of out-of-the-money calls and puts on the bond futures, and the risk-free return.

²One could argue that if one follows Black (1993), then the expected excess return of the long-term bond can also be obtained from the Capital Asset Pricing Model, as in $E_t(R_{t+1,\infty}) - R_{t+1,f} = \beta (E_t(R_{t+1,M}) - R_{t+1,f})$. Even when the beta of the bond can be estimated with some level of sophistication from the data (e.g., Ross, Westerfield, and Jaffe (2010, Chapter 13)), the difficulty is to infer the conditional expected excess return of the equity market (i.e., $E_t(R_{t+1,M}) - R_{t+1,f}$). Merton (1980) argues that estimating expected return can be challenging when it is believed that the expected return is shifting with economic conditions.

³Departing from our setting, Kazemi (1992, equation (5)) shows that $E_t(R_{t+1,\infty}) - R_{t+1,f} = \text{Var}_t(R_{t+1,\infty})$ (see Proposition 5, Kazemi (1992, equation (15))). This result is driven by the assumption that the state variables have a long-run stationary joint distribution and are governed by standard Brownian motions. In contrast, the theoretical lower bound in Proposition 1 does not require distributional assumptions.

The 30-year Treasury bond is used to proxy for the long-term bond in the empirical analysis (as also analyzed by Alvarez and Jermann (2005, Table 1)), as it is the longest maturity available, and the corresponding futures and options on the futures are actively traded in the market.

We compute the lower bound at the end of each month over the sample period of 1982:10 to 2013:12 (375 observations), which coincides with the introduction of options on the futures of the 30-year Treasury bond. This 31-year sample period encompasses many of the important events and developments in monetary and fiscal policy, and includes four recessions.

The construction of the options positioning underlying the square of the futures excess return, specifically, $E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)$, requires a careful handling of the options data on the bond futures, which are reported by the DataMine End-of-Day from the Chicago Mercantile Exchange. The options data is daily, and the contract months are the first three consecutive (two serial expirations and one quarterly expiration) months, plus the next two months in the March, June, September, and December quarterly cycle.

The futures data on the 30-year Treasury bond is also obtained from the Chicago Mercantile Exchange (CME). The data is daily, and the futures contract months are the first three consecutive contracts in the March, June, September, and December quarterly cycles.⁴ The Section I of the Internet Appendix describes how we convert the CME quotes on futures and options to the dollar prices.

Several criterion were applied to the daily options data to construct the end-of-month observations on calls and puts. First, we sample options at the end of each month and focus on options with maturity closest to 30 days. Let the corresponding closing price of the futures contract at the end-of-the-month be f_t . Second, we keep out-of-the-money calls, i.e., those with $f_t/K < 1$, and out-of-the-money puts, i.e., those with $K/f_t < 1$, where K is the strike price. Using out-of-the-money options also mitigates the impact of the early exercise feature of American options (as shown by Mueller, Vedolin, and Yen (2013, page 15)). Third, we omit 31 option quotes prior to 1993:07, at which point the strike price is more than \$4,000 away from the adjacent strike price. Moreover, we omit 175 option quotes after 1993:07, when the strike price is more than \$2,000 away from the adjacent strike price. Each of the omitted options is extremely deep out-of-the-money, has a settlement price of \$15.625 (the minimum tick size), and has zero trading volume.

The final options sample contains 9,471 observations, with 4,331 out-of-the-money calls, 5,140 out-of-the-money puts, and matched prices of the underlying bond futures. On average, our sample contains

⁴At the end of each month, we use the settlement price of the nearest maturity contract to compute the futures return, while taking into account the first notice day. For example, the March 2010 futures has a first notice day on February 26, 2010 (Friday). Accordingly, we use the June 2010 futures to compute the return at the end of March 2010.

11.55 calls and 13.71 puts in each month. The average maturity of options is 29.16 days. Table Internet-I provides a snapshot of the monthly average of the number of strikes and option open interest from 1982 to 2013.

Complementing our empirical analysis, we also construct the daily and monthly returns data on one-month Treasury bills and 30-year Treasury bonds. The source of the one-month returns is the data library of Ken French, whereas the 30-year Treasury bond data is taken from CRSP Fixed-Term Indexes.

4. Understanding the expected excess return of the long-term bond

The key step is to compute the lower bound in equation (16) of Proposition 2, hereby denoted as:

$$\text{LB}_{\{t \rightarrow t+1\}} \equiv \frac{2R_{t+1,f}^2}{f_t^2} \left(\int_{\{K > f_t\}} C_t[K] dK + \int_{\{K < f_t\}} P_t[K] dK \right). \quad (24)$$

The subscript notation $\{t \rightarrow t+1\}$ is meant to emphasize that the options positioning is computed at the end of month t and embodies expectation about excess returns over the subsequent month $t+1$. Hence, $\text{LB}_{\{t \rightarrow t+1\}}$ is the forward-looking component of expected excess returns between the end of month t and the end of month $t+1$, which we extract from options on the 30-year Treasury bond futures.

The integral representation of the lower bound in equation (24) is tractable and can be reliably approximated using the trapezoidal rule (e.g., Bakshi, Kapadia, and Madan (2003), Jiang and Tian (2005), and Carr and Wu (2009)). We calculate the integral of the call as (e.g., Lindfield and Penny (1995, page 125)):

$$\int_{\{K > f_t\}} C[K] dK \approx (C[K_{\min}] + 2C[K_{\min} + \Delta K] + 2C[K_{\min} + 2\Delta K] + \dots + C[K_{\max}]) \frac{\Delta K}{2}, \quad (25)$$

where $K_{\min} > f_t$. The analogous calculation for the integral of the put is:

$$\int_{\{K < f_t\}} P[K] dK \approx (P[K_{\max}] + 2P[K_{\max} - \Delta K] + 2P[K_{\max} - 2\Delta K] + \dots + P[K_{\min}]) \frac{\Delta K}{2}, \quad (26)$$

where $K_{\max} < f_t$. The options positioning that statically spans the square of the excess futures return differs from the one in Mueller, Vedolin, and Yen (2013, Proposition 2) and Mele and Obayashi (2013, equation (15)). This distinction emerges because these authors focus on the square of the log futures return.

4.1. *The estimated lower bound on the expected excess return is 1.18% on average*

Table 2 reports the average $LB_{\{t \rightarrow t+1\}}$ over the full 31-year sample and also across several subsamples. Our investment horizon for the expected excess return is monthly (with an average of 29.16 days), but the reported numbers are expressed in annualized percentage units.

The mean lower bound over the entire sample is 1.18% (118 basis points), and the Politis, White, and Patton (2009) 95% bootstrap confidence intervals are between 1.00% and 1.38%. Importantly, the lower bound reflects the investors' minimum expected excess return when they invest in the long-term bond.

Moreover, the mean lower bound ranges between 1.09% and 1.20% across the four subsamples. In particular, we obtain a mean lower bound of 1.09% prior to the financial crisis (i.e., 1982:10 to 2007:12). We are aware of a potential concern that there are fewer option strikes in the first few years of trading. If we compile our results starting in January 1985, the mean lower bound is 1.17%.

Whereas the average $LB_{\{t \rightarrow t+1\}}$ appears to be stable across the considered subsamples, there is substantial time variation in $LB_{\{t \rightarrow t+1\}}$, as depicted in Figure 1.⁵ The salient aspect of the lower bound appears to be its countercyclic nature. Notably, the lower bound increases following events of concern in the financial markets, reflecting an increase in the minimum expected excess return demanded by investors. Prominent among these events are the stock market crash in October 1987, the financial crisis, and the Federal Reserve lowering the discount rate from 1.25% to 1% (June 25, 2003). The method is flexible, allowing us to recover the minimum conditional expected excess return from options prices at each point in time.

The period of Federal Reserve policy of continued quantitative easing, namely, 2012:01 to 2013:12, is associated with a downward trending pattern of $LB_{\{t \rightarrow t+1\}}$, with an average of 0.89%. This observation can be understood from two angles. First, there was a decline in the number of sovereigns whose debt can be regarded as safe during this period (e.g., SEC (2014, page 3)). Second, the actions of the Federal Reserve lowered market uncertainty. Overall, $LB_{\{t \rightarrow t+1\}}$ lies between a low of 0.22% in 1991:09 (after a series of cuts in the federal funds rate) to a high of 6.07% in 2008:11 (in the aftermath of the Lehman collapse).

One question arises: how can we judge the reasonableness of the lower bound on the expected excess return of the long-term bond? We benchmark our estimates in three ways.

⁵Our confidence in the estimated lower bound is reinforced by the fact that the mean lower bound is 1.12%, using the second-nearest maturity options (with an average maturity of 84.31 days). See Table Internet-II and Section II of the Online Appendix.

	Nobs.	Annualized (%)				ACF ₁	Correlations	
		Mean	Std.	Min.	Max.		TB30	TB20
Excess return, 30-year bond futures	375	3.02	10.58	-10.17	13.39	0.02	0.94	0.97
Excess return, 30-year bond (TB30)	375	4.95	12.53	-14.75	17.41	0.07		0.98
Excess return, 20-year bond (TB20)	375	5.48	10.63	-10.60	14.45	0.03		

First, a fully collateralized long position in the 30-year Treasury bond futures delivers an average annual nominal return of 3.02%.⁶ Second, the coupon-inclusive average (nominal) returns of the 30-year and the 20-year Treasury bonds are 4.95% and 5.48%, respectively, with a correlation of 0.98. In sum, the average $LB_{\{t \rightarrow t+1\}}$ appears aligned with long-run average returns in the bond market while accounting for the effect of coupons. Finally, the full sample average $LB_{\{t \rightarrow t+1\}}$ of 1.18% corresponds to an average annualized risk-neutral futures return volatility of 10.27% (i.e., $\sqrt{E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)}$).

Two additional clarifications are in order. First, our approach moves away from using realized returns to capture the expected excess return. This is conceptually crucial since Elton (1999) argues that the past realized returns do not necessarily equate well with expected returns (see also Black (1993)). Second, our bound corresponds to a monthly holding period, and it is not to be compared with the promised return of a coupon-paying Treasury bond, which captures the entry yield and tends to vary over time with the prevailing interest rate environment.

What are the ways in which the option-inferred estimates of $LB_{\{t \rightarrow t+1\}}$ could prove useful in empirical work? For instance, one could test whether expected excess returns implied by a term structure model are consistent with our lower bound. In particular, the lower bound restriction on the expected excess return could complement the modeling approaches featured in Gurkaynak and Wright (2011).

Finally, Table 2 shows that the lower bound exhibits a first-order autocorrelation (reported in the column ACF₁) of 0.74. We also examine the persistence properties of $LB_{\{t \rightarrow t+1\}}$ and fit all ARMA(p, q) models with $p \leq 3$ and $q \leq 3$. Our findings from model selection indicate that ARMA(1,1) is the best model for $LB_{\{t \rightarrow t+1\}}$, according to the Bayesian information criterion. Relevant to our regression analysis that follows, this evidence shows that the time series of the lower bound $LB_{\{t \rightarrow t+1\}}$ is stationary.

⁶CME lowered the notional coupon rate of the Treasury bond underlying the futures contract from 8% to 6% in 1999. This change sharply reduced the futures prices of contracts maturing in and after March 2000, in comparison with those maturing in and before December 1999. We avoid the mechanical price drop by switching to the second-nearest maturity futures contract when computing the return in the month of December 1999. The option calculations were unaffected.

4.2. Sources of the variation in the lower bound

Asset pricing theories hypothesize that investors demand higher expected return on risky assets in states perceived to be bad, specifically those associated with high marginal utility of wealth (e.g., Merton (1973)). Such a hypothesis has generally proven difficult to implement in an empirical setting given that the conditional expected return is unobservable.

Our approach offers the advantage that the conditional expected excess return can be decomposed into two nonnegative parts, which helps to isolate possible drivers of the expected excess returns. We can do so by assessing which macroeconomic determinants could be underlying the time variation in $LB_{\{t \rightarrow t+1\}}$. Specifically, we employ a linear specification of the type:

$$LB_{\{t \rightarrow t+1\}} = \alpha + \mathbf{X}'_{\{t-1 \rightarrow t\}} \boldsymbol{\beta} + e_{\{t \rightarrow t+1\}}, \quad \text{with,} \\ \mathbf{X}_{\{t-1 \rightarrow t\}} = [\Delta \text{Spread}_{\{t-1 \rightarrow t\}} \quad r_{\{t-1 \rightarrow t\}}^{\text{SDF}} \quad \text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}}]', \quad (27)$$

where $\boldsymbol{\beta}$ is the vector of sensitivity coefficients and α is the intercept. Contained within $\mathbf{X}_{\{t-1 \rightarrow t\}}$ are variables of potentially different economic nature, which could reveal the possibly distinct determinants of expected excess returns. We motivate the choice of our explanatory variables $\mathbf{X}_{\{t-1 \rightarrow t\}}$ as follows:

Changes in yield spread ($\Delta \text{Spread}_{\{t-1 \rightarrow t\}}$): Fama (1990, page 1091) and Fama and French (1989, pages 26 and 27) have argued that the yield spread tracks expected returns. The yield spread is the differential between the 30-year Treasury bond yield and the one-month Treasury bill counterpart. We employ the first difference of the yield spread in our regression equation (27).

Return of a security with SDF payoff ($r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$): This variable captures the return of a security that pays the SDF. We adopt a specification where $m_{\{t-1 \rightarrow t\}} = 1/R_{\{t-1 \rightarrow t\}}^{\text{M}}$ and, hence, $m_{\{t-1 \rightarrow t\}}$ is the inverse of the gross return of the equity market (e.g., Cochrane (2005, equation 9.12, page 160)). The price of a security that pays the SDF is $E_{t-1}(m_{\{t-1 \rightarrow t\}} \times m_{\{t-1 \rightarrow t\}})$. Therefore, the return of a security with the SDF payoff is (see also Cochrane (2005, page 18)):

$$r_{\{t-1 \rightarrow t\}}^{\text{SDF}} \equiv \frac{m_{\{t-1 \rightarrow t\}}}{E_{t-1}(m_{\{t-1 \rightarrow t\}}^2)} - 1 = \frac{\frac{1}{R_{\{t-1 \rightarrow t\}}^{\text{M}}}}{E_{t-1}((R_{\{t-1 \rightarrow t\}}^{\text{M}})^{-2})} - 1. \quad (28)$$

To estimate $E_{t-1}((R_{\{t-1 \rightarrow t\}}^{\text{M}})^{-2})$ in each month, we use daily returns and the moment analog $\sum_{i=1}^{22} (R_{\{t+i-1 \rightarrow t+i\}}^{\text{M}})^{-2}$. The $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$ series so constructed has the feature that returns are positive during bad times, implying a pos-

itive relation between $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$ and $\text{LB}_{\{t \rightarrow t+1\}}$. We proxy the market by the value-weighted equity index.

Return of gold ($\text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}}$): Following Yellen (2012) and Barro and Misra (2013), we next explore the returns of an asset that investors rotate into when they are fearful about impending financial market catastrophe or economic tail risks. We consider in particular the monthly excess returns of a fully collateralized long futures position in gold (data source: CME):

$$\text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}} \equiv \frac{1}{f_{t-1}^{\text{gold}}} \left(f_t^{\text{gold}} - f_{t-1}^{\text{gold}} \right). \quad (29)$$

The hypothesis to consider is whether a rise in gold returns is associated with a rise in expected excess return of the long-term bond.

Tables 3 and 4 present results from univariate and multivariate regressions, respectively, while Table 5 presents results from multivariate regressions with controls. We focus on three controls: (i) industrial production growth, (ii) inflation, and (iii) log changes in the supply of long-term bonds. Reported are the estimates of individual slope coefficients along with the two-sided p -values $\text{NW}[p]$, based on the heteroskedasticity and autocorrelation consistent covariance estimator from Newey and West (1987). Our procedure relies on the Bartlett kernel and no prewhitening, with lag length selected automatically, according to Newey and West (1994).

Our results impart several insights about variations in the expected excess returns of the long-term bond. First, Table 3 shows that each of the three featured variables is positively correlated with $\text{LB}_{\{t \rightarrow t+1\}}$ over various sample periods. The pairwise correlation ranges between 0.07 to 0.15 for $\Delta\text{Spread}_{\{t-1 \rightarrow t\}}$; between 0.18 and 0.31 for $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$; and between 0.04 and 0.16 for $\text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}}$. At the same time, the explanatory variables do not share a large common covariation, as the maximum contemporaneous cross-correlation is 0.08 between $\Delta\text{Spread}_{\{t-1 \rightarrow t\}}$ and $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$. Figure 2 depicts the time-variation in these variables.

Second, our results indicate that the variables are significantly associated with $\text{LB}_{\{t \rightarrow t+1\}}$ over different samples. Consider the univariate regression results in Table 3, where (i) the $\text{NW}[p]$ value is below 0.1 in four out of five subsamples for $\Delta\text{Spread}_{\{t-1 \rightarrow t\}}$, (ii) the $\text{NW}[p]$ value is below 0.1 in three out of five subsamples for $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$, and (iii) $\text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}}$ retains individual significance in two out of five subsamples. The adjusted R^2 's of the univariate regressions are the highest with $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$, and range from 2.8% to 8.8%.

The impact of the featured variables appears economically nontrivial. In particular, a one-standard-deviation increase in $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$ is seen associated with an annualized increase of 0.18% (18 basis points)

in $LB_{\{t \rightarrow t+1\}}$. An analogous calculation reveals that a one-standard-deviation increase in $\Delta Spread_{\{t-1 \rightarrow t\}}$ ($er_{\{t-1 \rightarrow t\}}^{\text{gold}}$) is associated with an increase of 0.07% (0.10%) in $LB_{\{t \rightarrow t+1\}}$. Compared with the average $LB_{\{t \rightarrow t+1\}}$ of 1.18%, these impacts are economically meaningful. The interpretation is that a deterioration in the investor opportunity set is linked to an increase in the minimum expected excess returns, and this effect is the strongest in response to equity market declines.

Table 4 presents the multivariate regression results combining two or all three economic variables. The coefficient estimate for $\Delta Spread_{\{t-1 \rightarrow t\}}$ is uniformly positive and is statistically significant in 12 out of 15 regressions (i.e., $NW[p] \leq 0.1$). The impact of $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$ remains positive in the presence of other variables, and the coefficient estimate is statistically significant in 11 out of 15 regressions. In contrast, the coefficient on $gold_{\{t-1 \rightarrow t\}}$, while always positive, is statistically significant in five of 15 regressions, and is generally driven out by $\Delta Spread_{\{t-1 \rightarrow t\}}$ and $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$. For instance, over the entire sample, we obtain slope coefficient estimates of 0.09, 0.34%, and 0.16% in the multivariate regression with $\mathbf{X}_{\{t-1 \rightarrow t\}} = [\Delta Spread_{\{t-1 \rightarrow t\}} \ r_{\{t-1 \rightarrow t\}}^{\text{SDF}} \ er_{\{t-1 \rightarrow t\}}^{\text{gold}}]'$, with p -values of 0.05, 0.02, and 0.11, respectively. The three variables together track variations in expected excess returns with an adjusted R^2 of 7.8%.

What is the impact of including plausible controls? There are two points worth garnering from Table 5, which presents our results over several samples. First, each of the controls, namely, industrial production growth, inflation, and the log change in bond supply, are insignificant, with $NW[p]$ values higher than 0.05. Second, the statistical significance of the yield spread and the return of the SDF security does not diminish in the presence of controls. Thus, our evidence remains consistent with the view that a rise in the slope of the yield curve and the return of the SDF security tend to increase the expected excess return of the long-term bond. The focus on expected excess returns distinguishes our study from works that consider the possible role of economic variables for understanding the contemporaneous, or next-month, realized excess bond returns (e.g., Fama and French (1989), Ilmanen (1995), and Ludvigson and Ng (2009)).

To probe the importance of $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$ in explaining expected excess returns, we also conduct an exercise in the fashion of Lakonishok, Shleifer, and Vishny (1994). Specifically, we classify economic bad states by dividing the realizations of $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$ into four groups: (i) the 25 months with the highest positive returns (6.7% of the sample), (ii) the months with positive returns, excluding the 25 highest months (29.6%), (iii) the months with negative returns, excluding the 25 most negative months (57.1%), and (iv) the 25 months with the most negative returns (6.7%). We compute the average lower bound over the subsequent month within each of the four groups. The takeaway from Table 6 is that the lower bound averages 2.14%

during the highest positive 25 months, whereas it averages 1.29% during the most negative 25 months. Importantly, the hypothesis that the average $LB_{\{t \rightarrow t+1\}}$ during the highest positive 25 months equals the average during the lowest negative 25 months is rejected, with a p -value of 0.02. Overall, our evidence suggests that investors tend to reshape their expectation of excess returns of a long-term bond following negative shocks to their marginal utility of wealth.

5. Concluding remarks

Characterizing the expected excess return of a long-term bond is at the core of finance and economics. Variations in the expected excess return are thought to influence the functioning of financial markets and the valuation of risky cash flows. The expected excess return of a long-term bond can alter the strategic asset allocation decision between equities and long-term bonds. Moreover, it can also impact the corporate sector by changing the cost of corporate debt, incentives for corporate investment, and the holding of cash. Recognizing the possible pernicious effects of high bond risk premiums on the macroeconomy, the Federal Reserve has often sought to reduce the risk premium on the long-term bond through policy actions.

Our approach in this paper is to propose a theoretical lower bound on the expected excess return of a long-term Treasury bond. The economic interpretation of the lower bound is that it reflects the minimum expected excess return demanded by investors. We show that options on the 30-year Treasury bond futures – which is one of the most liquid markets – are informative about the expected excess return of the Treasury bond. We present an options positioning that extracts the theoretical lower bound on the expected excess return of the long-term bond from traded calls and puts on the 30-year Treasury bond futures.

The options data provide the insight that the annualized lower bound on the expected excess return of the long-term bond ranges between 0.22% and 6.07% with an unconditional average of 1.18%. In addition, our investigation shows that investors respond to increases in the slope of the Treasury yield curve by demanding a higher minimum compensation for holding the long-term bond. The empirical analysis also uncovers a positive relation between the expected excess return of a long-term bond and the return of a security that pays the stochastic discount factor. This finding conveys the insight that the lower bound increases during bad times, when the marginal utility of wealth is perceived to be high. Lastly, investors in the long-term bond market dislike increases in the return of gold futures over certain business conditions and price bonds with higher expected returns.

Finally, our Proposition 1 could be used to judge the reasonableness of conditional and unconditional estimates of expected excess returns from alternative models of the term structure of interest rates. Our work could also be connected to innovations in modeling risk premiums and expected returns, as in Aït-Sahalia, Cacho-Diaz, and Laeven (2015), Bollerslev and Todorov (2015), and Andersen, Fusari, and Todorov (2015). Overall, the pattern of options prices on the Treasury bond futures can help in understanding the market reaction to monetary policy, for example, forward guidance in a zero lower bound interest rate policy environment. We leave these ideas to follow-up research.

References

- Aït-Sahalia, Y., Cacho-Diaz, J., Laeven, R., 2015. Modeling financial contagion using mutually exciting jump processes. *Journal of Financial Economics* (forthcoming).
- Alvarez, F., Jermann, U., 2005. Using asset prices to measure the persistence of the marginal utility of wealth. *Econometrica* 73, 1977–2016.
- Andersen, T., Fusari, N., Todorov, V., 2015. Parametric inference and dynamic state recovery from option panels. *Econometrica* 83, 1081–1145.
- Bakshi, G., Chabi-Yo, F., 2012. Variance bounds on the permanent and transitory components of stochastic discount factors. *Journal of Financial Economics* 105, 191–208.
- Bakshi, G., Chabi-Yo, F., 2013. New entropy restrictions and the quest for better specified asset pricing models. Unpublished working paper. University of Maryland and Ohio State University.
- Bakshi, G., Kapadia, N., Madan, D., 2003. Stock return characteristics, skew laws, and the differential pricing of individual equity options. *Review of Financial Studies* 16, 101–143.
- Bakshi, G., Madan, D., 2000. Spanning and derivative-security valuation. *Journal of Financial Economics* 55, 205–238.
- Balakrishnan, N., Lai, C., 2009. *Continuous Bivariate Distributions*. Springer, New York, NY.
- Bansal, R., Lehmann, B., 1997. Growth-optimal portfolio restrictions on asset pricing models. *Macroeconomic Dynamics* 1, 333–354.
- Bansal, R., Yaron, A., 2004. Risks for the long run: A potential resolution of asset pricing puzzles. *Journal of Finance* 59, 1481–1509.
- Barro, R., Misra, S., 2013. Gold returns. Unpublished working paper. Harvard University.
- Black, F., 1993. Estimating expected returns. *Financial Analyst Journal* 49, 36–38.
- Bollerslev, T., Todorov, V., 2015. Tails, fears and risk premia. *Journal of Finance* 66, 2165–2211.
- Borovicka, J., Hansen, L., Scheinkman, J., 2014. Misspecified recovery. Unpublished working paper. New York University, University of Chicago, and Columbia University.
- Campbell, J., Cochrane, J., 1999. Force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy* 107, 205–251.

- Campbell, J., Sunderam, A., Viceira, L., 2013. Inflation bets or deflation hedges? The changing risks of nominal bonds. Unpublished working paper. Harvard University.
- Campbell, J., Viceira, L., 2001. Who should buy long-term bonds?. *The American Economic Review* 91, 99–127.
- Carr, P., Lee, R., 2008. Robust replication of volatility derivatives. Unpublished working paper. New York University and University of Chicago (<http://www.math.uchicago.edu/~rl/rrvd.pdf>).
- Carr, P., Madan, D., 2001. Optimal positioning in derivative securities. *Quantitative Finance* 1, 19–37.
- Carr, P., Wu, L., 2009. Variance risk premia. *Review of Financial Studies* 22, 1311–1341.
- Christensen, T., 2014. Nonparametric stochastic discount factor decomposition. Unpublished working paper. New York University.
- Cochrane, J., 2005. *Asset Pricing*. Princeton University Press, Princeton, NJ.
- Cochrane, J., Piazzesi, M., 2005. Bond risk premia. *American Economic Review* 95, 138–160.
- Constantinides, G. M., 1992. A theory of the nominal term structure of interest rates. *Review of Financial Studies* 5, 531–552.
- Cox, J., Ingersoll, J., Ross, S., 1981. The relation between forward prices and futures prices. *Journal of Financial Economics* 36, 321–346.
- Cox, J., Ingersoll, J., Ross, S., 1985. A theory of the term structure of interest rates. *Econometrica* 53, 385–407.
- Cuadras, C., 2002. On the covariance between functions. *Journal of Multivariate Analysis* 81, 19–27.
- Dybvig, P., Ingersoll, J., Ross, S., 1996. Long forward rates and zero-coupon rates can never fall. *Journal of Business* 69 (1), 1–25.
- Egozcue, M., Garcia, L., Wong, W., 2009. On some covariance inequalities for monotonic and non-monotonic functions. *Journal of Inequalities in Pure and Applied Mathematics* 10 (3), 1–7.
- Elton, E., 1999. Expected return, realized return and asset pricing tests. *Journal of Finance* 54, 1199–1220.
- Fama, E., 1990. Stock returns, expected returns, and real activity. *Journal of Finance* 45, 1089–1108.
- Fama, E., French, K., 1989. Business conditions and expected returns on stocks and bonds. *Journal of Financial Economics* 25, 23–89.

- Gabaix, X., 2012. Variable rare disasters: An exactly solved framework for ten puzzles in macro-finance. *Quarterly Journal of Economics* 127, 645–700.
- Gorton, G., Ordenez, G., 2013. The supply and demand for safe assets. Unpublished working paper. NBER # 18732.
- Gourinchas, P.-O., Jeanne, O., 2013. Global safe assets. Unpublished working paper. University of California-Berkeley and Johns Hopkins University.
- Greenwood, R., Vayanos, D., 2014. Bond supply and excess bond returns. *Review of Financial Studies* 27, 663–713.
- Gurkaynak, R., Wright, J., 2011. Macroeconomics and the term structure. *Journal of Economic Literature* 50, 331–367.
- Hansen, L., 2012. Dynamic value decomposition in stochastic economies. *Econometrica* 80, 911–967.
- Hansen, L., 2013. Risk pricing over alternative investment horizons. In: Engle, R. (Eds.), *Handbook of the Economics of Finance*. Elsevier, Amsterdam.
- Hansen, L., Jagannathan, R., 1991. Implications of security market data for dynamic economies. *Journal of Political Economy* 99, 225–261.
- Hansen, L., Scheinkman, J., 2009. Long-term risk: An operator approach. *Econometrica* 77, 177–234.
- Ilmanen, A., 1995. Time-varying expected returns in international bond markets. *Journal of Finance* 50, 481–506.
- Jarrow, R., 1978. The relationship between yield, risk, and return of corporate bonds. *Journal of Finance* 33, 1235–1240.
- Jiang, G., Tian, Y., 2005. The model-free implied volatility and its information content. *Review of Financial Studies* 18, 1305–1342.
- Kazemi, H., 1992. An intertemporal model of asset prices in a Markov economy with limiting stationary distribution. *Review of Financial Studies* 5, 85–104.
- Kimeldorf, G., Sampson, A., 1989. A framework for positive dependence. *Annals of the Institute of Statistical Mathematics* 41, 31–45.
- Lakonishok, J., Shleifer, A., Vishny, R., 1994. Contrarian investment, extrapolation, and risk. *Journal of Finance* 49, 1541–1578.

- Lehmann, E., 1966. Some concepts of dependence. *Annals of Mathematical Statistics* 37, 1137–1153.
- Lindfield, G., Penny, J., 1995. *Numerical Methods Using Matlab*. Ellis Horwood, New York, NY.
- Ludvigson, S., Ng, S., 2009. Macro factors in bond risk premia. *Review of Financial Studies* 22, 5027–5067.
- Mari, D., Kotz, S., 2004. *Correlation and Dependence*. Imperial College Press, London, UK.
- Martin, I., 2013. Simple variance swaps. Unpublished working paper. London School of Economics.
- Martin, I., 2015. What is the expected return of the market?. Unpublished working paper. London School of Economics.
- Martin, I., Ross, S., 2013. The long bond. Unpublished working paper. London School of Economics and MIT.
- Mele, A., Obayashi, Y., 2013. The price of government bond volatility. Unpublished working paper. University of Lugano.
- Merton, R., 1973. An intertemporal asset pricing model. *Econometrica* 41, 867–887.
- Merton, R., 1980. On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics* 8, 323–361.
- Mueller, P., Vedolin, A., Yen, Y.-M., 2013. Bond variance risk premiums. Unpublished working paper. London School of Economics.
- Newey, W., West, K., 1987. A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica* 55, 703–708.
- Newey, W., West, K., 1994. Automatic lag selection in covariance matrix estimation. *Review of Economic Studies* 61, 631–653.
- Pearson, N., Sun, T.-S., 1994. Exploiting the conditional density in estimating the term structure: An application to the cox, ingersoll, and ross model. *Journal of Finance* 49, 1279–1304.
- Piazzesi, M., 2010. Affine term structure models. In: Ait-Sahalia, Y., Hansen, L. (Eds.), *Handbook of Financial Econometrics*. Elsevier, Amsterdam.
- Politis, D., White, H., Patton, A., 2009. Correction: Automatic block-length selection for the dependent bootstrap. *Econometric Reviews* 28, 372–375.
- Ross, S., Westerfield, R., Jaffe, J., 2010. *Corporate Finance*. Princeton University Press, Princeton, NJ.

SEC, 2014. Demand and supply of safe assets in the economy. Unpublished working paper. Security Exchange Commission (<http://www.sec.gov/comments/s7-03-13/s70313-324.pdf>).

Singleton, K., 2006. Empirical Dynamic Asset Pricing. Princeton University Press, Princeton, NJ.

Wachter, J., 2013. Can time-varying risk of rare disasters explain aggregate stock market volatility?. *Journal of Finance* 68, 987–1035.

Yellen, J., 2012. Confirmation hearing. Transcripts. Federal Reserve Board.

Appendix A: Proof of Equation (14) of Lemma 1

Both m_{t+1}^P and m_{t+1}^T are positive random variables, hence, $\frac{1}{m_{t+1}^T}$ exists, and is well-defined. We suppose that the support of m_{t+1}^P and m_{t+1}^T are the intervals $[\underline{a}, \bar{a}]$ and $[\underline{b}, \bar{b}]$, respectively, where $\underline{a} > 0$, $\underline{b} > 0$ and $\bar{a} < +\infty$, $\bar{b} < +\infty$.

We know from Lehmann (1966, Lemma 2 and Lemma 3) that

$$\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) = \int \int \{\mathbb{H}[a, b] - \mathbb{F}[a]\mathbb{G}[b]\} da db \geq 0, \quad \text{when } (m_{t+1}^P, m_{t+1}^T) \in \mathcal{P}, \quad (30)$$

where \mathcal{P} denotes the class of bivariate distributions with positive quadrant dependency, $\mathbb{H}[a, b]$ is the cumulative distribution function of (m_{t+1}^P, m_{t+1}^T) , and $\mathbb{F}[a]$ and $\mathbb{G}[b]$ are the marginal distributions.

Then the proof of the lemma entails showing $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T}) \leq 0$ when $(m_{t+1}^P, m_{t+1}^T) \in \mathcal{P}$.

Observe that

$$\alpha[m_{t+1}^P] \equiv m_{t+1}^P \text{ is increasing, whereas } \beta[m_{t+1}^T] \equiv \frac{1}{m_{t+1}^T} \text{ is decreasing.} \quad (31)$$

When $\alpha[m_{t+1}^P]$ and $\beta[m_{t+1}^T]$ are, respectively, an increasing and decreasing function of their argument, and are of bounded variation, with $(m_{t+1}^P, m_{t+1}^T) \in \mathcal{P}$, then using the Cuadras (2002, equation (2)) identity $\text{Cov}_t(\alpha[m_{t+1}^P], \beta[m_{t+1}^T]) = \int \int \{\mathbb{H}[a, b] - \mathbb{F}[a]\mathbb{G}[b]\} d\alpha[a] d\beta[b]$, Egozcue, Garcia, and Wong (2009, Theorem 2.4, (2), page 5) establish that

$$\text{Cov}_t(\alpha[m_{t+1}^P], \beta[m_{t+1}^T]) \leq 0, \quad (32)$$

provided $E_t(\alpha[m_{t+1}^P])$ and $E_t(\beta[m_{t+1}^T])$ are finite (which is true under our assumptions in Section 2.2). Therefore, the conclusion $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T}) \leq 0$ follows. In our economic setting, $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) \geq 0$ is linked to $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T}) \leq 0$, under a weak notion of dependence between m_{t+1}^P and m_{t+1}^T (Mari and Kotz (2004, page 34)). ■

Appendix B: Proof of Equation (15) of Proposition 1

The degenerate case of $m_{t+1}^P = 1$, which furnishes the implication, from equation (3), that $m_{t+1} = m_{t+1}^T =$

$e^{\rho \frac{\phi[z_t]}{\phi[z_{t+1}]}}$, does not interest us. For conciseness, we set

$$R_{t+1,\infty} - R_{t+1,f} \equiv \mathbf{er}_{t+1,\infty}, \text{ and observe that } E_t(m_{t+1} \mathbf{er}_{t+1,\infty}) = 0. \quad (33)$$

Next, consider the expectation of the product of two mean-centered random variables:

$$E_t((m_{t+1} \mathbf{er}_{t+1,\infty} - 0) \times (\mathbf{er}_{t+1,\infty} - E_t(\mathbf{er}_{t+1,\infty}))) = \text{Cov}_t(m_{t+1} \mathbf{er}_{t+1,\infty}, \mathbf{er}_{t+1,\infty}). \quad (34)$$

The left-hand side of equation (34) has the representation:

$$E_t(m_{t+1} \mathbf{er}_{t+1,\infty} \times (\mathbf{er}_{t+1,\infty} - E_t(\mathbf{er}_{t+1,\infty}))) = E_t(m_{t+1} (R_{t+1,\infty} - R_{t+1,f})^2), \quad (35)$$

$$= \frac{1}{R_{t+1,f}} E_t^* \left((R_{t+1,\infty} - R_{t+1,f})^2 \right), \quad (36)$$

$$= R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right). \quad (37)$$

Equation (37) follows from equation (36), since $R_{t+1,\infty} - R_{t+1,f} = R_{t+1,f} \left(\frac{f_{t+1}}{f_t} - 1 \right)$ from equation (6).

Additionally, the right-hand side of equation (34) yields the representation:

$$\text{Cov}_t(m_{t+1} \mathbf{er}_{t+1,\infty}, \mathbf{er}_{t+1,\infty}) = \text{Cov}_t(m_{t+1} R_{t+1,\infty}, R_{t+1,\infty}) - \underbrace{R_{t+1,f} \text{Cov}_t(m_{t+1}, R_{t+1,\infty})}_{= -(E_t(R_{t+1,\infty}) - R_{t+1,f})}. \quad (38)$$

Combining equations (37) and (38) and using equation (34), we arrive at the identity:

$$E_t(R_{t+1,\infty}) - R_{t+1,f} = R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right) - \text{Cov}_t(m_{t+1} R_{t+1,\infty}, R_{t+1,\infty}), \quad (39)$$

$$= R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right) - \underbrace{\text{Cov}_t \left(m_{t+1}^P, \frac{1}{m_{t+1}^T} \right)}_{\leq 0 \text{ from equation (14)}}. \quad (40)$$

The negative sign assigned to $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$ in equation (40) follows by invoking the result from equation (14) of Lemma 1.

Hence, using $C \Leftrightarrow D$ to denote that C is mathematically equivalent to D , we have

$$\text{Cov}_t \left(m_{t+1}^P, \frac{1}{m_{t+1}^T} \right) \leq 0 \Leftrightarrow R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right) - (E_t(R_{t+1,\infty}) - R_{t+1,f}) \leq 0. \quad (41)$$

Equation (41) is also a statement that

$$\text{Cov}_t \left(m_{t+1}^P, \frac{1}{m_{t+1}^T} \right) \leq 0 \Leftrightarrow E_t(R_{t+1,\infty}) - R_{t+1,f} \geq R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right). \quad (42)$$

With this step, we have equation (15) of Proposition 1. ■

Appendix C: Proof of Equation (16) of Proposition 2

All contingent claims to be priced reflect uncertainty about interest rates beyond date time $t + 1$.

Any twice-continuously differentiable payoff function $H[f_{t+1}]$ with bounded expectation can be synthesized as per Bakshi and Madan (2000, Appendix A.3) and Carr and Madan (2001, equation (1)):

$$\begin{aligned} H[f_{t+1}] &= H[f_t] - f_t H_f[f_t] + f_{t+1} H_f[f_t] \\ &\quad + \int_{\{K > f_t\}} H_{ff}[K] (f_{t+1} - K)^+ dK + \int_{\{K < f_t\}} H_{ff}[K] (K - f_{t+1})^+ dK, \end{aligned} \quad (43)$$

where $a^+ \equiv \max(a, 0)$, $H_f[f_t]$ is the first-order derivative of the payoff $H[f_{t+1}]$ with respect to f_{t+1} evaluated at f_t , and $H_{ff}[K]$ is the second-order derivative, with respect to f_{t+1} evaluated at the strike price K .

To obtain general expressions, we consider the power contract and set $H[f_{t+1}] = \left(\frac{f_{t+1}}{f_t} - 1 \right)^n$. Hence, we obtain:

$$H[f_t] = \left(\frac{f_{t+1}}{f_t} - 1 \right)^n \Big|_{f_{t+1}=f_t} = 0, \quad (44)$$

$$H_f[f_t] = \frac{dH[f_{t+1}]}{df_{t+1}} \Big|_{f_{t+1}=f_t} = \frac{n}{f_t} \left(\frac{f_{t+1}}{f_t} - 1 \right)^{n-1} \Big|_{f_{t+1}=f_t} = 0, \quad (45)$$

$$H_{ff}[K] = \frac{d^2 H[f_{t+1}]}{df_{t+1}^2} \Big|_{f_{t+1}=K} = \frac{n(n-1)}{f_t^2} \left(\frac{f_{t+1}}{f_t} - 1 \right)^{n-2} \Big|_{f_{t+1}=K} = \frac{n(n-1)}{f_t^2} \left(\frac{K}{f_t} - 1 \right)^{n-2}. \quad (46)$$

Therefore, we can write the arbitrage-free value of $H[f_{t+1}]$ at date t as:

$$E_t(m_{t+1} H[f_{t+1}]) = \int_{\{K > f_t\}} H_{ff}[K] C_t[K] dK + \int_{\{K < f_t\}} H_{ff}[K] P_t[K] dK, \quad (47)$$

where the date- t value of the European call option, denoted by $C_t[K]$, and the European put option, denoted

by $P_t[K]$, with strike price K , satisfies:

$$C_t[K] = E_t(m_{t+1}(f_{t+1} - K)^+) \quad \text{and} \quad P_t[K] = E_t(m_{t+1}(K - f_{t+1})^+). \quad (48)$$

With our object of interest being the power contract with payoff $H[f_{t+1}] = (\frac{f_{t+1}}{f_t} - 1)^n$, we express:

$$\begin{aligned} E_t^*(H[f_{t+1}]) &= \frac{1}{E_t(m_{t+1})} E_t(m_{t+1} H[f_{t+1}]), \\ &= \frac{n(n-1)R_{t+1,f}}{f_t^2} \left(\int_{\{K > f_t\}} \left(\frac{K}{f_t} - 1\right)^{n-2} C_t[K] dK + \int_{\{K < f_t\}} \left(\frac{K}{f_t} - 1\right)^{n-2} P_t[K] dK \right). \end{aligned} \quad (49)$$

The right-hand side of equation (50) is computable at date t , given the observed futures price and the prices of out-of-the-money calls and puts written on the bond futures.

Going from the theoretical bound in equation (15) to the options-based model-free computation in equation (16) involves setting $n = 2$ in the pricing equation (50) of the power contract. This result holds under all martingale pricing measures (given that no assumptions were made about m_{t+1}).

The desired expression for $R_{t+1,f} E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)$ is as displayed in equation (16) of Proposition 2. ■

Table 1

Properties of the model in Case 3 and the consequences of the dependence between m_{t+1}^P and m_{t+1}^T

Featured are the properties of the model under two different parameterizations (Figure Internet-I provides additional corroborative evidence under alternative parameterizations):

$$\text{SET A: } (\varphi, \eta, \sigma_p, \sigma_\infty, \psi, \theta_z, \kappa_z, \sigma_z) = (-0.02, 0.80, 1.50, 0.08, -0.92, 0.03, 0.05, 0.02),$$

$$\text{SET B: } (\varphi, \eta, \sigma_p, \sigma_\infty, \psi, \theta_z, \kappa_z, \sigma_z) = (-0.02, 0.45, 2.05, 0.08, -0.90, 0.03, 0.05, 0.02).$$

Both of these parameterizations generate positive quadrant dependence between $(\log(m_{t+1}^P), \log(m_{t+1}^T))$, that is, a positive value of $\text{Cov}_t(\log(m_{t+1}^P), \log(m_{t+1}^T)) = \{c\sigma_z(\sigma_p\psi + \sigma_\infty) + c^2\sigma_z^2\}z_t$. As described in Section III of the Internet Appendix, we simulate the state variable z_t for $t = 1, \dots, 10,375$ months using a draw of the standard normal variate $\tilde{\omega}_{t+1}$ in equation (19). We discard the first 10,000 observations to match the sample length of the futures on the 30-year Treasury bond and options on the 30-year Treasury bond futures. Each month, we compute $R_{t+1,f}$, $E_t(R_{t+1,\infty})$, and $E_t(m_{t+1}^P/m_{t+1}^T)$ based on equations (20)–(22). We also compute $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$ and $R_{t+1,f}E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)$ in accordance with equations (10) and (11), respectively. For each of the *conditional variables*, we report the mean, the standard deviation (denoted by Std.), the 5th, 50th, 95% percentile values across the 10,000 simulations. The values under the column “Data” correspond to the *unconditional* values over the sample period 1982:10 to 2013:12, where the return of the long-term bond is inferred as $R_{t+1,\infty} = R_{t+1,f} \frac{f_{t+1}}{f_t}$. The source of the risk-free return is the data library of Ken French and the source of the futures and options data on the 30-year Treasury bond is the Chicago Mercantile Exchange (as discussed in Section 3).

	SET A					SET B					Data
	Mean	Std.	5th	50th	95th	Mean	Std.	5th	50th	95th	
$E_t(m_{t+1})$	0.9772	0.0031	0.9717	0.9731	0.9819	0.9759	0.0021	0.9723	0.9732	0.9791	
$E_t(m_{t+1}^T)$	0.9650	0.0045	0.9572	0.9591	0.9719	0.9560	0.0043	0.9484	0.9502	0.9626	
$\sqrt{\text{Var}_t(m_{t+1})}$	0.2407	0.0140	0.2185	0.2229	0.2644	0.3397	0.0206	0.3072	0.3134	0.3746	
$\sqrt{\text{Var}_t(m_{t+1}^T)}$	0.0992	0.0055	0.0905	0.0923	0.1084	0.0946	0.0052	0.0864	0.0879	0.1034	
$R_{t+1,f} - 1$	0.0235	0.0033	0.0185	0.0194	0.0292	0.0247	0.0022	0.0214	0.0220	0.0285	0.04
$E_t(R_{t+1,\infty}) - 1$	0.0480	0.0064	0.0384	0.0402	0.0589	0.0569	0.0061	0.0477	0.0493	0.0676	0.07
$E_t(R_{t+1,\infty}) - R_{t+1,f}$	0.0245	0.0030	0.0199	0.0208	0.0297	0.0322	0.0039	0.0262	0.0273	0.0391	0.03
$\sqrt{E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)}$	0.1031	0.0062	0.0933	0.0953	0.1136	0.0992	0.0059	0.0899	0.0917	0.1092	0.102
$R_{t+1,f}E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)$	0.0113	0.0014	0.0091	0.0096	0.0138	0.0104	0.0013	0.0085	0.0089	0.0127	0.011
$\text{Cov}_t(m_{t+1}^P, m_{t+1}^T)$	0.0122	0.0014	0.0100	0.0105	0.0146	0.0199	0.0023	0.0164	0.0171	0.0239	
$\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$	-0.0132	0.0016	-0.016	-0.0153	-0.0107	-0.0218	0.0027	-0.0264	-0.0253	-0.0177	

Table 2

Lower bound on the expected excess returns of a long-term bond (annualized, in percentage)

Reported are the lower bounds on the expected excess returns of a long-term bond, computed as:

$$\text{LB}_{\{t \rightarrow t+1\}} = \underbrace{R_{t+1,f} E_t^* \left(\left(\frac{f_{t+1}}{f_t} - 1 \right)^2 \right)}_{\text{Lower bound}} = \frac{2R_{t+1,f}^2}{f_t^2} \left(\int_{\{K > f_t\}} C_t[K] dK + \int_{\{K < f_t\}} P_t[K] dK \right),$$

where $C_t[K]$ ($P_t[K]$) are the prices of the out-of-the-money calls (puts) written on the 30-year Treasury bond futures, reported at the end of month t . The futures price is f_t and $R_{t+1,f}$ is the gross return of the one-month Treasury bill that is known at the end of month t . The average maturity of the options in our sample is 29.16 days, and the sample period is 1982:10 to 2013:12, with a total of 375 monthly observations. The lower and upper 95% confidence intervals are constructed following the stationary bootstrap procedure of Politis, White, and Patton (2009), with optimal block size, and are based on 10,000 bootstrap draws. All the statistics are annualized and expressed in percentages. For example, over the full sample, the mean lower bound on the expected excess return of the long-term bond is 1.18% per year. ACF_1 is the first-order autocorrelation.

Sample	Mean	Std.	Bootstrap 95% CI		Min.	Max.	ACF ₁	Nobs.
			Lower	Upper				
<i>Panel A: Full sample</i>								
1982:10- 2013:12	1.18	0.71	1.00	1.38	0.22	6.07	0.74	375
<i>Panel B: Subsamples</i>								
1982:10-1997:12	1.20	0.62	0.99	1.45	0.22	4.12	0.66	183
1998:01-2013:12	1.15	0.79	0.91	1.49	0.33	6.07	0.79	192
1990:01-2013:12	1.10	0.69	0.92	1.34	0.22	6.07	0.74	288
1982:10-2007:12	1.09	0.57	0.93	1.29	0.22	4.12	0.68	303

Table 3

Univariate regression analysis of the lower bound on the expected excess return

We perform univariate regressions of the type: $LB_{\{t \rightarrow t+1\}} = \alpha + \beta X_{\{t-1 \rightarrow t\}} + e_{\{t \rightarrow t+1\}}$, where $LB_{\{t \rightarrow t+1\}}$ is the estimated lower bound on the expected excess return of the long-term bond (computed at the end of month t) and $X_{\{t-1 \rightarrow t\}}$ is an economic variable. Each variable is annualized and in percentage terms. Reported are the coefficient estimates, as well as the two-sided p -values, denoted by $NW[p]$, based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994), reported in the “optimal lag” column. The adjusted R^2 (in %) is denoted by \bar{R}^2 , DW is the Durbin-Watson statistic, and CORR is the correlation coefficient between $LB_{\{t \rightarrow t+1\}}$ and $X_{\{t-1 \rightarrow t\}}$.

	Constant		Slope		\bar{R}^2 (%)	DW	CORR.	Optimal lag
	α	NW[p]	β	NW[p]				
<i>Panel A: Full sample, 1982:10–2013:12</i>								
$\Delta Spread_{\{t-1 \rightarrow t\}}$	1.17	0.00	0.12	0.02	0.8	0.49	0.10	13
$r_{\{t-1 \rightarrow t\}}^{SDF} * 100$	1.20	0.00	0.35	0.02	6.1	0.51	0.25	13
$er_{\{t-1 \rightarrow t\}}^{gold} * 100$	1.17	0.00	0.17	0.08	1.5	0.51	0.13	13
<i>Panel B: Subsample, 1982:10–1997:12</i>								
$\Delta Spread_{\{t-1 \rightarrow t\}}$	1.19	0.00	0.12	0.06	1.6	0.62	0.14	9
$r_{\{t-1 \rightarrow t\}}^{SDF} * 100$	1.24	0.00	0.41	0.01	8.8	0.65	0.31	9
$er_{\{t-1 \rightarrow t\}}^{gold} * 100$	1.20	0.00	0.14	0.23	0.6	0.67	0.11	9
<i>Panel C: Subsample, 1998:01–2013:12</i>								
$\Delta Spread_{\{t-1 \rightarrow t\}}$	1.15	0.00	0.12	0.20	-0.1	0.42	0.07	10
$r_{\{t-1 \rightarrow t\}}^{SDF} * 100$	1.17	0.00	0.33	0.13	4.5	0.44	0.22	10
$er_{\{t-1 \rightarrow t\}}^{gold} * 100$	1.14	0.00	0.20	0.09	2.0	0.42	0.16	10
<i>Panel D: Subsample, 1990:01–2013:12</i>								
$\Delta Spread_{\{t-1 \rightarrow t\}}$	1.10	0.00	0.14	0.01	0.6	0.51	0.10	12
$r_{\{t-1 \rightarrow t\}}^{SDF} * 100$	1.12	0.00	0.31	0.09	4.4	0.54	0.22	12
$er_{\{t-1 \rightarrow t\}}^{gold} * 100$	1.09	0.00	0.17	0.15	1.4	0.52	0.13	12
<i>Panel E: Subsample, 1982:10–2007:12</i>								
$\Delta Spread_{\{t-1 \rightarrow t\}}$	1.08	0.00	0.13	0.02	1.8	0.60	0.15	12
$r_{\{t-1 \rightarrow t\}}^{SDF} * 100$	1.10	0.00	0.21	0.11	2.8	0.60	0.18	12
$er_{\{t-1 \rightarrow t\}}^{gold} * 100$	1.08	0.00	0.04	0.60	-0.2	0.62	0.04	12

Table 4

Multivariate regression analysis of the lower bound on the expected excess return

We perform multivariate regressions of the type: $LB_{\{t \rightarrow t+1\}} = \alpha + \mathbf{X}'_{\{t-1 \rightarrow t\}} \boldsymbol{\beta} + e_{\{t \rightarrow t+1\}}$, where $\mathbf{X}_{\{t-1 \rightarrow t\}} = [\Delta \text{Spread}_{\{t-1 \rightarrow t\}} \ r_{\{t-1 \rightarrow t\}}^{\text{SDF}} \ \text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}}]'$. Each variable is annualized and in percentage terms. Reported are the coefficient estimates, as well as the two-sided p -values, denoted by $NW[p]$, based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994), reported in the “optimal lag” column. The adjusted R^2 (in %) is denoted by \bar{R}^2 . Reported are the results from the three bivariate regressions, and all three variables are included.

α	NW[p]	$\Delta \text{Spread}_{\{t-1 \rightarrow t\}}$		$r_{\{t-1 \rightarrow t\}}^{\text{SDF}} * 100$		$\text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}} * 100$		\bar{R}^2 (%)	Optimal lag
		Coef.	NW[p]	Coef.	NW[p]	Coef.	NW[p]		
<i>Panel A: Full sample, 1982 to 2013</i>									
1.20	0.00	0.10	0.04	0.35	0.02			6.5	11
1.17	0.00	0.11	0.02			0.16	0.09	2.2	12
1.20	0.00			0.35	0.02	0.16	0.09	7.5	11
1.20	0.00	0.09	0.05	0.34	0.02	0.16	0.11	7.8	11
<i>Panel B: Subsample, 1982 to 1997</i>									
1.24	0.00	0.06	0.14	0.39	0.01			8.8	5
1.20	0.00	0.11	0.06			0.11	0.23	1.8	6
1.25	0.00			0.40	0.01	0.06	0.50	8.6	5
1.24	0.00	0.06	0.18	0.38	0.02	0.05	0.53	8.5	3
<i>Panel C: Subsample, 1998 to 2013</i>									
1.17	0.00	0.19	0.05	0.35	0.09			5.0	9
1.13	0.00	0.14	0.17			0.21	0.09	2.0	10
1.15	0.00			0.36	0.13	0.24	0.10	7.3	8
1.15	0.00	0.21	0.04	0.38	0.09	0.25	0.10	8.1	9
<i>Panel D: Subsample, 1990 to 2013</i>									
1.12	0.00	0.15	0.00	0.31	0.07			5.2	12
1.09	0.00	0.14	0.01			0.17	0.16	2.1	12
1.12	0.00			0.32	0.10	0.18	0.16	6.1	10
1.12	0.00	0.16	0.01	0.32	0.08	0.18	0.17	6.9	11
<i>Panel E: Subsample, 1982 to 2007</i>									
1.10	0.00	0.11	0.01	0.19	0.11			4.1	8
1.08	0.00	0.12	0.01			0.03	0.71	1.5	10
1.10	0.00			0.21	0.12	0.02	0.75	2.5	10
1.10	0.00	0.11	0.01	0.19	0.12	0.01	0.86	3.8	8

Table 5

Multivariate regression analysis of the lower bound, with controls

This table shows multivariate regressions of the form: $LB_{\{t \rightarrow t+1\}} = \alpha + \mathbf{X}'_{\{t-1 \rightarrow t\}} \boldsymbol{\beta} + e_{\{t \rightarrow t+1\}}$, where $\mathbf{X}_{\{t-1 \rightarrow t\}} = [\Delta \text{Spread}_{\{t-1 \rightarrow t\}} \ r_{\{t-1 \rightarrow t\}}^{\text{SDF}} \ \text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}} \ \text{Control}_{\{t-1 \rightarrow t\}}]'$. Reported are the coefficient estimates, as well as the two-sided p -values for the null hypothesis $\boldsymbol{\beta} = 0$, denoted by $\text{NW}[p]$, based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994), reported in the “optimal lag” column. The adjusted R^2 (in %) is denoted by \bar{R}^2 . We consider three control variables, namely, the industrial production growth, the inflation, and the log change in bond supply. The monthly time series of industrial production growth and inflation are taken from Datastream. Following Greenwood and Vayanos (2014), the bond supply is the long-term debt divided by the nominal GDP. The long-term debt is the entire face value of Treasury bonds with maturity between 10 and 30 years, plus the sum of remaining coupon payments (source: CRSP Treasury data). We do not report the intercept in the regressions to save space.

	$\Delta \text{Spread}_{\{t-1 \rightarrow t\}}$		$r_{\{t-1 \rightarrow t\}}^{\text{SDF}} * 100$		$\text{er}_{\{t-1 \rightarrow t\}}^{\text{gold}} * 100$		$\text{Control}_{\{t-1 \rightarrow t\}}$		\bar{R}^2 (%)	Optimal lag
	Coef.	NW[p]	Coef.	NW[p]	Coef.	NW[p]	Coef.	NW[p]		
<i>Panel A: Control is the industrial production growth</i>										
1982:10-2013:12	0.09	0.03	0.32	0.01	0.16	0.11	-0.18	0.28	10.1	11
1982:10-1997:12	0.06	0.17	0.37	0.02	0.06	0.50	0.05	0.46	8.2	6
1998:01-2013:12	0.23	0.09	0.31	0.08	0.28	0.08	-0.36	0.08	17.7	9
1990:01-2013:12	0.17	0.01	0.28	0.05	0.19	0.15	-0.26	0.17	12.4	12
1982:10-2008:12	0.11	0.01	0.20	0.10	0.01	0.87	0.08	0.19	4.1	9
<i>Panel B: Control is the inflation</i>										
1982:10-2013:12	0.09	0.04	0.31	0.01	0.18	0.13	-0.03	0.13	9.5	11
1982:10-1997:12	0.06	0.15	0.39	0.01	0.05	0.55	-0.03	0.14	8.9	7
1998:01-2013:12	0.21	0.05	0.30	0.05	0.30	0.10	-0.04	0.08	11.5	9
1990:01-2013:12	0.16	0.00	0.26	0.03	0.22	0.17	-0.04	0.11	9.8	11
1982:10-2008:12	0.11	0.01	0.19	0.12	0.01	0.85	0.00	0.65	3.5	10
<i>Panel C: Control is the log change in bond supply</i>										
1982:10-2013:12	0.10	0.02	0.35	0.02	0.16	0.11	2.42	0.23	8.0	11
1982:10-1997:12	0.05	0.24	0.38	0.02	0.05	0.56	-1.21	0.47	8.2	4
1998:01-2013:12	0.18	0.07	0.39	0.07	0.26	0.09	7.82	0.10	10.2	9
1990:01-2013:12	0.16	0.00	0.33	0.08	0.19	0.17	2.81	0.48	7.0	11
1982:10-2008:12	0.11	0.01	0.19	0.12	0.01	0.86	0.33	0.86	3.4	8

Table 6

Lower bound on the expected excess returns (annualized, in percentages) during periods of the highest and lowest returns of the SDF security

We first divide the returns of the SDF security, $r_{\{t-1 \rightarrow t\}}^{\text{SDF}}$, into two groups containing positive and negative returns. Then we divide the sample of positive (negative) returns into two parts: (i) the months with the highest (lowest) 25 monthly returns, and (ii) the remainder of positive (negative) returns. Reported are the properties of the lower bound $\text{LB}_{\{t \rightarrow t+1\}}$. Also shown is the difference between the lower bounds during months with the 25 most positive returns and the 25 most negative returns, with the p -value in parentheses. The highest 25 positive returns to the SDF security correspond to bad times with highest marginal utility of wealth.

	Positive returns		Negative returns		25 highest minus 25 lowest (p -value)
	Highest 25 positive	Remaining positive	Lowest 25 negative	Remaining negative	
Lower bound on the expected excess return					
Mean	2.14	1.12	1.29	1.08	0.85
Std.	1.52	0.52	0.66	0.56	(0.02)
Minimum	0.61	0.22	0.51	0.33	
Maximum	6.07	2.85	2.86	4.21	

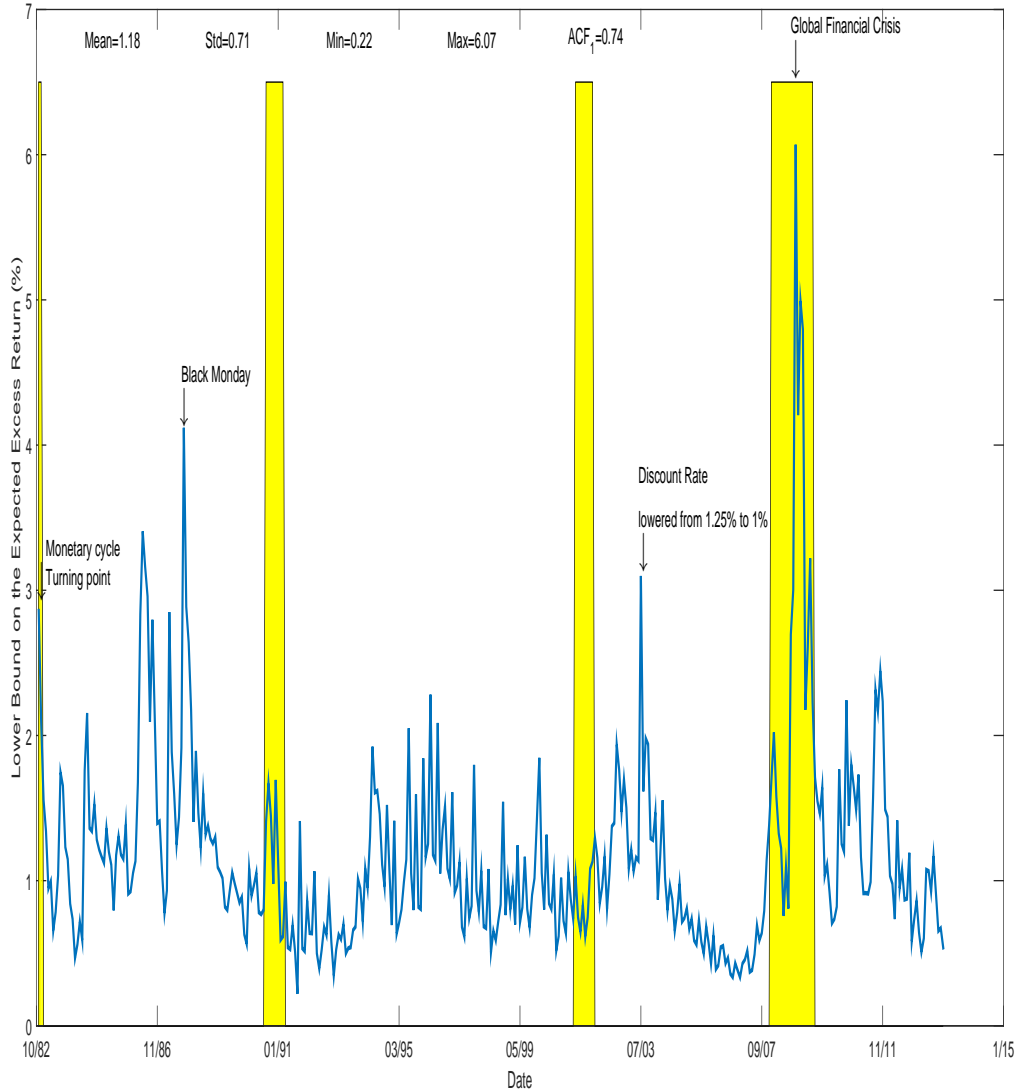


Figure 1. **Variation in the lower bound on the expected excess return of a long-term bond**

Plotted is the time series of the lower bound on the expected excess return of a long-term bond. The lower bound is expressed in *annualized percentage terms* and is computed following equation (16) as:

$$LB_{\{t \rightarrow t+1\}} = \frac{2R_{t+1,f}^2}{f_t^2} \left(\int_{\{K > f_t\}} C_t[K] dK + \int_{\{K < f_t\}} P_t[K] dK \right),$$

where $C_t[K]$ ($P_t[K]$) are the prices of the out-of-the-money calls (puts) written on the 30-year Treasury bond futures at the end of month t . The futures price is f_t , and $R_{t+1,f}$ is the gross return of the one-month Treasury bill that is known at the end of month t . The yellow-shaded regions are the recession months as classified by the NBER. The sample period is October 1982 to December 2013.

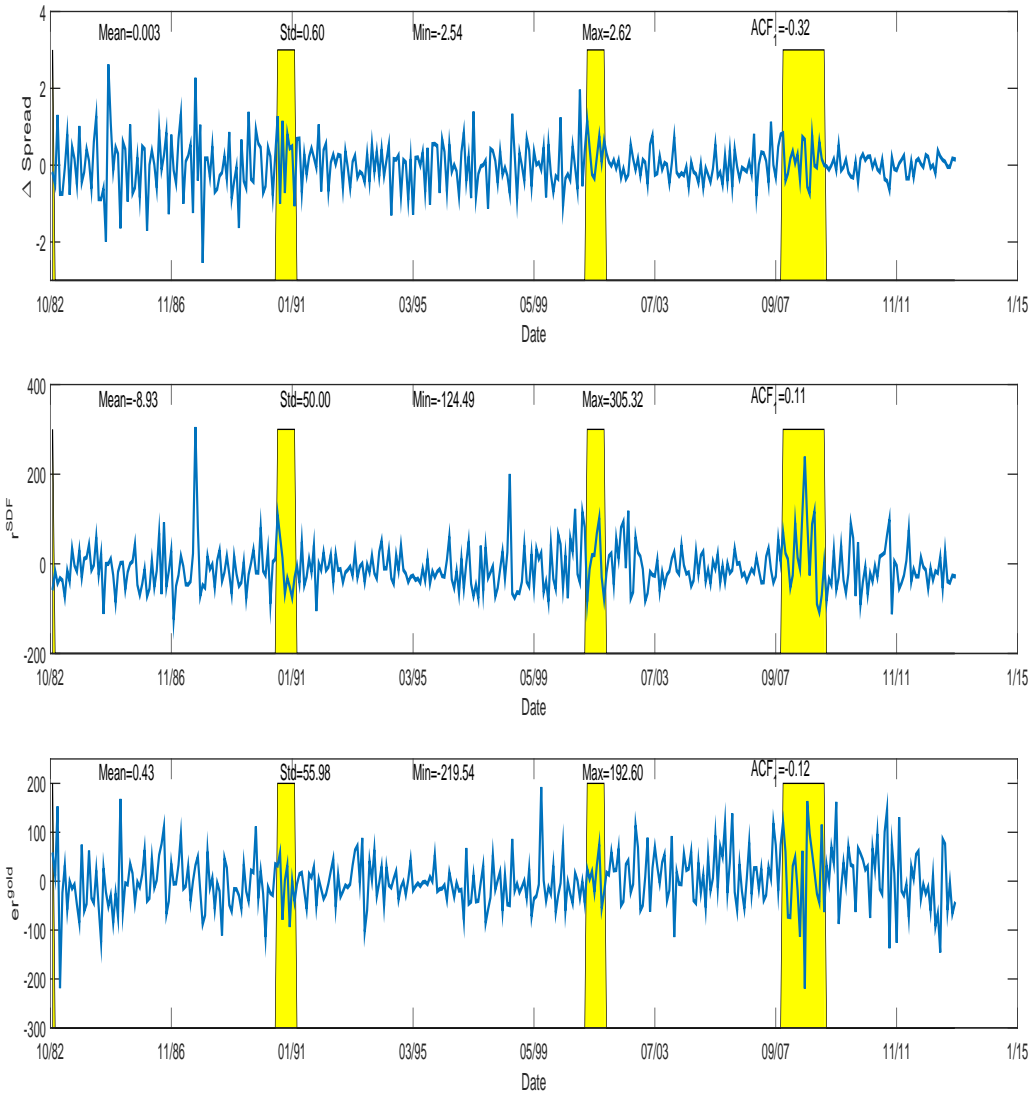


Figure 2. **Variation in the explanatory variables**

We plot the variation in the three featured explanatory variables: (i) changes in yield spread (ΔSpread), (ii) return of the SDF security (r^{SDF}), and (iii) excess returns of gold futures (er^{gold}). The yield spread is the difference between the yield of the 30-year Treasury bond and the yield of the one-month Treasury bill. The return of the SDF security is as described in equation (28), while the excess return of gold futures is the return of a fully collateralized long futures position. The yellow shaded regions are the recession months as classified by the NBER. The three monthly series are all expressed in annualized percentage terms.

An Inquiry into the Nature and Sources of Variation in the Expected Excess Return of a Long-Term Bond

Gurdip Bakshi^{a*} Fousseni Chabi-Yo^{b†} Xiaohui Gao^{c‡}

^a*Smith School of Business, University of Maryland, College Park, MD 20742, USA*

^b*Fisher College of Business, Ohio State University, Columbus, OH 43210, USA*

^c*Smith School of Business, University of Maryland, College Park, MD 20742, USA*

Internet Appendix: Not for Publication

Abstract

This internet appendix provides additional empirical and theoretical results. Section 1 describes the quoting convention in the 30-year Treasury bond futures and options markets, and how we convert quotes to dollar prices. Section 2 provides the estimates of the theoretical lower bound based on the second-nearest maturity options on the 30-year Treasury bond futures. Section 3 describes a model that parameterizes the evolution of the stochastic discount factor and solves the eigenfunction problem to derive the transitory and the permanent components. Our illustrative calculations and simulation analysis indicate that when the conditional covariance between the permanent component and the transitory component is negative, the model fails to reproduce many of the salient features of the economy.

*Tel.: +1-240-464-8137. E-mail address: gbakshi@rhsmith.umd.edu

†Tel.: +1-614-292-8477. E-mail address: chabi-yo_1@fisher.osu.edu

‡Tel.: +1-240-507-9877. E-mail address: xiaohui@rhsmith.umd.edu

I. Conversion of quotes to dollar prices on options and futures on the 30-year Treasury bond

In what follows, we provide an example of how we convert quotes to dollar prices in the options and futures markets. On April 30, 2003, we first retain the nearest maturity options and then the five nearest out-of-the-money calls and puts, which all expire on May 23, 2003. The underlying futures is the June 2003 contract.

Date	Put/Call	Strike		Options contract		Futures quote	Futures price	Futures contract		Option expiration date
		quote	price	quote	price			year	month	
4/30/2003	P	110-00	110000	9	140.625	114-01	114031.3	2003	6	5/23/2003
4/30/2003	P	111-00	111000	16	250.000	114-01	114031.3	2003	6	5/23/2003
4/30/2003	P	112-00	112000	26	406.250	114-01	114031.3	2003	6	5/23/2003
4/30/2003	P	113-00	113000	44	687.500	114-01	114031.3	2003	6	5/23/2003
4/30/2003	P	114-00	114000	105	1078.125	114-01	114031.3	2003	6	5/23/2003
4/30/2003	C	115-00	115000	42	656.250	114-01	114031.3	2003	6	5/23/2003
4/30/2003	C	116-00	116000	22	343.750	114-01	114031.3	2003	6	5/23/2003
4/30/2003	C	117-00	117000	11	171.875	114-01	114031.3	2003	6	5/23/2003
4/30/2003	C	118-00	118000	5	78.125	114-01	114031.3	2003	6	5/23/2003
4/30/2003	C	119-00	119000	2	31.250	114-01	114031.3	2003	6	5/23/2003

The futures prices are quoted in points and 1/32 of a point. Each point is \$1,000. Therefore, the June 2003 futures closing quote of 114-01 translates to $(114 + 01/32) * 1000 = 11,4031.3$.

Next, we consider options on the futures, where the strikes are quoted in points with increment of one point. So a strike price of 110-00 maps to a strike price of $110 * 1000 = 110,000$.

The minimum tick size of the settlement price of options is 1/64 of a point (\$15.625). If the settlement price quote is lower than 100, then the option price is the quote multiplied by \$15.625. For example, if the option settlement quote is 9, then the price is $9 * 15.625 = 140.625$. If the settlement price quote is higher than 100, then the option price is one full point plus the remainder multiplied by the minimum tick size. Specifically, for the option settlement quote of 105, the price is $(1 * 64 + 5) * 15.625 = \$1,078.125$. ■

II. Robustness of the results using second-nearest maturity options

The paper features the results on the lower bound of the expected excess return of the long-term bond based on the nearest-maturity options with an average maturity of 29.16 days. To assess robustness, Table Internet-II presents the properties of the lower bound using the second-nearest maturity options. These options have an average maturity of 84.31 days, and there are on average, 11.37 calls and 13.42 puts at the end of each month.

We note that the time series of the lower bound, obtained using the second-nearest maturity options, mimics many of the properties reported in Table 2 (of the main body of the paper). For instance, the mean (annualized) lower bound on the expected excess return of the long-term bond is 1.12%. Moreover, the two series have a correlation of 0.92. ■

III. Model economy of Case 3

We outline an economy where the permanent and the transitory components of the SDF are obtained from solving the eigenfunction problem. The saliency of this economy lies in its ability to produce $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T}) < 0$ when $(\log(m_{t+1}^P), \log(m_{t+1}^T)) \in \mathcal{P}$, the class of bivariate distributions with positive quadrant dependency. Moreover, suitable parameterizations can reproduce features of the long-term bond market together with those of the risk-neutral return volatility of futures on the long-term bond.

III.1. Specification of the stochastic discount factor

Suppose the log stochastic discount factor evolves as

$$\log(m_{t+1}) = \varphi + \eta(\theta_z - z_t) - \frac{1}{2}\sigma_p^2 z_t - \sigma_p \sqrt{z_t} \tilde{\boldsymbol{\epsilon}}_{t+1} - \sigma_\infty \sqrt{z_t} \tilde{\boldsymbol{\omega}}_{t+1}, \quad \text{and}, \quad (\text{A1})$$

$$z_{t+1} - z_t = \kappa_z(\theta_z - z_t) + \sigma_z \sqrt{z_t} \tilde{\boldsymbol{\omega}}_{t+1}, \quad (\text{A2})$$

where the shocks $(\tilde{\boldsymbol{\epsilon}}_{t+1}, \tilde{\boldsymbol{\omega}}_{t+1})'$ are bivariate standard normal with correlation ψ , and $z_t > 0$ describes the state of the economy.

The model has nine parameters and we assume $2\kappa_z\theta_z \geq \sigma_z^2$. Our specification is flexible in parameterizing the dynamics of the permanent and the transitory components of m_{t+1} .

III.2. Solution to the eigenfunction problem

The decomposition of m_{t+1} in its m_{t+1}^P and m_{t+1}^T components is unique (Hansen and Scheinkman (2009, Proposition 2)) when the solution of the

$$\text{eigenfunction problem } E_t \left(m_{t+1} \frac{\phi[z_{t+1}]}{\phi[z_t]} \right) = e^\rho \text{ is of the form } \phi[z_t] = \exp(cz_t). \quad (\text{A3})$$

The solution technique entails inserting the dynamics of m_{t+1} in equations (A1)–(A2) in conjunction with the conjectured form of $\phi[z_t]$ in equation (A3), and choosing solutions consistent with stationary $R_{t+1,\infty}$.

Exploiting the moment generating function of the normal distribution and using the method of undetermined coefficients, we derive the first restriction that

$$-\left(\eta + \frac{1}{2} \sigma_p^2 + c \kappa_z \right) + \frac{1}{2} (c \sigma_z - \sigma_\infty)^2 + \frac{1}{2} \sigma_p^2 - (c \sigma_z - \sigma_\infty) \sigma_p \psi = 0. \quad (\text{A4})$$

Expanding on the squared term $(c \sigma_z - \sigma_\infty)^2$ and solving the quadratic equation in c , the potential solution admits two roots:

$$c_\pm = \frac{(\sigma_z (\psi \sigma_p + \sigma_\infty) + \kappa_z) \pm \sqrt{(\sigma_z (\psi \sigma_p + \sigma_\infty) + \kappa_z)^2 - 2 \sigma_z^2 (\frac{1}{2} \sigma_\infty^2 - \eta + \sigma_\infty \sigma_p \psi)}}{\sigma_z^2}. \quad (\text{A5})$$

Among the two roots c_- and c_+ , we specifically choose the one satisfying

$$c \equiv c_- < 0. \quad (\text{A6})$$

We emphasize that our choice is consistent with $\rho < 0$ (Hansen and Scheinkman (2009, pages 204 and Section 7.2.1)) and the stationarity of $R_{t+1,\infty}$, where the solution of the eigenfunction problem also imposes the second restriction:

$$\rho = \varphi + \eta \theta_z + c \kappa_z \theta_z. \quad (\text{A7})$$

Note that if $\theta_z = 0$, $\rho = \varphi$, therefore, we restrict $\varphi < 0$.

Aided by the solution $\phi[z_t] = \exp(cz_t)$, we can consequently derive

$$m_{t+1}^T = e^\rho \frac{\phi[z_t]}{\phi[z_{t+1}]} = \exp(\rho - c \kappa_z \theta_z + c \kappa_z z_t - c \sigma_z \sqrt{z_t} \tilde{\omega}_{t+1}), \quad (\text{A8})$$

along with $m_{t+1}^P = m_{t+1}/m_{t+1}^T$. The derived solution for c and ρ in equations (A6) and (A7) also ensure that m_{t+1}^P is a martingale, as required.

III.3. Closed-form characterizations

The analytical solutions to $R_{t+1,f} = (E_t(m_{t+1}))^{-1}$, $E_t(R_{t+1,\infty})$, and $E_t\left(\frac{m_{t+1}^P}{m_{t+1}^T}\right)$ are as presented in equations (19) through (21) of the main body of the paper. Armed by these expressions, we can also analytically compute (i) $\text{Cov}_t\left(m_{t+1}^P, \frac{1}{m_{t+1}^T}\right) = E_t\left(\frac{m_{t+1}^P}{m_{t+1}^T}\right) - E_t(R_{t+1,\infty})$ and (ii) $R_{t+1,f} E_t^*\left(\left(\frac{f_{t+1}}{f_t} - 1\right)^2\right) = E_t(R_{t+1,\infty}) - R_{t+1,f} + \text{Cov}_t\left(m_{t+1}^P, \frac{1}{m_{t+1}^T}\right)$, according to equations (9) and (10) of the paper.

We reiterate that equation (11) of the paper identifies two sources of variation in the expected excess return of the long-term bond. One source can be identified by computing the lower bound (as in our Proposition 1), whereas the contribution of the covariance source can only be determined through model parametrization.

The economy so constructed offers tractability since the forecasting equations are linked to the state variable z_t . Given that the joint moment generating function of $(\log(m_{t+1}^P), \log(m_{t+1}^T))$ is bivariate normal, we can derive conditional expectations and conditional variances in closed-form.

III.4. Simulation procedure and model properties

In this model, we may derive

$$\begin{aligned} \text{Cov}_t(\log(m_{t+1}^P), \log(m_{t+1}^T)) &= \text{Cov}_t(\log(m_{t+1}) - \log(m_{t+1}^T), \log(m_{t+1}^T)), \\ &= c\sigma_z(c\sigma_z + \psi\sigma_p + \sigma_\infty)z_t. \end{aligned} \quad (\text{A9})$$

Under our parameterizations $c < 0$, and the pair $(\log(m_{t+1}^P), \log(m_{t+1}^T))$ displays positive quadrant dependency provided $c\sigma_z + \psi\sigma_p + \sigma_\infty < 0$. The latter is satisfied when $\psi\frac{\sigma_p}{\sigma_\infty} + 1 < 0$ (i.e., intuitively, when the size of the permanent component is kept large).

We also obtain the expressions below:

$$E_t(m_{t+1}^2) = \exp(2\phi + 2\eta(\theta_z - z_t) + \sigma_p^2 z_t + 2\sigma_\infty^2 z_t + 4\psi\sigma_p\sigma_\infty z_t), \quad (\text{A10})$$

$$E_t((m_{t+1}^T)^2) = \exp(2\rho - 2c\kappa_z\theta_z + (2c\kappa_z + 2c^2\sigma_z^2)z_t), \quad (\text{A11})$$

which allows us to compute the conditional variance of the SDF as: $\text{Var}_t(m_{t+1}) = E_t(m_{t+1}^2) - (E_t(m_{t+1}))^2$, and the conditional variance of the transitory component as: $\text{Var}_t(m_{t+1}^T) = E_t((m_{t+1}^T)^2) - (E_t(m_{t+1}^T))^2$.

The simulation is performed as follows:

1. We simulate the path of z_t for $t = 1, \dots, 10,375$ months, with starting value $z_0 = \theta_z$ and using draws of the standard normal variate $\tilde{\omega}_{t+1}$.
2. Next, we discard the first 10,000 months to match the length of the times-series of the futures on the 30-year Treasury bond, and options on the futures of the 30-year Treasury bond.
3. For each of the 375 months, we compute $R_{t+1,f}$, $E_t(R_{t+1,\infty})$, $E_t^*((\frac{f_{t+1}}{f_t} - 1)^2)$, $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$, $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T)$, and some other relevant quantities.
4. The procedure is repeated 10,000 times to obtain a panel of dimension $375 \times 10,000$ for each variable.

Table 1 in the main body of the paper presents the summary statistics across the 10,000 copies of the simulated paths under two different parameterizations:

$$\text{SET A : } (\varphi, \eta, \sigma_p, \sigma_\infty, \psi, \theta_z, \kappa_z, \sigma_z) = (-0.02, 0.80, 1.50, 0.08, -0.92, 0.03, 0.05, 0.02),$$

$$\text{SET B : } (\varphi, \eta, \sigma_p, \sigma_\infty, \psi, \theta_z, \kappa_z, \sigma_z) = (-0.02, 0.45, 2.05, 0.08, -0.90, 0.03, 0.05, 0.02), \quad (\text{A12})$$

Several aspects of our results are worthy of comments. Focus on SET B. First, the parameterizations of $(\sigma_p, \sigma_\infty, \sigma_z)$ ensure that the model generated volatility of the SDF is high and the volatility of the transitory component is low (Alvarez and Jermann (2005)). Second, accompanying the 34% volatility of m_{t+1} , the risk-neutral futures return volatility is 9.92%. The latter mimics the data counterparts from options on the futures of the 30-year Treasury bond. Third, reinforcing our analysis, $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T)$ is always positive, while $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$ is always negative. Fourth, the model simultaneously produces a reasonable risk-free return (2.47%) and expected excess return of the long-term bond (3.22%).

One fundamental question not yet addressed is: How plausible are economies that support a negative value for $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T)$ and a positive value of $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$? To investigate this issue, we keep the parameters in SET B, and vary ψ (the correlation of the shocks between $\tilde{\epsilon}_{t+1}$ and $\tilde{\omega}_{t+1}$) between -1.0 and 1.0 , in increments of 0.10 .

For each of the 20 parameterizations, we simulate 10,000 times the path of z_t over 10,375 months. For

each ψ , we plot in Figure Internet-I, the *mean* across the 10,000 realizations of the path of z_t for (i) the conditional volatility of m_{t+1} , (ii) the risk-neutral return volatility of futures of the long-term bond, (iii) the risk-free return, (iv) the conditional excess return of the long-term bond, (v) the covariance between m_{t+1}^P and m_{t+1}^T , and (vi) the covariance between m_{t+1}^P and $1/m_{t+1}^T$.

The crux of this exercise is that when $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T)$ switches sign from a positive value to a negative value, $\text{Cov}_t(m_{t+1}^P, \frac{1}{m_{t+1}^T})$ moves in the opposite direction, switching sign from negative to positive. Moreover, economies with $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) < 0$ have certain counterfactual attributes. For example, the expected excess return of the long-term bond is slightly negative. At the same time, the risk-neutral return volatility of the futures is below 5%. Taking $\psi = 0.5$ as an example, the expected excess return of the long-term bond is -0.04% and the corresponding risk-neutral return volatility of futures is 1.698%.

Overall, choosing parameterizations that preserve $E_t(m_{t+1}) \geq E_t(m_{t+1}^T)$ appears useful in reconciling the data properties and is consistent with our Proposition 1. ■

References

Alvarez, F., Jermann, U., 2005. Using asset prices to measure the persistence of the marginal utility of wealth. *Econometrica* 73, 1977–2016.

CME, 2012. U.S. Treasury futures and options. Unpublished working paper. Chicago Mercantile Exchange (<http://www.cmegroup.com/trading/interest-rates/files/us-treasury-futures-and-options-fact-card.pdf>).

Hansen, L., Scheinkman, J., 2009. Long-term risk: An operator approach. *Econometrica* 77, 177–234.

Politis, D., White, H., Patton, A., 2009. Correction: Automatic block-length selection for the dependent bootstrap. *Econometric Reviews* 28, 372–375.

Table Internet-I

Summary statistics for out-of-the-money call and put options on the 30-year Treasury bond futures

The underlying unit of one options contract is one 30-year U.S. Treasury bond with a face value at maturity of \$100,000. CME (2012) provides a more detailed description of the contract specifications for the Treasury bond futures and options. Among all available option strikes and maturities at the end of each month, we first retain options with maturity closest to 30 days. Next, we keep out-of-the-money calls, i.e., $f_t/K < 1$, and out-of-the-money puts, i.e., $K/f_t < 1$, where f_t is the futures price and K is the strike price. Reported is a snapshot of (i) the number of strikes for calls, (ii) the open interest for calls, (iii) the number of strikes for puts, and (iv) the open interest for puts. The average maturity of the out-of-the-money options is 29.16 days. The sample period is 1982:10 to 2013:12, for a total of 375 monthly observations.

	Calls on 30-year Treasury bond futures		Puts on 30-year Treasury bond futures	
	Number of Calls (end of month)	Open Interest (end of month)	Number of Puts (end of month)	Open Interest (end of month)
1982	2	3037	5	3310
1983	5	21534	4	9718
1984	7	74676	5	42544
1985	4	64761	9	97126
1986	5	91288	16	147991
1987	10	177260	9	137888
1988	10	136272	10	169757
1989	7	136763	11	168581
1990	8	166863	8	156011
1991	4	69913	6	85077
1992	4	80629	6	69070
1993	6	80448	8	107432
1994	10	129677	9	105819
1995	7	69905	12	122296
1996	9	114496	10	98043
1997	8	118954	10	131747
1998	9	134286	14	180984
1999	14	166896	11	131621
2000	9	77871	12	111384
2001	11	74699	13	68160
2002	12	77541	15	94089
2003	16	107373	19	89873
2004	14	97925	19	115324
2005	11	96873	16	139977
2006	11	121840	13	165790
2007	15	123813	15	144760
2008	24	139791	24	158342
2009	24	89588	26	70196
2010	20	83934	25	79399
2011	24	92277	26	89657
2012	23	91638	26	116481
2013	21	110622	20	110417

Table Internet-II

Lower bound extracted from second-nearest maturity options (annualized, in percentages)

Reported are the lower bounds on the expected excess returns of the long-term bond, computed using second-nearest maturity options:

$$\text{LB}_{\{t \rightarrow t+1\}} = \frac{2R_{t+1,f}^2}{f_t^2} \left(\int_{\{K > f_t\}} C_t[K] dK + \int_{\{K < f_t\}} P_t[K] dK \right),$$

where $C_t[K]$ ($P_t[K]$) are the prices of the out-of-the-money calls (puts) written on the 30-year Treasury bond futures, reported at the end of month t . The futures price is f_t and $R_{t+1,f}$ is the gross return of the one-month Treasury bill that is known at the end of month t . *The average maturity is 84.31 days*, and the sample period is 1982:10 to 2013:12, with a total of 375 monthly observations. On average, there are 11.37 calls and 13.42 puts in each month. The lower and upper 95% confidence intervals are constructed following the stationary bootstrap procedure of Politis, White, and Patton (2009), with optimal block size, and are based on 10,000 bootstrap draws. All the statistics are annualized and expressed in percentages. ACF_1 is the first-order autocorrelation.

Sample	Mean	Std.	Bootstrap 95% CI		Min.	Max.	ACF_1	Nobs.
			Lower	Upper				
<i>Panel A: Full sample</i>								
1982:10- 2013:12	1.12	0.65	0.93	1.33	0.29	6.07	0.76	375
<i>Panel B: Subsamples</i>								
1982:10-1997:12	1.09	0.51	0.88	1.35	0.33	3.58	0.75	183
1998:01-2013:12	1.15	0.76	0.89	1.48	0.29	6.07	0.76	192
1990:01-2013:12	1.04	0.65	0.85	1.30	0.29	6.07	0.77	288
1982:10-2007:12	1.02	0.50	0.85	1.22	0.29	3.58	0.74	303

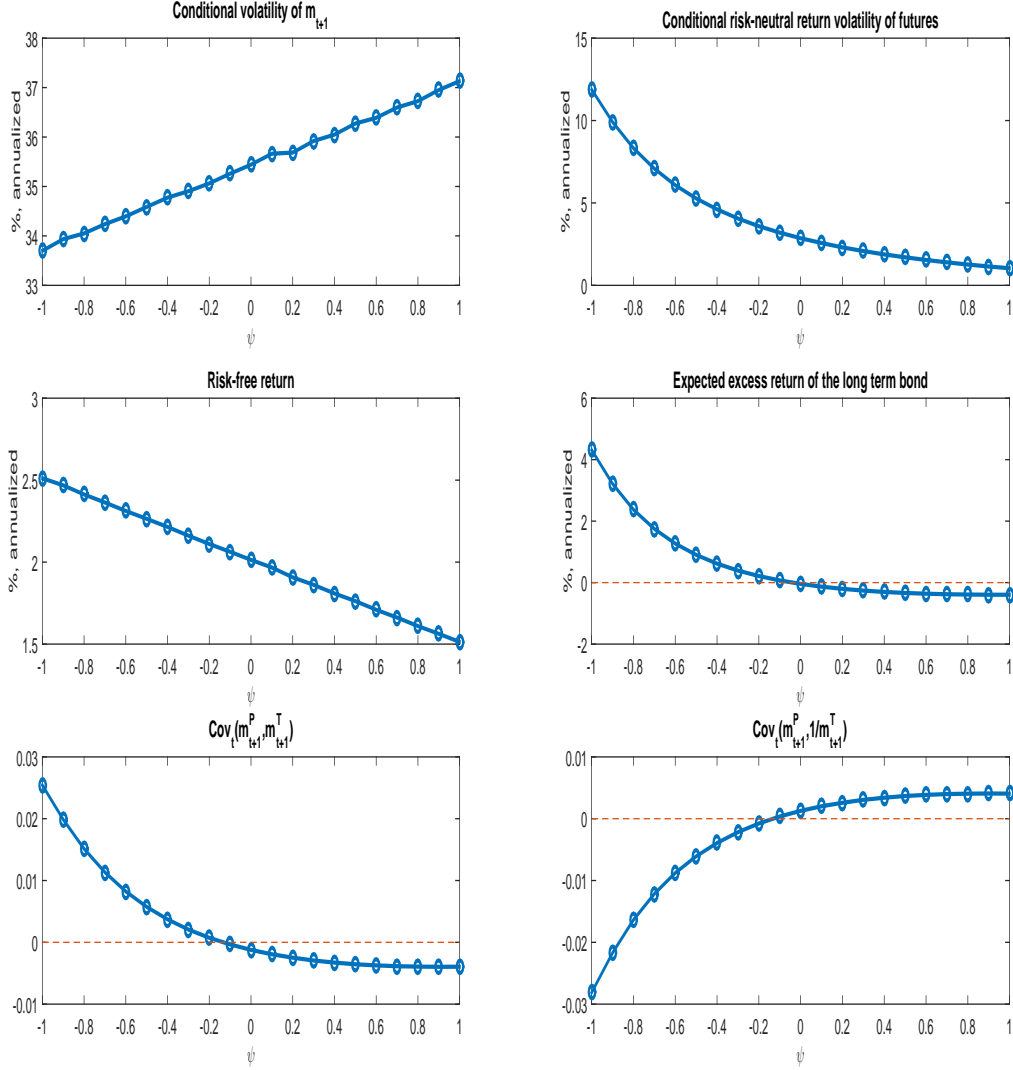


Figure Internet-I. **The impact of changing the dependence between m_{t+1}^P and m_{t+1}^T**

This plot evaluates the impact of altering the dependence between m_{t+1}^P and m_{t+1}^T components of the SDF by changing ψ , the correlation of the shocks between $\tilde{\boldsymbol{\epsilon}}_{t+1}$ and $\tilde{\boldsymbol{\omega}}_{t+1}$. For our exercise,

$$(\varphi, \eta, \sigma_p, \sigma_\infty, \psi, \theta_z, \kappa_z, \sigma_z) = (-0.02, 0.45, 2.05, 0.08, \psi_j, 0.03, 0.05, 0.02).$$

where $\psi_j = \{-1.00, -0.90, \dots, 0.1, 0, 0.1, \dots, 0.90, 1.00\}$. For each of the 20 parameter sets, we simulate the state variable z_t for $t = 1, \dots, 10,375$ months using the standard normal variate $\tilde{\omega}_{t+1}$ (as in equation (A2)). We discard the first 10,000 observations to match the sample length of our data. Each month, we compute the conditional variables of interest and its *mean* across the 10,000 simulations. For each of the *conditional variables*, we plot its mean in relation to ψ . Our illustrative calculations show that ψ values that result in $\text{Cov}_t(m_{t+1}^P, m_{t+1}^T) < 0$ yield the wrong order of magnitudes for both the expected excess return of the long-term bond and the risk-neutral return volatility of the futures of the long-term bond.