

An Approach to Measure the Expectation of Generic Functions of the Market Return*

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First draft, January 2016

September 19, 2017

Abstract

This paper proposes an approach that associates the risk-neutral probability measure with option prices and then computes the expectation of quantities under the real world probability measure, exploiting the form of the stochastic discount factor. This approach deviates from foundational approaches that embrace assumptions about preferences and economic primitives to propose formulas for prices and risk premiums. Our method is analytically tractable, absolved of distributional assumptions, and we use the approach to elaborate on empirical questions regarding disaster probabilities, conditional return moments, and conditional return asymmetries.

*The authors are indebted to Ngoc-Khanh Tran for his constructive and generous written feedback (March 2016) that led to many improvements in the paper. Wei Zhou suggested improvements to the proofs. This paper has undergone many revisions, and earlier versions were presented at Chinese University of Hong Kong (May 2016), University of Massachusetts–Amherst (March 2016), University of Maryland (February 2016), and University of North Carolina–Chapel Hill. We are grateful for discussions with Nikolaos Artavanis, Jay Cao, Peter Carr, Vincent Campasano, Fousseni Chabi-Yo, Ric Colacito, Robert Connolly, Jennifer Conrad, John Crosby, Julien Cujean, Robert Engle, Mila Getmansky, Mustafa Gultekin, Steve Heston, Griffin Jiang, Nenjiu Ju, Nikunj Kapadia, Hossein Kazemi, Camelia Kuhnen, Bing Liang, Matthew Linn, Mark Loewenstein, Dilip Madan, Sanjay Nawalkha, Daniela Osterrieder, George Panayotov, Alberto Rossi, Shrihari Santosh, Georgios Skoulakis, Liuren Wu, and Lai Xu. We welcome comments, including references to related papers we have inadvertently overlooked. All errors are our responsibility. All computer codes are available from any of the authors. An earlier version of this paper was circulated under the title “Characterizing Disaster Probabilities and Disaster Risk Premiums from Put Option Prices.”

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1. Introduction

This paper proposes an approach that inverts the Girsanov theorem to arrive at the solution for conditional expectation of generic functions of the market return. The linchpin of the approach is to associate the risk-neutral probability measure with option prices and then compute the conditional expectation of quantities under the real world probability measure, exploiting the form of the stochastic discount factor. This approach deviates from foundational approaches that embrace assumptions about preferences and economic primitives to propose formulas for prices and risk premiums. Our method is analytically tractable, absolved of distributional assumptions, and can be applied to a broad class of functions of market return that satisfy certain smoothness properties.

Our approach can address a set of conceptually important, but not fully resolved, questions: How should one characterize the real world (physical) probability of wealth disasters? What are the salient features of the disaster risk premiums? What are the economic consequences of variations in disaster risk premiums? We synthesize the real world probability of wealth disasters using information that is adapted to forward-looking option prices, paving the way for extracting the disaster risk premiums from option prices.

The aversion to wealth losses permeates works on risk premiums (e.g., Markowitz (1952)). In implementation, we define a wealth disaster to be a greater than 5% (or 7%) decline in the S&P 500 index, measured over a window that matches the number of days in the nearest-maturity option expiration cycles (an average of 26.45 calendar days). The economic magnitude of the wealth declines is nontrivial, given the 2015 year-end market capitalization of \$17.8 trillion for firms in the S&P 500 index (relative to the GDP of \$18 trillion).

Our formulaic treatment to compute real world probability of disasters and risk premiums deviates from Aït-Sahalia, Wang, and Yared (2001), Bollerslev and Todorov (2011), Wachter

(2013), and Seo and Wachter (2015). A different approach is taken by Kelly and Jiang (2014), who extract a tail risk measure from the cross-section of stock returns, assuming that the conditional lower tail distribution of returns obeys a power law. Distinct from others, we show how to analytically compute real world disaster probabilities from put option prices.^{1 2}

Envisioned by the options market, the real world probability of a disaster of size greater than 5% (7%) has an average value of 9% (6%) and an average conditional disaster risk premium of -28% (-37%). We provide bootstrap-based evidence and establish statistical significance. Our framework simultaneously produces sensible estimates of the conditional (log) expected return, variance, and variance risk premium. Additionally, we consider the Mincer-Zarnowitz regression, which shows that conditional variance provides an *unbiased* forecast of realized variance with an adjusted R^2 of 41%. Rejecting the null of no out-of-sample predictability, the adjusted mean-squared prediction error statistic of Clark and West (2007) yields a p -value of 0.004.

Salient to our approach is the measurement of disaster uncertainty.³ Speaking to the relevance of our formula for disaster probabilities, our predictive exercises show that more negative estimates of disaster risk premiums are associated with an adverse macroeconomic outlook.

The toolkit of “inverting the Girsanov theorem” is also handy when studying empirically motivated questions involving conditional expected return (e.g., Merton (1980), Black (1993),

¹Du and Kapadia (2013), Gao and Song (2015), and Siriwardane (2015) focus on quantifying *risk-neutral tails* from option prices. On the other hand, we show how to parsimoniously extract real world probability of wealth disasters and disaster risk premiums from put option prices, and the proposed mechanism is novel to our paper. Weller (2015) relies on bid and ask prices from TAQ data to recover an expected tail measure at high frequency.

²Our goals are conceptually different from a strand of knowledge based on models of structural uncertainty and consumption disasters, and their macroeconomic implications. For example, Weitzman (2007), Bakshi and Skoulakis (2010), Backus, Chernov, and Martin (2011), Collin-Dufresne, Johannes, and Lochstoer (2016), and Dew-Becker and Bidder (2016) study unconditional probabilities of a consumption disaster. In contrast, Chen, Joslin, and Tran (2012), Gabaix (2012), Wachter (2013), Farhi and Gabaix (2016), and Tsai and Wachter (2016) highlight the relevance of variable-rate consumption disasters.

³This work is prompted by the literature on identifying, measuring, and understanding sources of uncertainty (e.g., Hansen (2007), Bloom (2009), Chen, Dou, and Kogan (2015), Jurado, Ludvigson, and Ng (2015), Orlik and Veldkamp (2015), Mecikovsky and Meier (2015), and Manela and Moreira (2016)).

Lewellen and Shanken (2002), and Greenwood and Shleifer (2014)), conditional return asymmetries (e.g., Engle (2004)), and conditional leverage (e.g., Aït-Sahalia, Fan, and Li (2013)). For one, our novelty is to construct estimates of the expected excess return of the market, together with estimates of conditional return variance of market (which we show are aligned with estimates of realized return variance). This attribute of our analysis is a stepping stone for constructing a counterexample that shows that the conditional lower bound on the expected excess return of the market proposed in Martin (2017) is *not* approximately tight. We formulate the null hypothesis that the conditional lower bound is tight, which is rejected with a p -value of 0.000. Additionally, we show that the lower bound is on average about 36% of the size of the expected excess return, undercutting the practical relevance and usefulness of the lower bound of Martin.

Moreover, we develop measures of upside and downside semivariance under the real world probability measure and operationalize a measure of conditional asymmetry of returns from option prices. Our return asymmetry formula fills a gap, in light of the challenges described in Kim and White (2004), Engle (2011), Neuberger (2012), Barndorff-Nielsen, Kinnebrock, and Shephard (2013), Chang, Christoffersen, and Jacobs (2013), and Ghysels, Plazzi, and Valkanov (2016).

Finally, we emphasize that our methodology departs from Ross (2015), Borovička, Hansen, and Scheinkman (2016), Bakshi, Chabi-Yo, and Gao (2017), Jackwerth and Menner (2017), Jensen, Lando, and Pedersen (2017), Qin, Linetsky, and Nie (2016), and Schneider and Trojani (2017). For instance, Schneider and Trojani (2017) obtain the minimum variance projection of SDF on the market return (i.e., a polynomial in gross market returns), whereas our formulaic treatment yields the conditional expectation of a generic function of market return under the physical measure using option prices, given a form of the SDF. In particular, our methodology to obtain physical probabilities of wealth disasters using put spread positions is new and is a central

differentiating element with macrofinance implications.

2. The theoretical building blocks

The time- t price of the equity market index (i.e., wealth) is denoted by $S_t > 0$, $r_{t+1} \equiv \log(\frac{S_{t+1}}{S_t})$ is the logarithmic return, and $\mathbb{E}_t^{\mathbb{P}}(\cdot)$ is the conditional expectation under the physical probability measure. Define $\Omega \equiv \{S_{t+1} > 0\}$ and let the physical return density be represented by $p[r_{t+1}]$.

Suppose one is interested in computing the conditional expectation of *some* function of r_{t+1} represented by $\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}])$, with $|\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}])| < +\infty$.

Additionally, we suppose that the absence of arbitrage guarantees the existence of a stochastic discount factor (SDF), $\Lambda[r_{t+1}] > 0$, or equivalently, $\Lambda[S_{t+1}] > 0$, with probability 1. Assume that $\Lambda[r_{t+1}] \in \mathcal{C}^2$, the space of twice-continuously differentiable functions. The gross interest rate on a default-free discount bond is $R_{t,t+1} = 1/\mathbb{E}_t^{\mathbb{P}}(\Lambda[r_{t+1}])$.

Let $\mathbb{E}_t^{\mathbb{Q}}(\cdot)$ denote expectation under the risk-neutral measure. We assume that the measure \mathbb{Q} , corresponding to the density $q[r_{t+1}]$, is absolutely continuous with respect to \mathbb{P} (e.g., Harrison and Kreps (1979)). When the measure \mathbb{Q} is absolutely continuous with respect to \mathbb{P} , all \mathbb{P} -zero sets are also \mathbb{Q} -zero sets and the two measures are equivalent.

Then by the Radon-Nikodym theorem, we can write the change-of-measure density of \mathbb{Q} with respect to \mathbb{P} as $\mathcal{L}^{-1}[r_{t+1}] = \frac{d\mathbb{Q}}{d\mathbb{P}}$ or the change-of-measure density of \mathbb{P} with respect to \mathbb{Q} as $\mathcal{L}[r_{t+1}] = \frac{d\mathbb{P}}{d\mathbb{Q}}$. Hence, the change-of-measure density, or the likelihood function, can be represented as

$$\mathcal{L}[r_{t+1}] = \frac{p[r_{t+1}]}{q[r_{t+1}]} \tag{1}$$

With $\mathcal{L}[r_{t+1}] = \frac{\int_{\Omega} \Lambda[r_{t+1}] p[r_{t+1}] dr_{t+1}}{\Lambda[r_{t+1}]}$, which rearranges the expression for the density function

$q[r_{t+1}] = \frac{\Lambda[r_{t+1}]p[r_{t+1}]}{\int_{\Omega} \Lambda[r_{t+1}]p[r_{t+1}]dr_{t+1}}$, and at the same time ensures that $q[r_{t+1}]$ is a proper density function satisfying $1 = \int_{\Omega} q[r_{t+1}]dr_{t+1}$, we can then express, considering $\mathcal{L}[r_{t+1}] > 0$,

$$\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}]) = \int_{\Omega} f[r_{t+1}] \mathcal{L}[r_{t+1}] q[r_{t+1}] dr_{t+1}, \quad (2)$$

$$= \mathbb{E}_t^{\mathbb{Q}}(f[r_{t+1}] \mathcal{L}[r_{t+1}]), \quad (\text{substituting } \mathcal{L}[r_{t+1}] = \frac{\int_{\Omega} \Lambda[r_{t+1}] p[r_{t+1}] dr_{t+1}}{\Lambda[r_{t+1}]}) \quad (3)$$

$$= \underbrace{\int_{\Omega} \Lambda[r_{t+1}] p[r_{t+1}] dr_{t+1}}_{R_{\mathbf{f},t+1}^{-1}} \mathbb{E}_t^{\mathbb{Q}}\left(\frac{f[r_{t+1}]}{\Lambda[r_{t+1}]}\right), \quad (4)$$

$$= R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(g[r_{t+1}]), \quad \text{where defining } g[r_{t+1}] \equiv \frac{f[r_{t+1}]}{\Lambda[r_{t+1}]}. \quad (5)$$

The formula in equation (2) is associated with reweighting, whereby the likelihood function $\mathcal{L}[r_{t+1}]$ transforms the measure \mathbb{P} into \mathbb{Q} . The implication is that the expected value of $f[r_{t+1}]$ is unaltered because $\mathcal{L}[r_{t+1}]$ is included in the expectation with respect to the measure \mathbb{Q} .

Remark 1 *Options on S_{t+1} of all strikes are traded. The function $g[r_{t+1}]$ can be synthesized with a portfolio of options and the market for claims on S_{t+1} is complete.*

Remark 2 *We assume $|\mathbb{E}_t^{\mathbb{Q}}(g[r_{t+1}])| < +\infty$ and so $\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}])$ in equation (5) is well-defined. Our constructions do not require distributional assumptions about r_{t+1} .*

Remark 3 *The moment generating function of $\mathcal{L}[r_{t+1}]$ under the \mathbb{P} and \mathbb{Q} probability measures are related as $\mathbb{E}_t^{\mathbb{P}}(\mathcal{L}^{\kappa}[r_{t+1}]) = \mathbb{E}_t^{\mathbb{Q}}(\mathcal{L}^{\kappa+1}[r_{t+1}])$, for $\kappa \in \mathbb{R}$ (setting $f[r_{t+1}] = \mathcal{L}^{\kappa}[r_{t+1}]$ in equation (2)). In contrast to $\mathbb{E}_t^{\mathbb{P}}(\mathcal{L}^{-1}[r_{t+1}]) = \mathbb{E}_t^{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}) = 1$, Jensen's inequality implies $\mathbb{E}_t^{\mathbb{P}}(\mathcal{L}[r_{t+1}]) \geq 1$.*

We call the method in equation (5) as ‘‘inverting the Girsanov theorem’’ approach for obtaining the expectation of generic functions of market return.⁴

⁴The internal consistency of our approach manifests in other ways. For example, $\Lambda[r_{t+1}] = 1/(1+r_0)$, for constant r_0 , results in zero risk premiums. Additionally, if $f[r_{t+1}] = \Lambda[r_{t+1}]$, the discount bond price coincides with $R_{\mathbf{f},t+1}^{-1}$.

The Girsanov theorem involves reweighting of the original diffusion process to obtain a different diffusion process that modifies the drift, but not the diffusion coefficient. In contrast, the counterpart for semimartingales can potentially alter all moments of the \mathbb{Q} measure when moving from \mathbb{P} to \mathbb{Q} (e.g., Øksendal and Sulem (2007, Section 4.1 and Theorem 1.30)).

Instead, we operate in a discrete-time environment and propose going in the reverse, from \mathbb{Q} to \mathbb{P} , and using option prices to identify \mathbb{Q} . This, in turn, can help to analytically characterize \mathbb{P} -measure conditional expectations without taking a stand on how to model the structure of uncertainty.

Remark 4 *Does the lack of state dependencies in the SDF $\Lambda[r_{t+1}]$ detract from generality? Rosenberg and Engle (2002, Section 2.2) and Bakshi, Madan, and Panayotov (2010, equation (2), page 132) show that when there are contingent claims on market return for which there is data, one can model change-of-measure densities that are conditioned only on the market return.*

Remark 5 *What is the broadest class of functions for which one could compute $\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}])$, given $\Lambda[r_{t+1}] \in \mathcal{C}^2$? Our answer is that the framework is tractable when $f[r_{t+1}]$ is of the class C^{-1} , where the space C^{-1} is the set of functions for which $\int f[r_{t+1}]dr_{t+1}$ is continuous (e.g., Reid (2015, Table 1)).*

Our formulaic treatment shows next that $\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}])$ is adapted to the price of the spanning securities (i.e., the bond, equity, and options on the equity), regardless of the specification of $\Lambda[r_{t+1}] \in \mathcal{C}^2$. It further reveals the mechanism by which theory assigns mathematical weights to the spanning securities in terms of the parameters of $\Lambda[r_{t+1}]$.

While in theory the \mathbb{P} -measure expectations are not uniquely inferred, these assigned weights are quantitatively close across the various choices of $\Lambda[r_{t+1}]$ in our implementations.

3. Developing the expectation of the Heaviside function

The first centerpiece of our efforts is the development of the conditional expectation of a wealth disaster under the *physical* probability measure, corresponding to some breakpoint K^* :

$$\text{Prob}_t(S_{t+1} < K^*) \equiv \mathbb{E}_t^{\mathbb{P}}(\theta[K^* - S_{t+1}]), \quad (6)$$

$$= \mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}]), \quad (7)$$

where $\theta[x] \in C^{-1}$ is the Heaviside theta function, defined as (Kanwal (1998, Chapter 1)):

$$\theta[x] \equiv \frac{d}{dx} \max(x, 0) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases} \quad (8)$$

We define the threshold $r^* \equiv \log(\frac{K^*}{S_t}) < 0$. Therefore, the set $\{r_{t+1} < r^*\}$ captures a disaster over $[t, t + 1]$.

For example, if the level of the S&P 500 index at the start of the options expiration cycle on 11/23/2015 is 2086.59, then choosing $r^* = -0.07257$ implies the breakpoint $K^* = 1940.53$, and $\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$ measures the conditional expectation of a wealth disaster of size greater than 7% (i.e., $\frac{K^*}{S_t} - 1 = -0.07$) by the end of the expiration cycle on 12/19/2015.

For the results to follow, we let $C_t[K]$ and $P_t[K]$ be the time- t prices of European calls and puts written on S_{t+1} with strike price K . Moreover, $g'[S_{t+1}]$ and $g''[S_{t+1}]$ are the first and second derivatives of $g[S_{t+1}]$ (defined in equation (5)) with respect to S_{t+1} . The function $g[S_{t+1}]$ inherits some of its smoothness properties from $f[S_{t+1}]$. Let $g[K^*] \equiv g[S_{t+1}]|_{K^*}$, $g'[K^*] \equiv g'[S_{t+1}]|_{K^*}$, and $g''[K] \equiv g''[S_{t+1}]|_{K < K^*}$.

Next, we show how to compute $\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}])$ from option prices.

3.1. Real world (physical) probability of wealth disasters

To illustrate the approach, we focus on a wealth disaster $\{r_{t+1} < r^*\}$ over $[t, t+1]$, and express $\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}])$ in equation (5) as:⁵

$$\mathbb{E}_t^{\mathbb{P}}(f[S_{t+1}]) = R_{\mathbf{f}, t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(g[S_{t+1}]), \quad \text{where}$$

$$\left. \begin{aligned} f[S_{t+1}] &= \theta[K^* - S_{t+1}] \\ g[S_{t+1}] &\equiv \theta[K^* - S_{t+1}] \frac{1}{\Lambda[S_{t+1}]} \geq 0 \end{aligned} \right\}. \quad (9)$$

The physical probability of disasters can be analytically computed as follows:

$$\underbrace{\mathbb{E}_t^{\mathbb{P}}(\theta[K^* - S_{t+1}])}_{\text{Disaster probability}} = g[K^*] \underbrace{\mathbb{B}_t[K^*]}_{\text{Binary put}} + \underbrace{\int_{K < K^*} g''[K] P_t[K] dK}_{\text{Collection of puts with strike } K < K^*} - g'[K^*] P_t[K^*] \quad (10)$$

where $\mathbb{B}_t[K^*] \equiv R_{\mathbf{f}, t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(\theta[K^* - S_{t+1}])$ is the time- t price of the binary put, computed as

$$\mathbb{B}_t[K^*] \approx \frac{P_t[K^* + \Delta K] - P_t[K^* - \Delta K]}{2\Delta K}. \quad (11)$$

The proof of equations (10) and (11) is provided in the Online Appendix (Sections A and B).

The development in equation (10) is that the physical probability of a wealth disaster over $[t, t+1]$ can be computed from out-of-the-money (OTM) put option prices. The analyticity of the formulation can be traced to the feature that $g[S_{t+1}] = \theta[K^* - S_{t+1}] \frac{1}{\Lambda[S_{t+1}]}$ can be synthesized from elementary payoffs of the type $\theta[K - S_{t+1}]$ and $\max(K - S_{t+1}, 0)$.

⁵The Heaviside function is a building block for a number of considered objects. Suppose $f[S_{t+1}] = (\frac{S_{t+1}}{S_t} - 1)\theta[K^* - S_{t+1}]$, then $\mathbb{E}_t^{\mathbb{P}}(f[S_{t+1}])$ represents expected return to the downside. Similarly, $\mathbb{E}_t^{\mathbb{P}}(\{\log(\frac{S_{t+1}}{S_t})\}^2 \theta[(K^d - S_{t+1})])$ represents semivariance to the downside, whereas $\mathbb{E}_t^{\mathbb{P}}(\{\log(\frac{S_{t+1}}{S_t})\}^2 \theta[-(K^u - S_{t+1})])$ represents semivariance to the upside (e.g., see among others, Barndorff-Nielsen, Kinnebrock, and Shephard (2013), Feunou, Jahan-Parvar, and Tedongap (2016), Feunou and Okou (2016), and Kilic and Shaliastovich (2017)).

The expected return of the claim with payoff $f[r_{t+1}] = \theta[r^* - r_{t+1}]$ is $1 + \mu_t[r^*] \equiv \frac{\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])}{R_{f,t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(\theta[r^* - r_{t+1]})}$.

Accordingly, we can define the disaster risk premium (in log form) as

$$\text{drp}_t[r^*] \equiv \log(1 + \mu_t[r^*]) - \log(R_{f,t+1}) = \log\left(\frac{\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])}{\mathbb{E}_t^{\mathbb{Q}}(\theta[r^* - r_{t+1}])}\right), \quad (12)$$

where $\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$ can be inferred based on equation (10) and $\mathbb{E}_t^{\mathbb{Q}}(\theta[r^* - r_{t+1}]) = R_{f,t+1} \mathbb{B}_t[K^*]$.

Equation (10) quantifies the disaster probability under the \mathbb{P} measure, and can be viewed as an analog to Bollerslev and Todorov (2011, equation (29)). Their idea is to measure the realized jumps with high-frequency data and translate that information into forward-looking \mathbb{P} measure quantities using the Extreme Value Theory. Our innovation (in equations (10) and (12)) is to characterize the physical probability of wealth disasters and disaster risk premiums by relying on options data and without modeling wealth.

There may be some choices of $\Lambda[S_{t+1}]$ for which disaster probabilities and disaster risk premiums may reflect a subset of the relevant pricing information on put options. For example, if $\Lambda[S_{t+1}] = (\frac{S_{t+1}}{S_t})^{-1}$, in which case, $g[S_{t+1}] = \frac{\theta[K^* - S_{t+1}]}{(\frac{S_{t+1}}{S_t})^{-1}}$, implying that $g[K^*] = \frac{K^*}{S_t} = \exp(r^*)$ (which is a pre-specified constant), $g'[K^*] = \frac{1}{S_t}$, and $g''[K] = 0$. Hence, $\text{drp}_t[r^*] = \log(\frac{K^*}{S_t} - \frac{P_t[K^*]}{S_t \mathbb{B}_t[K^*]}) - \log(R_{f,t+1})$, which is deprived of a contribution from put prices with strikes lower than K^* .

In light of the above, we focus on $\Lambda[S_{t+1}]$ specifications that incorporate a nonzero curvature of the function $g[S_{t+1}]$ (i.e., $g''[S_{t+1}]$ is not zero):

Case 1 Suppose $\Lambda[r_{t+1}] = e^{-\gamma r_{t+1}}$, or equivalently, $\Lambda[S_{t+1}] = (\frac{S_{t+1}}{S_t})^{-\gamma}$ (e.g., Ait-Sahalia and Lo

(2000, Section 6.3)). Computing $g[S_{t+1}]|_{K^*}$, $g'[S_{t+1}]|_{K^*}$, and $g''[S_{t+1}]|_{K < K^*}$, we obtain

$$\mathbb{E}_t^{\mathbb{P}}(\theta[K^* - S_{t+1}]) = \left(\frac{K^*}{S_t}\right)^\gamma \mathbb{B}_t[K^*] + \frac{\gamma(\gamma-1)}{S_t^2} \int_{K < K^*} \left(\frac{K}{S_t}\right)^{\gamma-2} P_t[K] dK - \frac{\gamma}{S_t} \left(\frac{K^*}{S_t}\right)^{\gamma-1} P_t[K^*]. \quad (13)$$

Case 2 Consider $\Lambda[r_{t+1}] = (1 + \lambda r_{t+1})^{-\gamma}$ for $\lambda > 0$ and $\gamma > 0$, which generalizes Case 1, since $-\Lambda'/\Lambda''$ is linear in r_{t+1} (analogous to Sharpe (2007, equation (25))). It then follows that

$$\mathbb{E}_t^{\mathbb{P}}(\theta[K^* - S_{t+1}]) = \left(1 + \lambda \log\left(\frac{K^*}{S_t}\right)\right)^\gamma \mathbb{B}_t[K^*] + \int_{K < K^*} g''[K] P_t[K] dK - \frac{\lambda\gamma}{K^*} \left(1 + \lambda \log\left(\frac{K^*}{S_t}\right)\right)^{\gamma-1} P_t[K^*], \quad (14)$$

$$\text{where } g''[K] = \frac{\lambda^2(\gamma-1)\gamma\left(1 + \lambda \log\left(\frac{K}{S_t}\right)\right)^{\gamma-2}}{K^2} - \frac{\lambda\gamma\left(1 + \lambda \log\left(\frac{K}{S_t}\right)\right)^{\gamma-1}}{K^2}.$$

Case 3 Let $\Lambda[S_{t+1}] = \psi_a(\log(\frac{S_{t+1}}{S_t}))^2 + \psi_b \log(\frac{S_{t+1}}{S_t}) + \psi_c$. We assume $\psi_b < 0$. To maintain the positivity of $\Lambda[S_{t+1}]$, we impose $\psi_a > 0$ and $\frac{\psi_b^2}{4\psi_a} < \psi_c$. Then $\mathbb{E}_t^{\mathbb{P}}(\theta[K^* - S_{t+1}]) = g[K^*]\mathbb{B}_t[K^*] + \int_{K < K^*} g''[K] P_t[K] dK - g'[K^*]P_t[K^*]$, $g[K^*] = \frac{1}{\Lambda[K^*]}$,

$$g'[K^*] = -\frac{1}{K^*} \left(2\psi_a \log\left(\frac{K^*}{S_t}\right) + \psi_b\right) (\Lambda[K^*])^{-2}, \text{ and} \quad (15)$$

$$g''[K] = \frac{2}{K^2} \left(2\psi_a \log\left(\frac{K}{S_t}\right) + \psi_b\right)^2 (\Lambda[K])^{-3} - \frac{2}{K^2} \left(\psi_a - \psi_a \log\left(\frac{K}{S_t}\right) - \frac{\psi_b}{2}\right) (\Lambda[K])^{-2}. \quad (16)$$

The $\Lambda[r_{t+1}]$ serves to assign weights to the spanning instruments: (i) a *long* binary put position with strike K^* , (ii) a collection of *long* put positions with strikes deeper OTM than K^* , and (iii) a *short* put position with strike K^* . These cases illustrate the analyticity of our approach. Moreover, the specifications in our Cases 1, 2, and 3 are in line with using polynomials of market returns to represent $\Lambda[r_{t+1}]$, and is consistent with the derived form of minimum variance SDFs (i.e.,

$a_0 + \sum_{j=1}^J a_j (\frac{S_{t+1}}{S_t})^j$) in Schneider and Trojani (2017, footnote 14, equation (7)).

3.2. The estimation framework without imposing distributional assumptions

Our identification of physical disaster probabilities and risk premiums is based on the information contained in options on the S&P 500 index. The horizon over which we measure disaster probabilities is the number of days in the options expiration cycle. This number varies between 23 and 32 calendar days, with an average of 26.45 days. The first expiration cycle started on 1/22/1990, and the last one on 11/23/2015. Thus, we consider the 26-year history with 311 option expiration cycles. The index options data is described in the Online Appendix (Section C).

Equation (10) is instrumental to our construction of physical probability of disasters but requires an input for Θ , the parameter vector that governs the slope and curvature of $\Lambda[S_{t+1}; \Theta]$. We estimate Θ through the minimization problem (we refer the reader to the discussions on loss functions in Christoffersen and Jacobs (2004) and Patton and Timmermann (2007))

$$\hat{\Theta} \equiv \arg \min_{\Theta} \sum_{t=1}^T \left(\text{var}_t^{\mathbb{P}}(r_{t+1}; \Theta) - \text{rv}_{\{t \rightarrow t+1\}} \right)^2, \quad (17)$$

where $\text{var}_t^{\mathbb{P}}(r_{t+1}; \Theta)$ is the (options-based) conditional return variance, $\text{rv}_{\{t \rightarrow t+1\}}$ is realized return variance, and $T=311$. We calculate $\text{rv}_{\{t \rightarrow t+1\}}$ as the sum of daily squared returns:

$$\text{rv}_{\{t \rightarrow t+1\}} = r_{t+\Delta, \Delta}^2 + r_{t+2\Delta, \Delta}^2 + \dots + r_{t+h\Delta, \Delta}^2, \quad \text{with } r_{t+h\Delta, \Delta} \equiv \log\left(\frac{S_{t+h\Delta}}{S_{t+(h-1)\Delta}}\right), \quad (18)$$

where Δ is one day and h is the number of trading days over an options expiration cycle.

Our rationale for the procedure in equation (17) is that the ex-post value of the realized return variance is an unbiased estimator of the conditional return variance, that is, $\mathbb{E}_t^{\mathbb{P}}(\text{rv}_{\{t \rightarrow t+1\}})$ equates conditional return variance (Andersen, Bollerslev, Diebold, and Labys (2003, Corollary

1))). While the literature often uses the realized return variance to estimate the conditional return variance, we estimate Θ such that the conditional return variance based on our approach is a good forecast of the subsequent realized return variance.

The conceptually important step (in the vein of equation (5)) is that our methods allow us to derive an analytical expression for the conditional return variance that incorporates the price information from options market:

$$\text{var}_t^{\mathbb{P}}[r_{t+1}; \Theta] = \underbrace{R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(r_{t+1}^2 / \Lambda[r_{t+1}; \Theta])}_{\mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2)} - \underbrace{\{R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(r_{t+1} / \Lambda[r_{t+1}; \Theta])\}^2}_{\{\mathbb{E}_t^{\mathbb{P}}(r_{t+1})\}^2}. \quad (19)$$

Recognize that $g[S_{t+1}] = (\log(\frac{S_{t+1}}{S_t}))^n / \Lambda[S_{t+1}; \Theta] \in \mathcal{C}^2$, for $n = 1$ and $n = 2$, which can be synthesized from a positioning in puts and calls.

Consider Case 1 in which $\Lambda[S_{t+1}; \Theta] = (\frac{S_{t+1}}{S_t})^{-\gamma}$. Then

$$\mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2) = \int_{K > S_t} a[K] C_t[K] dK + \int_{K < S_t} a[K] P_t[K] dK \quad \text{and} \quad (20)$$

$$\mathbb{E}_t^{\mathbb{P}}(r_{t+1}) = 1 - R_{\mathbf{f},t+1}^{-1} + \int_{K > S_t} b[K] C_t[K] dK + \int_{K < S_t} b[K] P_t[K] dK, \quad (21)$$

where

$$\left. \begin{aligned} a[K] &= \frac{\left(\frac{K}{S_t}\right)^\gamma \left(2 + (4\gamma - 2) \log\left(\frac{K}{S_t}\right) + \gamma(\gamma - 1) \left\{\log\left(\frac{K}{S_t}\right)\right\}^2\right)}{K^2} \\ b[K] &= \frac{1}{K^2} \left(\frac{K}{S_t}\right)^\gamma \left(2\gamma - 1 + (\gamma - 1) \gamma \log\left(\frac{K}{S_t}\right)\right) \end{aligned} \right\}. \quad (22)$$

The $a[K]$ and $b[K]$ for Cases 2 and 3 are displayed in Online Appendix (Section D).

Later, we elaborate on the goodness of fit metrics and compare forecast accuracy of conditional return variances (via Mincer-Zarnowitz regression and out-of-sample adjusted mean-squared prediction error statistic) that result from employing the loss function in equation (17).

3.3. Evidence on real world probability of disasters and disaster risk premiums

In our implementations and discussions to follow, we define a wealth disaster as the event:

$$\{r_{t+1} < r^*\}, \text{ for } r^* = -0.05129 \text{ or } r^* = -0.07257. \quad (23)$$

This treatment reflects a greater than 5% or 7% decline in the S&P 500 index over an option expiration cycle, that is, $\frac{K^*}{S_t} - 1 = \exp(r^*) - 1$.

The featured breakpoints are compatible with Goldman Sachs EQMOVE. Their model calculates the probabilities of a 5% return movement on the downside (and the upside) in 30 days, based on free-cash-flow yield, return on equity, Institute for Supply Management data, and capacity utilization (Bloomberg (March 7, 2016)). In contrast, our idea in equation (10) is to study physical probability of disasters that are adapted to information in put option prices.

Given a 4.53% ($15.7\%/\sqrt{12}$ monthly) standard deviation of the S&P 500 index returns over the expiration cycles (as shown in Table Internet Appendix-IV), a 7% decline can be viewed statistically as a 1.54 sigma event. Bollerslev and Todorov (2011, Figures 1 and 5, footnote 28) consider a left-tail event to be a breakpoint of less than a 10% decline over 14 days (median).⁶

3.3.1. Salient features of disaster probabilities and disaster risk premiums

Table 1, Table Internet Appendix-I, and Table Internet Appendix-II display the estimates of Θ across the considered $\Lambda[S_{t+1}]$ (as in Cases 1 through 3). The estimation results with the different

⁶The at-the-money (averaged across a call and put) Black-Scholes implied volatility ranges between 7.77% and 57.63% annualized. A tail event, if defined as a three-sigma return movement, would then range between 6.7% (obtained as $3 * 7.77\%/\sqrt{12}$) and 49.9% over option expiration cycles. It is not feasible to construct three-sigma dependent strike price breakpoints consistently across our sample.

$\Lambda[S_{t+1}]$ illustrate the robustness of our findings. For example, the estimated $\widehat{\Theta}$ result in estimates of $\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$ and $\text{drp}_t[r^*]$ that are consistent across our featured $\Lambda[S_{t+1}]$'s.

We focus first on Table 1 (Panel A), which shows that the estimate of γ is 2.93 when $\Lambda[S_{t+1}] = (\frac{S_{t+1}}{S_t})^{-\gamma}$. The γ is reliably estimated with a 95% lower and upper bootstrap confidence interval of 1.60 and 4.27, respectively. Drawing on recent work, our point estimate of γ is consistent with the risk aversion of 3.0 employed in Wachter (2013, Table I), and a value of 4.0 employed in Gabaix (2012, Table I). Moreover, our bootstrap confidence intervals and point estimates of γ are in line with Barro and Jin (2011, Table 1) under their double power law specification.

To compute the lower and upper 95% confidence intervals on γ , we proceed as follows. First, we randomly draw (with replacement) daily returns to generate a time-series of bootstrapped realized return variance over expiration cycles. Next, we estimate γ via equation (17). With 1,000 bootstrap trials, we obtain the 1,000 estimates of γ . The idea of the bootstrap is to establish that the estimates of γ are robust to perturbations in realized variance.

To measure the goodness of fit of our estimation with $\gamma = 2.93$, we perform the following Mincer and Zarnowitz (1969) regression (e.g., Patton and Timmermann (2007)):

$$\text{rv}_{\{t \rightarrow t+1\}} = \alpha + \underbrace{\beta \text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma)}_{\text{from options}} + \varepsilon_{t+1}. \quad (24)$$

A properly specified $\text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma)$ would yield estimates of α close to zero and β close to one. Table 2 reports that the estimates of α and β are -0.00048 and 1.071 , respectively, with an adjusted R^2 equal to 41.0%. The p -value for the null hypothesis of $\alpha = 0$ is 0.28, and the p -value for the null hypothesis of $\beta = 1$ is 0.76, based on Newey and West (1987), with lag automatically selected as in Newey and West (1994). Furthermore, we evaluate predictability of $\text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma)$ in an out-of-sample framework, using an expanding window with 120 initial observations. In this

regard, the adjusted mean-squared prediction error (MSPE) statistic of Clark and West (2007) generates a p -value of 0.004, where the null is that $\text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma)$ carries no predictability.

[Fig. 1 about here.]

Table 3 elaborates on our theoretical analysis and provides the estimates of physical probability of disasters (i.e., $\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$) and disaster risk premiums (i.e., $\text{drp}_t[r^*]$).⁷ Figure 1 depicts the time variation in the physical probability of disasters.⁸

There are a number of takeaways. First, the sample average of the physical probability of a 5% (7%) disaster is 0.092 (0.060). The lower and upper 95% bootstrap confidence intervals for the average of $\mathbb{E}_t^{\mathbb{P}}(\theta[-0.05129 - r_{t+1}])$ are 0.081 and 0.104, respectively. To gauge the sensibility of the reported physical probability of disasters, we compute the occurrence probabilities based on return realizations over the 311 expiration cycles. We detect 23 (14) declines of magnitude $>5\%$ ($>7\%$), resulting in disaster probabilities in the data of 0.074 (0.045). Thus, the unconditional disaster probabilities generated by our formulaic treatment seem aligned with the data attributes.⁹

Our characterization of $\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$ from put options indicates that the probability of a disaster of a fixed size is fluctuating, which is akin to the notion of variable-rate disasters in Gabaix (2012) and Wachter (2013). The disaster probabilities are countercyclical. For example, the first principal component of the 17 output and income series in McCracken and Ng (2015, Appendix,

⁷Options markets imply an average price of a binary put of a greater-than 5% (7%) decline of \$0.124 (\$0.089), with a standard deviation of 0.061 (0.055). Probing further, the bootstrap lower (upper) 95% confidence interval for the average difference between the prices of a 5% and 7% binary put (over the 5/19/1997 to 11/23/2015 period; 223 observations) is 0.0451 (0.050).

⁸The accuracy of the calculations in equations (10)–(11) hinges on the number of put strikes $K < K^*$. The sparsity of put strikes deeper OTM than 7% precludes us from reliably constructing the price of a 7% binary put prior to 5/19/1997 (often 2 to 4 strikes). In contrast, there is an average of 28.78 put strikes deeper OTM than 7% (the minimum number of put strikes is 6) after the expiration cycle of 5/19/1997.

⁹There is a stream of research that uses parameterized models of diffusive volatility and/or jumps to study the economic implications of discontinuities in returns (e.g., Carr and Wu (2003a, 2003b), Huang and Wu (2004), Santa-Clara and Yan (2010), Bates (2012), Andersen, Fusari, and Todorov (2015a, 2015b), Bollerslev, Todorov, and Xu (2015), and Carr and Wu (2016)). The central differentiating element is that we operate in a discrete-time environment and our framework is free of distributional assumptions about market returns.

Group 1), which includes real income and industrial production, has a correlation of -0.28 , with the disaster probability of size 5%. The first principal component inherits the cyclicity of the output series and captures 51.46% of their variance.

The evidence reported in Table 3 and Figure 2 provides a perspective on the disaster risk premiums. Intuitively, the security with the disaster payout is a hedging asset, with $\mathbb{E}_t^{\mathbb{Q}}(\theta[r^* - r_{t+1}])$ dominating $\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$ at each r^* . The sample average of $\text{drp}_t[r^*]$ is -28.4% (-37.2%) for a 5% (7%) disaster. The statistical significance of average disaster risk premiums can be judged by the 95% bootstrap confidence intervals, which do not bracket zero.

[Fig. 2 about here.]

3.3.2. Gauging the sensibility of our findings

Where do we stand on the reasonableness of our findings? We take a multidimensional view and quantify (i) the conditional expected (log) return, (ii) the conditional return variance (its square root), and (iii) the conditional variance risk premium, implied by our approach. Each construct admits a representation in terms of option prices, and can be computed given γ .

The message from our analysis (Panel B of Table 1) is that the average conditional mean (volatility) is 14.3% (18.2%) annualized, and the average conditional variance risk premium is -23.5% . Additionally, the bootstrapped confidence intervals on the respective averages are reasonable.¹⁰ We further find that a higher level of the macroeconomic uncertainty index of Jurado, Ludvigson, and Ng (2015, Figure 1) is associated with higher disaster probabilities (correlation of 0.50) and more negative $\text{drp}_t[r^*]$ (correlation of -0.44). Moreover, $\text{drp}_t[r^*]$ exhibits a correlation of 0.82 with Investor Fears index of Bollerslev and Todorov (2011, equation (32), Figure 5).

¹⁰For example, the average conditional expected return of 14.3% lies between the bootstrapped 95% lower and upper confidence interval of 2.3% and 16.4% of the average *realized* returns (Table Internet Appendix-IV).

3.3.3. Point estimates of conditional expectations are similar across SDFs

Are there material differences in the disaster probabilities, conditional expected (log) return, and conditional return volatility among the three adopted SDFs? To this end, we conduct a bootstrap experiment, in which we set the random number seed in the resampled series of daily index returns. For each bootstrap draw, we compute the parameter vector Θ for each of the adopted SDFs and generate the objects of interest. We repeat this exercise across 1,000 trials and report the *average differences* (multiplied by 100) and the 95% confidence intervals in Table 4.

The result is that the generated values are in a similar economic ballpark, particularly for disaster probabilities and the conditional return volatility with tight bootstrap intervals. While we do observe some deviations in conditional expected (log) return, these deviations are generally not statistically significant, as the bootstrap confidence intervals often straddle zero.

At the root of our findings is that the level of the weighting functions (for example, $g[S_{t+1}]|_{K^*}$, $g'[S_{t+1}]|_{K^*}$, and $g''[S_{t+1}]|_{K < K^*}$ in equation (10)) are quantitatively close among different SDFs at each t , so that we do not observe economically meaningful differences. Next, the weighting functions are stable across time, as K^*/S_t is a constant by construction, resulting in 0.99 correlation between disaster probabilities across the featured SDFs.

Finally, we evaluate the specification errors of each $\Lambda[r_{t+1}]$ that arise when pricing a set of assets: gross return of the risk-free bond, gross return of the equity market index, and gross returns of 5%, 3%, and 1% out-of-the-money put options, as well as 1%, 3%, and 5% out-of-the-money call options on the S&P 500 index. For this purpose, we use the methodology of Hansen and Jagannathan (1997) and report their distance measure in Table 5 for each of the three adopted SDFs. The distance measure can be interpreted as the maximum pricing error when the norm of the gross portfolio return is one. Our results indicate that the three SDFs imply

similar point estimates of the distance measure. Considering $\Lambda[r_{t+1}] = (\frac{S_{t+1}}{S_t})^{-\gamma}|_{\gamma=2.93}$, the lower (upper) bootstrap 95% confidence interval is 0.318 (0.816). The substantive finding here is that models exhibit similar pricing ability when their parameters ensure consistency with forecasting the realized return variance.

3.3.4. *Relevance of disaster uncertainty*

This section is motivated by arguments (see footnote 3) that various forms of uncertainty can percolate through the economy. For example, there is a constituency favoring the view that heightened levels of tail uncertainty can impact risk premiums in financial markets and also stem the flow of capital investment and hiring in the labor markets. Adherents of this constituency have implored the Federal Reserve to act in a timely fashion to remedy tail risk concerns through forward guidance and macroprudential policies (e.g., Financial Times (May 9, 2014), Brunnermeier and Sannikov (2014, Section 2), and Adrian and Liang (2014, Table 3)).

Do negative shifts in the disaster risk premiums portend worsening macroeconomy? We explore this question in the context of the predictive regressions below:

$$y_{\{t \rightarrow t+j\}} = \Pi_0 + \Pi \text{drp}_t[r^*] + \varepsilon_{t+j}, \quad \text{for } j = 1, 3, 6, 12 \text{ months.} \quad (25)$$

We employ the database maintained by McCracken and Ng (2015, Appendix) and initially consider the 17, 32, and 14 (properly transformed) variables categorized as output and income (Group 1); labor markets (Group 2); and consumption, orders, and inventories (Group 4), respectively. We need not worry about the non-stationary of drp_t , as the best model is ARMA(1,1), selected according to the Bayesian information criterion.

For compactness, we feature three dependent variables in our investigation: (i) log change in IP index from “output and income,” (ii) log change in all employees: total nonfarm from

“labor markets,” and (iii) level of the ISM: PMI composite index from “consumption, orders, and inventories.” We feature the variables above on the grounds that they exhibit high (absolute) correlation with the first principal component of the respective group, with an adjusted R^2 of 94%, 84%, and 67%. If $\text{drp}_t[r^*]$ is inferred at the start of the expiration cycle on 6/18/2007, then in equation (25) we are predicting $y_{\{t \rightarrow t+j\}}$ over July 2007 and beyond.

Table 6 indicates that the slope coefficients are positive and statistically significant.¹¹ For example, the Π estimate is 0.096 (the correlation is 0.35) with IP index regression over the three-month horizon with a $\text{NW}[p]$ of 0.01 and an adjusted R^2 of 12.1%. The $\text{H}[p]$ is 0.001, and is corrected for overlapping observations. Additionally, a one standard deviation decrease in disaster risk premiums raises the level of IP index by 1.92% ($400 \times 0.096 \times 0.05$) annualized. Finally, more negative drp_t are associated with higher disaster probabilities (the correlation is -0.56). If we replace the regressor drp_t with $\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$, the significance is preserved (not reported).

Our work implies that efforts by the Federal Reserve to mitigate disaster risk premiums (and tail risk concerns in the equity market) can exert a beneficial effect on economic outlook. This positive effect can last up to 12 months on the IP index and total nonfarm.

Internet Appendix (Section E) considers another application and shows that the worsening of disaster risk premiums imply higher next-period average return correlations between stocks.

4. A blind spot on the lower bound of the expected return

The second centerpiece of our efforts is to exploit our framework to address a contention (i.e., abstract, and pages 368, 383, 386, 387, 414, 415) in Martin (2017) that the lower bound on the

¹¹Reported throughout are the two-sided p -values $\text{H}[p]$ and $\text{NW}[p]$, based (i) on the Hodrick (1992) 1B covariance matrix estimator under the null of no predictability, and (ii) on the HAC estimator from Newey and West (1987), with lag ℓ^* automatically selected as in Newey and West (1994), respectively.

expected (excess) return is *approximately tight*.

Wading into the threshold question, we define the gross return as $R_{t+1} \equiv \frac{S_{t+1}}{S_t} = e^{r_{t+1}}$ and consider the conditional covariance between $\Lambda[R_{t+1}]R_{t+1}$ and R_{t+1} , as in Martin (2017):

$$\underbrace{-\text{cov}_t(\Lambda[R_{t+1}]R_{t+1}, R_{t+1})}_{\text{Conditional covariance}} = \underbrace{-\mathbb{E}_t^{\mathbb{P}}(\Lambda[R_{t+1}]R_{t+1}^2)}_{=R_{\mathbf{f},t}^{-1}\mathbb{E}_t^{\mathbb{Q}}(R_{t+1}^2)} + 1 \times \mathbb{E}_t^{\mathbb{P}}(R_{t+1}), \quad (26)$$

$$= \underbrace{-R_{\mathbf{f},t}^{-1}\text{var}_t^{\mathbb{Q}}(R_{t+1})}_{\text{Martin lower bound}} + \{\mathbb{E}_t^{\mathbb{P}}(R_{t+1} - 1) - (R_{\mathbf{f},t} - 1)\}. \quad (27)$$

With the understanding that $-\text{cov}_t(\Lambda[R_{t+1}]R_{t+1}, R_{t+1}) \geq 0$, our key step is to infer an estimate of $-\text{cov}_t(\Lambda[R_{t+1}]R_{t+1}, R_{t+1})$. Guided by our empirical considerations, we define

$$\eta_t \equiv \frac{-\text{cov}_t(\Lambda[R_{t+1}]R_{t+1}, R_{t+1})}{R_{\mathbf{f},t}^{-1}\text{var}_t^{\mathbb{Q}}(R_{t+1})} = \frac{\mathbb{E}_t^{\mathbb{P}}(R_{t+1} - 1) - (R_{\mathbf{f},t} - 1)}{R_{\mathbf{f},t}^{-1}\text{var}_t^{\mathbb{Q}}(R_{t+1})} - 1. \quad (28)$$

For the conditional lower bound on the expected (excess) return of the market to be approximately tight, it must hold that $\mathbb{E}_t^{\mathbb{P}}(R_{t+1} - 1) - (R_{\mathbf{f},t} - 1) \approx R_{\mathbf{f},t}^{-1}\text{var}_t^{\mathbb{Q}}(R_{t+1})$. This entails the restriction, from equation (28), that

$$\eta_t \text{ is a small nonnegative number close to zero (point-by-point)}. \quad (29)$$

Using the fact that η_t is identically zero when $\Lambda[R_{t+1}] = \left(\frac{S_{t+1}}{S_t}\right)^{-1} = \frac{1}{R_{t+1}}$ (since $\text{cov}_t\left(\frac{1}{R_{t+1}}R_{t+1}, R_{t+1}\right) = 0$), we can test whether $\eta_t = 0$ by employing candidate SDF specifications that nest $\Lambda[R_{t+1}] = \frac{1}{R_{t+1}}$ as a special case. It suffices to establish a counterexample that falsifies the contention of an approximately tight conditional lower bound on the expected (excess) return of the market.

To provide point estimates of η_t for each date t and to test whether the restriction $\eta_t = 0$ is supported in the data from options market, we proceed as follows:

- (a) Infer $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})$, as in Martin (2017, page 381), via $\frac{2}{S_t^2} \int_{K < R_{f,t} S_t} P_t[K] dK + \frac{2}{S_t^2} \int_{K > R_{f,t} S_t} C_t[K] dK$.
- (b) Next, we consider $\Lambda[R_{t+1}] = (\frac{S_{t+1}}{S_t})^{-\gamma} = R_{t+1}^{-\gamma}$ and apply our approach to compute the conditional expectation of the market return, $\mathbb{E}_t^{\mathbb{P}}(R_{t+1} - 1)$, using option prices. This involves computing

$$\mathbb{E}_t^{\mathbb{P}}\left(\frac{S_{t+1}}{S_t} - 1\right) = 1 - R_{f,t}^{-1} + \int_{K < S_t} w[K] P_t[K] dK + \int_{K > S_t} w[K] C_t[K] dK, \quad (30)$$

where $w[K] = \gamma K^{-3} (\frac{K}{S_t})^{\gamma+1} \{\gamma K + K - \gamma S_t + S_t\}$. The SDF $\Lambda[S_{t+1}] = (\frac{S_{t+1}}{S_t})^{-\gamma}$ is monotonically declining and convex in S_{t+1} ($-\Lambda'[S_{t+1}]/\Lambda[S_{t+1}] = \gamma$ is a constant) and subsumed within the example arguments of Martin (2017, Examples 3a and 3b, pages 375–377).

- (c) Construct the time-series of η_t , relying on equation (28) while setting γ equal to 2.93 (as in Table 1).
- (d) Perform the regression $\eta_t = \bar{\eta} + e_t$. The empirical test of whether the intercept coefficient $\bar{\eta} = 0$ is equivalent to assessing whether the unconditional mean of η_t is zero.

[Fig. 3 about here.]

Table 7 presents our findings and Figure 3 plots the time-series of η_t . Our evidence shows that the average value of η_t is 1.76 and is statistically different from zero with the Newey and West (1987) two-sided p -value of 0.000. Additionally, the value of $\eta_t = 0$ is never achieved at any date t with the minimum of 1.37, and a maximum of 1.94. The Wald test for the hypothesis $\bar{\eta} = 1$ is also rejected with a two-sided p -value of 0.000, affirming that $-\text{cov}_t(\Lambda[R_{t+1}]R_{t+1}, R_{t+1})$ is, on average, at least as big as the Martin lower bound of $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})$.

While other $\gamma > 1$ can be used to infer the size of $-\text{cov}_t(\Lambda[R_{t+1}]R_{t+1}, R_{t+1})$, we feature γ estimate of 2.93, because Table 2 shows that estimates of conditional return variance based on

our approach provide an unbiased forecast of realized return variance. Furthermore, $\gamma = 1$ does not bracket the 95% bootstrap confidence interval of [1.60 4.27].

Remark 6 In view of η_t and $\mathbb{E}_t^{\mathbb{P}}(\frac{S_{t+1}}{S_t} - 1)$ in equations (28) and (30), we further observe that $\eta_t|_{\gamma=1} = 0$. Thus, we evaluate $\frac{d\eta_t}{d\gamma}|_{\gamma=1} = \frac{1}{R_{t,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})} (\int_{K < S_t} \ell[K] P_t[K] dK + \int_{K > S_t} \ell[K] C_t[K] dK)$, where $\ell[K] \equiv \frac{dw[K;\gamma]}{d\gamma}|_{\gamma=1} = \frac{K-S_t}{KS_t^2} + \frac{2\log(\frac{K}{S_t})}{S_t^2} + \frac{2}{S_t}$. The derivative $\frac{d\eta_t}{d\gamma}|_{\gamma=1}$ is positive at each t in our implementation, implying that a higher γ raises η_t .

The analytical tractability of our counterexample reveals that $-\text{cov}_t(\Lambda[R_{t+1}]R_{t+1}, R_{t+1})$ is not approximately zero with the considered SDF specification. Putting the theoretical and empirical pieces together, our evidence shows that the conditional lower bound on the expected (excess) return of the market is *not* approximately tight, countering a contention in Martin (2017).

5. Characterizing conditional return asymmetries

The third, and final, centerpiece of our efforts is the development of an ex-ante measure of conditional return asymmetries under the physical probability measure. Consider thresholds (K^d , K^u), centered around S_t , satisfying $\log(\frac{K^d}{S_t}) = r^* \leq 0$ and $\log(\frac{S_t}{K^u}) = r^* \leq 0$ and define (e.g., Feunou, Jahan-Parvar, and Tedongap (2016, Section 2)):

$$\text{Asymmetry}_t^{\mathbb{P}}[K^d, K^u] \equiv \frac{\mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2 \theta[S_{t+1} - K^u]) - \mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2 \theta[K^d - S_{t+1}])}{\mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2)}. \quad (31)$$

Germane to capturing the semivariances in equation (31), we consider

$$f[S_{t+1}] = \underbrace{r_{t+1}^2 \theta[S_{t+1} - K^u]}_{\equiv I[S_{t+1}]} - \underbrace{r_{t+1}^2 \theta[K^d - S_{t+1}]}_{\equiv J[S_{t+1}]}, \quad (32)$$

$$\text{and correspondingly } \hat{I}[S_{t+1}] \equiv \frac{I[S_{t+1}]}{\Lambda[S_{t+1}]} \text{ and } \hat{J}[S_{t+1}] \equiv \frac{J[S_{t+1}]}{\Lambda[S_{t+1}]} \quad (33)$$

The following calculation is revealing about conditional return asymmetries:

$$\text{Asymmetry}_t^{\mathbb{P}}[K^d, K^u] = \frac{R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}} \left(\widehat{I}[S_{t+1}] - \widehat{J}[S_{t+1}] \right)}{R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}} \left(\left\{ \log \left(\frac{S_{t+1}}{S_t} \right) \right\}^2 / \Lambda[S_{t+1}] \right)}. \quad (34)$$

Under the specification that $\Lambda[S_{t+1}] = \left(\frac{S_{t+1}}{S_t} \right)^{-\gamma}$, we consider a set of calculations for the expectation of the Heaviside function $\theta[S_{t+1} - K^u]$, that reflect the upside probabilities, shown in the Online Appendix (Section F).

The numerator in equation (34) admits the representation

$$\begin{aligned} R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}} \left(\widehat{I}[S_{t+1}] - \widehat{J}[S_{t+1}] \right) &= \left\{ \log \left(\frac{K^u}{S_t} \right) \right\}^2 \left(\frac{K^u}{S_t} \right)^{\gamma} \mathbb{C}_t[K^u] - \left\{ \log \left(\frac{K^d}{S_t} \right) \right\}^2 \left(\frac{K^d}{S_t} \right)^{\gamma} \mathbb{B}_t[K^d] \\ &+ \widehat{I}[K^u] \mathbb{C}_t[K^u] + \widehat{J}[K^d] \mathbb{P}_t[K^d] \\ &+ \int_{K > K^u} a[K] \mathbb{C}_t[K] dK - \int_{K < K^d} a[K] \mathbb{P}_t[K] dK, \end{aligned} \quad (35)$$

where $\mathbb{C}_t[K^u]$ ($\mathbb{B}_t[K^d]$) is the price of the binary call (put). The price of the binary call is

$$\mathbb{C}_t[K^u] = R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}} \left(\theta[S_{t+1} - K^u] \right) \approx \frac{C[K^u - \Delta K] - C[K^u + \Delta K]}{2\Delta K}, \quad (36)$$

$$\widehat{I}[K^u] = \frac{\left(\frac{K^u}{S_t} \right)^{\gamma} \log \left(\frac{K^u}{S_t} \right) \left(\gamma \log \left(\frac{K^u}{S_t} \right) + 2 \right)}{K^u}, \quad \text{and} \quad (37)$$

$$\widehat{J}[K^d] = \frac{\left(\frac{K^d}{S_t} \right)^{\gamma} \log \left(\frac{K^d}{S_t} \right) \left(\gamma \log \left(\frac{K^d}{S_t} \right) + 2 \right)}{K^d}. \quad (38)$$

The denominator in equation (34), i.e., $R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}} \left(\left\{ \log \left(\frac{S_{t+1}}{S_t} \right) \right\}^2 / \Lambda[S_{t+1}] \right)$ is presented in equations (20) and (21), while $a[K]$ is displayed in equation (22).¹²

While alternative measures of conditional asymmetries have been adopted by, among others,

¹²We note that when $r^* = 0$ and, hence, $K^u = K^d = S_t$, the first four terms in equation (35) are zero, and the numerator for $\text{Asymmetry}_t^{\mathbb{P}}$ becomes $R_{\mathbf{f},t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}} \left(\widehat{I}[S_{t+1}] - \widehat{J}[S_{t+1}] \right) = \int_{K > S_t} a[K] \mathbb{C}_t[K] dK - \int_{K < S_t} a[K] \mathbb{P}_t[K] dK$.

Harvey and Siddique (2000), Dittmar (2002), and Ghysels, Plazzi, and Valkanov (2016) to understand economic phenomena, we explore an application where $\text{Asymmetry}_t^{\mathbb{P}}[K^d, K^u]$ helps to predict variance of the VIX index over the option expiration cycles, computed as

$$\text{vov}_{\{t \rightarrow t+1\}} = z_{t+\Delta, \Delta}^2 + z_{t+2\Delta, \Delta}^2 + \dots + z_{t+h\Delta, \Delta}^2, \text{ where } z_{t+h\Delta, \Delta} \equiv \log(\text{VIX}_{t+h\Delta} / \text{VIX}_{t+(h-1)\Delta}).$$

The predictor $\text{Asymmetry}_t^{\mathbb{P}}[K^d, K^u]$ is forward looking.

Setting the playing field for our analysis, we let $Z_{t+1} = \log(\text{vov}_{\{t \rightarrow t+1\}})$ and examine ARMAX models of the type:

$$Z_{t+1} = \phi_0 + \phi \text{Asymmetry}_t^{\mathbb{P}}[K^d, K^u] + \sum_{i=1}^p \vartheta_i Z_{t-i+1} + \sum_{j=1}^q \xi_j \varepsilon_{t-j+1} + \varepsilon_{t+1}, \quad (39)$$

for $p \leq 3$ and $q \leq 3$. Such models are suitable for investigating the predictability of persistent variables, as its own past values may contain predictive content (e.g., Stock and Watson (2003)).

Our approach incorporates an $\text{ARMA}(p, q)$ structure and then selects the best of these models using the Bayesian information criterion. This turns out to be an AR-1 model. Our interest lies in whether the slope coefficient ϕ is statistically significant beyond the own past values of the predicted variable. We consider two time-series of $\text{Asymmetry}_t^{\mathbb{P}}[K^d, K^u]$ that respectively set $r^* = (\log(\frac{K^d}{S_t}), \log(\frac{S_t}{K^u}))$ equal to 0.00 or -0.01 .

The conclusion to draw from Table 8 is that lower values of $\text{Asymmetry}_t^{\mathbb{P}}$ are significantly associated with higher $\log(\text{vov}_{t \rightarrow t+1})$. The ϕ estimates are negative and statistically significant. Our focus on forecasting $\log(\text{vov}_{t \rightarrow t+1})$ is guided by its relative novelty, and is complementary to the interests of Huang and Shaliastovich (2014), Song (2014), and Park (2015).

6. Concluding remarks

The idea of this paper is to present an approach that inverts the Girsanov theorem to derive representations for the conditional expectation of generic functions of the market return. Intrinsic to the approach is the link between inferring the risk-neutral probability measure from option prices, while accommodating the form of the stochastic discount factor. Developed in a discrete-time environment, our method for obtaining conditional expectations is ingrained in established theory and is tractable for a broad class of functions of market return.

Our theoretical enrichments differ from approaches that start with assumptions about both preferences and primitives to propose analytical relations between the price of the contingent claims and the underlying primitive state variables and their parameterizations. We illustrate the workings of the method using option prices on the S&P 500 index and in the context of conditional probability of wealth disasters, conditional expectation of return moments, upside and downside semivariances, and conditional asymmetries. Our work squares well with research that assigns a role to the information in traded options prices and their connections to economic phenomena.

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Table 1

Estimation results and the derived values of conditional expected (log) return, conditional volatility, and conditional variance risk premium

Reported results are based on $\Lambda[S_{t+1}; \gamma] = (\frac{S_{t+1}}{S_t})^{-\gamma} = e^{-\gamma r_{t+1}}$, for $r_{t+1} = \log(\frac{S_{t+1}}{S_t})$, and the analytical representation of conditional return variance $\text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma)$ (as in equation (19)). The estimation of γ entails

$$\hat{\gamma} \equiv \arg \min_{\gamma} \sum_{t=1}^T \left(\text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma) - \text{rv}_{\{t \rightarrow t+1\}} \right)^2,$$

where $\text{rv}_{\{t \rightarrow t+1\}}$ is the realized return variance, as calculated using equation (18). To obtain the bootstrap distribution of γ , we perform an i.i.d. bootstrap on daily returns and generate the time-series of realized variance $\text{rv}_{\{t \rightarrow t+1\}}^b$, $b = 1, \dots, 1000$. In each bootstrap iteration, we estimate γ , yielding the reported mean, standard deviation, lower and upper bootstrap confidence intervals, and percentiles. We additionally report the bootstrapped values of $\mathbb{E}_t^{\mathbb{P}}(r_{t+1})$ (annualized, in decimals), $\sqrt{\text{var}_t^{\mathbb{P}}}$ (annualized, in decimals), and the variance risk premium vrp_t . We compute

$$\text{vrp}_t \equiv \log(\text{var}_t^{\mathbb{P}}(r_{t+1})) - \log(\text{var}_t^{\mathbb{Q}}(r_{t+1})),$$

where $\text{var}_t^{\mathbb{P}}(r_{t+1})$ is obtained from equations (20)–(22) and $\text{var}_t^{\mathbb{Q}}(r_{t+1})$ is inferred from call and put option prices (e.g., Bakshi, Kapadia, and Madan (2003, Section 1.2 and equation (7))).

Across the 1,000 bootstraps

	Mean	Std.	95% CI [Lower Upper]	5th	25th	50th	75th	95th
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Panel A: Estimates of γ when $\Lambda[S_{t+1}; \gamma] = (\frac{S_{t+1}}{S_t})^{-\gamma}$

	Estimated								
γ	2.93	2.91	0.68	[1.60 4.27]	1.69	2.44	2.92	3.39	4.01

Panel B: Conditional expected (log) return, volatility, and variance risk premium

	Average								
$\mathbb{E}_t^{\mathbb{P}}(r_{t+1})$	0.143	0.142	0.029	[0.087 0.200]	0.090	0.123	0.143	0.162	0.188
$\sqrt{\text{var}_t^{\mathbb{P}}}$	0.182	0.182	0.005	[0.172 0.191]	0.174	0.178	0.182	0.185	0.191
vrp_t	-0.235	-0.232	0.049	[-0.331 -0.139]	-0.309	-0.267	-0.234	-0.199	-0.143

Table 2

Goodness of fit metrics and forecast accuracy

Reported are the results from the following Mincer-Zarnowitz regression

$$rv_{\{t \rightarrow t+1\}} = \alpha + \beta \text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma) + \varepsilon_{t+1}.$$

The realized return variance $rv_{\{t \rightarrow t+1\}}$ is calculated as the sum of daily squared returns:

$$rv_{\{t \rightarrow t+1\}} = r_{t+\Delta, \Delta}^2 + r_{t+2\Delta, \Delta}^2 + \dots + r_{t+h\Delta, \Delta}^2, \quad \text{with } r_{t+h\Delta, \Delta} \equiv \log\left(\frac{S_{t+h\Delta}}{S_{t+(h-1)\Delta}}\right),$$

where Δ is one day and h is the number of trading days over an options expiration cycle. To correct for autocorrelation and heteroscedasticity, we use the Newey and West (1987) estimator with automatically selected lag l^* (in our case $l^* = 7$), as in Newey and West (1994), and the reported two-sided p -values are denoted by $NW[p]$. We also report the p -value corresponding to the Wald test of the hypothesis of $\beta = 1$. The adjusted R^2 is reported as \bar{R}^2 . The p -value for the MSPE statistic is computed by regressing f_{t+1} on a constant, where

$$f_{t+1} = (rv_{\{t \rightarrow t+1\}} - \bar{rv}_{\{t \rightarrow t+1\}})^2 - [(rv_{\{t \rightarrow t+1\}} - \hat{rv}_{\{t \rightarrow t+1\}})^2 - (\hat{rv}_{\{t \rightarrow t+1\}} - \bar{rv}_{\{t \rightarrow t+1\}})^2].$$

$\hat{rv}_{\{t \rightarrow t+1\}}$ is an estimate of $rv_{\{t \rightarrow t+1\}}$, using information up to period t , based on equation (24), and $\bar{rv}_{\{t \rightarrow t+1\}}$ is obtained from a restricted model imposing $\beta = 0$. Low one-sided p -values associated with the adjusted MSPE test statistic indicate that $\text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma)$ generates additional predictive power. The third term is an adjustment that leads to better small-sample properties of the test.

The out-of-sample R_{OOS}^2 is calculated as $R_{\text{OOS}}^2 \equiv 1 - \frac{\sum_{t=0}^{T-1} (rv_{\{t \rightarrow t+1\}} - \hat{rv}_{\{t \rightarrow t+1\}})^2}{\sum_{t=0}^{T-1} (rv_{\{t \rightarrow t+1\}} - \bar{rv}_{\{t \rightarrow t+1\}})^2}$.

α	NW[p]	β	NW[p] $\beta = 0$	Wald		Out-of-sample	
				NW[p] $\beta = 1$	\bar{R}^2 (%)	MSPE[p]	R_{OOS}^2 (%)
-0.00048	0.28	1.07	0.00	0.76	41.0	0.004	37.2

Table 3

Disaster probabilities under the physical measure and the associated disaster risk premiums

The reported results for the disaster probabilities rely on equation (10). Specifically, when $\Lambda[S_{t+1};\gamma] = (\frac{S_{t+1}}{S_t})^{-\gamma}$, we determine

$$\mathbb{E}_t^{\mathbb{P}}(\theta[K^* - S_{t+1}]) = \left(\frac{K^*}{S_t}\right)^\gamma \mathbb{B}_t[K^*] + \frac{\gamma(\gamma-1)}{S_t^2} \int_{K < K^*} \left(\frac{K}{S_t}\right)^{\gamma-2} P_t[K] dK - \frac{\gamma}{S_t} \left(\frac{K^*}{S_t}\right)^{\gamma-1} P_t[K^*].$$

The calculation of the disaster risk premiums uses equation (12), that is,

$$\text{drp}_t[r^*] \equiv \log(1 + \mu_t[r^*]) - \log(R_{f,t+1}) = \log\left(\frac{\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])}{\mathbb{E}_t^{\mathbb{Q}}(\theta[r^* - r_{t+1}])}\right).$$

The wealth disasters of size 5% (7%) correspond to r^* of -0.05129 (-0.07257) over the expiration cycles of the S&P 500 index options. The results are reported across the 1,000 bootstraps (as explained in Table 1).

Across the 1,000 bootstraps

Average	Mean	Std.	95% CI [Lower Upper]	5th	25th	50th	75th	95th
Panel A: Wealth disaster of size 5% (311 expiration cycles, 1/22/1990–11/23/2015)								
$\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$	0.092	0.093	0.006 [0.081 0.104]	0.083	0.088	0.092	0.096	0.104
$\text{drp}_t[r^*]$	-0.284	-0.281	0.062 [-0.403 -0.159]	-0.380	-0.325	-0.283	-0.239	-0.169
Panel B: Wealth disaster of size 7% (223 expiration cycles, 5/19/1997–11/23/2015)								
$\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$	0.060	0.061	0.005 [0.051 0.071]	0.053	0.057	0.060	0.064	0.071
$\text{drp}_t[r^*]$	-0.372	-0.367	0.082 [-0.527 -0.207]	-0.497	-0.425	-0.371	-0.312	-0.221

Table 4

Differences in generated values across the specifications of $\Lambda[S_{t+1}]$

This table compares the values of disaster probabilities, conditional expected return, and conditional volatility across the different $\Lambda[S_{t+1}]$, as in Cases 1, 2, and 3 and Table Internet Appendix-I and Table Internet Appendix-II. The procedure is as follows. First, we set the random number seed in the resampled dataset of daily returns. Second, for each bootstrap draw, we compute the parameter vector Θ across each of the three $\Lambda[S_{t+1}]$, allowing us to generate the objects of interest. We repeat this exercise across 1,000 trials. Reported are the average differences multiplied by 100 (in bold). The 95% lower and upper bootstrap confidence intervals are displayed in square brackets. In our illustrations, we set $r^* = -0.05129$.

	Case 1 vs.		Case 2 ($\lambda = 1$) vs.		Case 2 ($\lambda = 0.8$) vs.
Case 2 ($\lambda = 1$)	Case 2 ($\lambda = 0.8$)	Case 3	Case 2 ($\lambda = 0.8$)	Case 3	Case 3
<i>Differences in disaster probabilities (multiplied by 100)</i>					
-0.3 [-0.4 -0.2]	-0.2 [-0.3 -0.2]	0.7 [0.1 1.2]	0.1 [0.0 0.1]	1.0 [0.4 1.5]	0.9 [0.3 1.5]
<i>Differences in conditional expected return (annualized, %)</i>					
1.9 [1.6 2.2]	1.5 [1.4 1.7]	-0.9 [-5.0 3.2]	-0.4 [-0.5 -0.3]	-2.8 [-7.1 1.4]	-2.5 [-6.6 1.7]
<i>Differences in conditional volatility (annualized, %)</i>					
-0.1 [-0.2 0.0]	-0.1 [-0.2 0.0]	0.4 [-0.1 1.0]	0.00 [0.0 0.1]	0.5 [0.0 1.1]	0.5 [0.0 1.0]

Table 5

Specification errors of $\Lambda[r_{t+1}]$ using the Hansen-Jagannathan (1997) distance measure

We evaluate the ability of an SDF, $\Lambda[r_{t+1}]$, to price eight assets (or a subset of them):

- Gross return of the risk-free bond ($R_{\mathbf{f},t+1}$);
- Gross return of the equity market index ($R_{t+1} = \frac{S_{t+1}}{S_t}$);
- Gross return of the $\ell\%$ out-of-the-money (OTM) put option, calculated as $R_{t+1,\ell\%,\text{put}} \equiv \max(S_t e^{-\ell} - S_{t+1}, 0) / P_t[S_t e^{-\ell}]$;
- Gross return of the $\ell\%$ out-of-the-money call option, calculated as $R_{t+1,\ell\%,\text{call}} \equiv \max(S_{t+1} - S_t e^{+\ell}, 0) / C_t[S_t e^{+\ell}]$.

Thus, the gross return vector employed in our calculations is defined as follows:

$$\mathbf{R}_{t+1} = \begin{pmatrix} R_{\mathbf{f},t+1} \\ R_{t+1} \\ R_{t+1,5\%,\text{put}} \\ R_{t+1,3\%,\text{put}} \\ R_{t+1,1\%,\text{put}} \\ R_{t+1,1\%,\text{call}} \\ R_{t+1,3\%,\text{call}} \\ R_{t+1,5\%,\text{call}} \end{pmatrix}.$$

We report the Hansen and Jagannathan (1997) distance as

$$\text{HJ}^{\text{dist}} \equiv \sqrt{\mathbf{e}'_{t+1} \{ \mathbb{E}(\mathbf{R}_{t+1} \mathbf{R}'_{t+1}) \}^{-1} \mathbf{e}_{t+1}} \quad \text{where,} \quad \mathbf{e}_{t+1} \equiv \mathbb{E}(\Lambda[r_{t+1}] \mathbf{R}_{t+1}) - \mathbf{1}.$$

The reported lower and upper bootstrap confidence intervals are based on an i.i.d. bootstrap. There are 311 options expiration cycles over the sample period of January 1990 to December 2015.

	All eight assets			Excluding 5% OTM options			Excluding calls					
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3			
	$\lambda = 1$	$\lambda = 0.8$		$\lambda = 1$	$\lambda = 0.8$		$\lambda = 1$	$\lambda = 0.8$				
HJ ^{dist}	0.445	0.445	0.445	0.452	0.278	0.279	0.278	0.288	0.410	0.411	0.410	0.420
Lower 95% CI	0.318	0.317	0.316	0.329	0.171	0.170	0.170	0.186	0.280	0.276	0.278	0.292
Upper 95% CI	0.816	0.817	0.815	0.818	0.542	0.545	0.546	0.546	0.756	0.761	0.760	0.761

Table 6

Disaster risk premiums and subsequent output, unemployment, and orders/inventories

We report the results from the predictive regressions with $r^* = -0.05129$:

$$y_{\{t \rightarrow t+j\}} = \Pi_0 + \Pi \text{drp}_t[r^*] + \varepsilon_{t+1}, \quad \text{for } j = 1, 3, 6, 12 \text{ months.}$$

In our specification, we take the dependent variable, $y_{\{t \rightarrow t+j\}}$, to be (i) the log change in the IP index (FRED mnemonic, INDPRO) from “output and income,” (ii) the log change in all employees: total nonfarm (FRED mnemonic, PAYEMS) from “labor markets,” and (iii) the level of ISM: PMI composite index (FRED mnemonic, NAPM) from “orders and inventories.” Each of the three featured variables exhibit high (absolute) correlation with the first principal component of the respective group with an R^2 of 94%, 84%, and 67%. The data is described in McCracken and Ng (2015, Appendix). To correct for autocorrelation and heteroscedasticity, we use the Newey and West (1987) estimator with automatically selected lag, denoted by ℓ^* , as in Newey and West (1994), and the reported two-sided p -values are denoted by $\text{NW}[p]$. Shown also are Hodrick (1992) two-sided p -values, denoted by $\text{H}[p]$. The Hodrick p -values are adjusted for overlapping observations when $j = 3$, $j = 6$, and $j = 12$. The adjusted R^2 is reported as \bar{R}^2 . The pairwise correlation coefficient between the dependent variable and drp_t is denoted by CORR . The intercept Π_0 is not reported. The computation of disaster risk premiums are based on the expression for disaster probabilities in equation (13).

$y_{\{t \rightarrow t+j\}}$	j	Slope coefficient			\bar{R}^2 (%)	ℓ^*	CORR
		Π	$\text{NW}[p]$	$\text{H}[p]$			
Log change of IP index (INDPRO)	1	0.033	0.015	0.003	6.5	5	0.26
	3	0.096	0.010	0.001	12.1	4	0.35
	6	0.133	0.012	0.002	7.2	2	0.27
	12	0.165	0.003	0.007	3.7	5	0.20
Log change of total nonfarm (PAYEMS)	1	0.015	0.004	0.000	21.1	10	0.46
	3	0.044	0.005	0.000	22.6	10	0.48
	6	0.083	0.004	0.000	21.1	10	0.46
	12	0.137	0.001	0.000	16.1	9	0.40
ISM index (NAPM)	1	39.822	0.005	0.000	16.1	11	0.40
	3	33.445	0.009	0.000	11.2	10	0.34
	6	16.030	0.072	0.007	2.3	7	0.16
	12	1.564	0.869	0.747	-0.00	12	0.02

Table 7

Tightness of the lower bound on the expected (excess) return of market

Reported are the values corresponding to

$$\eta_t \equiv \frac{\mathbb{E}_t^{\mathbb{P}}(R_{t+1} - 1) - (R_{f,t} - 1)}{R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})} - 1.$$

The lower bound on the expected (excess) return of the market is approximately tight if $\mathbb{E}_t^{\mathbb{P}}(R_{t+1} - 1) - (R_{f,t} - 1) \approx R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})$. Hence, the testable restriction is that

η_t is a small positive number close to zero (point-by-point).

We consider the null hypothesis that $\eta_t = 0$, and we proceed as follows:

- Infer $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})$ (as in Martin (2017)) via $\frac{2}{S_t^2} \int_{K < F_{t,T}} P_t[K] dK + \frac{2}{S_t^2} \int_{K > F_{t,T}} C_t[K] dK$.
- We consider $\Lambda[R_{t+1}] = (\frac{S_{t+1}}{S_t})^{-\gamma}$ and set $\gamma = 2.93$. In this case, we apply our approach to compute the conditional expectation of the market return using option prices and without imposing any distributional assumptions. This involves computing

$$\mathbb{E}_t^{\mathbb{P}}\left(\frac{S_{t+1}}{S_t} - 1\right) = 1 - R_{f,t}^{-1} + \int_{K < S_t} w[K] P_t[K] dK + \int_{K > S_t} w[K] C_t[K] dK,$$

where $w[K] = \gamma K^{-3} (\frac{K}{S_t})^{\gamma+1} \{\gamma K + K - \gamma S_t + S_t\}$.

- For each t , construct η_t from option prices, relying on the expressions in equation (28).
- Perform the regression $\eta_t = \bar{\eta} + e_t$. The test of whether the intercept coefficient $\bar{\eta} = 0$ is equivalent to assessing whether the unconditional mean of η_t is zero. We also do the Wald test of $\bar{\eta} = 1$.

To correct for autocorrelation and heteroscedasticity, we use the Newey and West (1987) estimator with automatically selected lag, as in Newey and West (1994), and the reported two-sided p -values are denoted by $\text{NW}[p]$. The reported $\mathbb{E}_t^{\mathbb{P}}(R_{t+1} - R_{f,t})$, $R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})$, and $\sqrt{\text{var}_t^{\mathbb{Q}}(R_{t+1})}$ are annualized (in decimals).

	Mean	Std.	5th	50th	95th	Min.	Max.	$\eta_t = \bar{\eta} + e_t$	
								$\text{NW}[p]$ $\bar{\eta} = 0$	$\text{NW}[p]$ $\bar{\eta} = 1$
η_t	1.76	0.07	1.64	1.77	1.85	1.37	1.94	0.00	0.00
$\mathbb{E}_t^{\mathbb{P}}(R_{t+1} - R_{f,t})$	0.129	0.139	0.025	0.085	0.373	0.016	1.186		
$R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})$	0.047	0.052	0.009	0.031	0.136	0.006	0.445		
$\sqrt{\text{var}_t^{\mathbb{Q}}(R_{t+1})}$	0.197	0.092	0.095	0.177	0.37	0.075	0.667		

Table 8

Predicting realized variance of variance with the conditional asymmetry measure

The predicted variable is the log of the realized variance of variance, denoted by $Z_{t+1} \equiv \log(\text{vov}_{\{t \rightarrow t+1\}})$, with

$$\text{vov}_{\{t \rightarrow t+1\}} = z_{t+\Delta, \Delta}^2 + z_{t+2\Delta, \Delta}^2 + \dots + z_{t+h\Delta, \Delta}^2, \text{ where } z_{t+h\Delta, \Delta} \equiv \log(\text{VIX}_{t+h\Delta} / \text{VIX}_{t+(h-1)\Delta}).$$

We consider results with the ARMAX specification

$$Z_{t+1} = \phi_0 + \phi \text{Asymmetry}_t^{\mathbb{P}}[K^d, K^u] + \sum_{i=1}^p \vartheta_i Z_{t-i+1} + \sum_{j=1}^q \xi_j \varepsilon_{t-j+1} + \varepsilon_{t+1}.$$

We use maximum likelihood to fit all ARMAX models with $p \leq 3$ and $q \leq 3$ and show results for the best model, selected according to the Bayesian information criterion. This model turned out to be AR-1. Reported are the coefficient estimates and their two-sided p -values (in square brackets). The expression of conditional asymmetry, $\text{Asymmetry}_t^{\mathbb{P}}[K^d, K^u]$, is as per the calculation in equation (34). Here $r^* = (\log(\frac{K^d}{S_t}), \log(\frac{S_t}{K^u}))$ equals either 0.00 or -0.01 .

r^*	0.00	-0.01
ϕ_0	-2.151	-2.056
p -value	[0.000]	[0.000]
ϕ	-0.471	-0.716
p -value	[0.018]	[0.010]
AR-1 coef.	0.272	0.247
p -value	[0.000]	[0.000]

An Approach to Measure the Expectation of Generic Functions of the Market Return

Internet Appendix: Not for Publication

Abstract

Section A provides the proof of the analytical representation of the real world probability of wealth disasters in equation (10), while Section B provides the proof of the expression for the binary put option price in equation (11). Section C describes the data on interest rates and options on the S&P 500 index. Section D presents the expressions for the conditional return variance. Section E provides an additional empirical exercise, where we show that forward looking disaster risk premiums are useful for predicting equicorrelation between stocks. Finally, Section F develops the conditional expectation of the Heaviside function to the upside of the market return (i.e., $\theta[S_{t+1} - K^u]$, where the threshold K^u satisfies $\frac{S_t}{K^u} - 1 < 0$).

I. Internet Appendix

A. Representation of the real world probability of disasters in equation (10)

For the proof to follow¹, we make our analysis self-contained and clarify some intermediate steps in Carr and Madan (2002, Appendix 1, page 20).

For an SDF, $\Lambda[S_{t+1}]$, that is strictly positive and twice-continuously differentiable, we consider the function

$$g[S_{t+1}] = \frac{\theta[K^* - S_{t+1}]}{\Lambda[S_{t+1}]}, \quad (\text{A1})$$

where we recall that $\theta[x]$ is the Heaviside theta function:

$$\theta[x] \equiv \frac{d}{dx} \max(x, 0) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

The derivative of $\theta[x]$, denoted by $\theta'[x]$, is the Dirac delta function:

$$\delta[x] = \theta'[x], \quad \text{implying} \quad \theta[x] = \int_{-\infty}^x \delta[s] ds, \quad (\text{A2})$$

where the Dirac delta function, $\delta[x]$, is defined as

$$\delta[x] = \begin{cases} 0 & x < 0 \\ +\infty & x = 0. \\ 0 & x > 0 \end{cases} \quad (\text{A3})$$

¹In his private communications with us, Ngoc-Khanh Tran provided an alternative way to think about the proof and suggested refinements that led to many improvements.

$\delta[x]$ satisfies $1 = \int_{-\infty}^{+\infty} \delta[x] dx$. The sifting property of the delta function asserts that

$$\int_{-\infty}^{+\infty} g[x] \delta[x - x_0] dx = g[x_0]. \quad (\text{A4})$$

For concreteness, suppose $x = K - S_{t+1}$. Then

$$\theta[K - S_{t+1}] = \frac{d}{dK} \max(K - S_{t+1}, 0) \quad \text{and} \quad (\text{A5})$$

$$\delta[K - S_{t+1}] = \frac{d}{dK} \theta[K - S_{t+1}] = \frac{d^2}{dK^2} \{\max(K - S_{t+1}, 0)\}. \quad (\text{A6})$$

Now consider a test function $g[S_{t+1}]$. By the sifting property in equation (A4):

$$g[S_{t+1}] = \int_0^{+\infty} g[K] \delta[K - S_{t+1}] dK, \quad (\text{A7})$$

$$= \int_0^{+\infty} g[K] \delta[S_{t+1} - K] dK, \quad (\text{symmetry } \delta[x] = \delta[-x]) \quad (\text{A8})$$

$$= \int_0^{K^*} g[K] \delta[K - S_{t+1}] dK + \underbrace{\int_{K^*}^{+\infty} g[K] \delta[S_{t+1} - K] dK}_{=0, \text{ for claims with } S_{t+1} > K^*}, \quad (\text{A9})$$

$$= \int_0^{K^*} g[K] d\theta[K - S_{t+1}] \quad (\text{since } \delta[K - S_{t+1}] dK = d\theta[K - S_{t+1}]). \quad (\text{A10})$$

Setting $u = g[K]$ and $v = \theta[K - S_{t+1}]$, integration by parts (i.e., $\int u dv = uv - \int v du$) implies that

$$g[S_{t+1}] = \int_0^{K^*} g[K] d\theta[K - S_{t+1}], \quad (\text{A11})$$

$$= g[K] \theta[K - S_{t+1}] \Big|_0^{K^*} - \int_0^{K^*} \theta[K - S_{t+1}] g'[K] dK. \quad (\text{A12})$$

Substituting $u = g'[K]$ and $v = \max(K - S_{t+1}, 0)$, a repeated application of integration by parts to

$\int_0^{K^*} \theta[K - S_{t+1}] g'[K] dK$ implies that

$$\int_0^{K^*} \underbrace{g'[K]}_u \underbrace{\theta[K - S_{t+1}] dK}_{dv} = g'[K] \max(K - S_{t+1}, 0) \Big|_0^{K^*} - \int_0^{K^*} \max(K - S_{t+1}, 0) g''[K] dK. \quad (\text{A13})$$

The steps above allow us to write $g[S_{t+1}] = \int_0^{K^*} g[K] \delta[K - S_{t+1}] dK$ as

$$g[S_{t+1}] = g[K] \theta[K - S_{t+1}] \Big|_0^{K^*} - g'[K] \max(K - S_{t+1}, 0) \Big|_0^{K^*} + \int_0^{K^*} \max(K - S_{t+1}, 0) g''[K] dK. \quad (\text{A14})$$

Now suppose $\Lambda[S_{t+1}] = \left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}$. Then, using equation (F9), and the calculus of singularity functions (e.g., Beatty (1986, Chapter 1.9.3.3)), we conclude that

$$g[S_{t+1}] = \frac{\theta[K^* - S_{t+1}]}{\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}} = \left(\frac{K^*}{S_t}\right)^\gamma \theta[K^* - S_{t+1}] - \frac{\gamma}{S_t} \left(\frac{K^*}{S_t}\right)^{\gamma-1} \max(K^* - S_{t+1}, 0) + \frac{\gamma(\gamma-1)}{S_t^2} \int_{K < K^*} \left(\frac{K}{S_t}\right)^{\gamma-2} \max(K - S_{t+1}, 0) dK. \quad (\text{A15})$$

Equation (10) then follows as $P_t[K] = R_{t,t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(\max(K - S_{t+1}, 0))$. ■

B. Proof of the expression for the price of the binary put in equation (11)

We wish to calculate the price of the binary put. Using the Heaviside theta function,

$$\begin{aligned} \theta[K^* - S_{t+1}] &= \frac{d}{dK^*} \overbrace{\max(K^* - S_{t+1}, 0)}^{\equiv h[K^*]}, \\ &= \frac{\max(K^* + \Delta K - S_{t+1}, 0) - \max(K^* - \Delta K - S_{t+1}, 0)}{2\Delta K}. \end{aligned} \quad (\text{B1})$$

The step in (B1) follows (Stefanica (2011, page 207)) since $h'[K] = \frac{h[K+\Delta K] - h[K-\Delta K]}{2\Delta K} + O[(\Delta K)^2]$.

Thus,

$$R_{f,t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(\theta[K^* - S_{t+1}]) \approx \frac{P[K^* + \Delta K] - P[K^* - \Delta K]}{2\Delta K}. \quad (\text{B2})$$

The binary put with threshold strike K^* can be synthesized via two adjacent put options. ■

C. Interest rates and options data on the S&P 500 index

S&P 500 index options (ticker, SPX): The data on S&P 500 index options are daily and constructed over the sample period of January 1990 to December 2015. The call and put option data are extracted from the Optsum historical database (Market Data Express), maintained by the Chicago Board of Options Exchange (CBOE).

The data contains the end-of-the-day information, including the strike price, the expiration date, the trading volume, the open interest, and the open, high, low, and last trade prices. Recorded also in the database are the last bid and ask quotes and the closing level of the S&P 500 index. The S&P 500 index options are of the European style and expire on the third Friday of the expiration month. We focus on the nearest maturity options and construct the data by expiration cycle dates. The start date for the first (final) expiration cycle is January 22, 1990 (November 23, 2015).

Our calculation of the number of days during each expiration cycle takes into account the fact that these options expire at the market close prior to August 24, 1992, and expire at the market open afterward. Accordingly, the Friday on which the option expires is included in the expiration cycle prior to August 24, 1992, and is excluded afterward. The option price in our calculations is the midpoint of the bid and ask quotes, and we only keep out-of-the-money options (i.e., $K/S_t < 1$ for puts and $S_t/K < 1$ for calls). We screen the raw data in a number of ways:

- We omit quotes with bid price lower than or equal to \$0.10. These quotes are close to the

minimum tick of \$0.05 for options trading below \$3.00, and the minimum tick of \$0.10 for other options.

- Next, we detect, and eliminate, 39 recording errors. Specifically, 35 option observations have Black-Scholes implied volatility higher than 450%, and there are four observations for which the bid is \$998 and the ask is \$999.
- Finally, we exclude deep out-of-the-money options that exhibit excessive percentage bid-ask spreads (e.g., if the bid was \$0.15 and the ask was \$1.25).

There are 4,831 calls and 10,757 puts, with an average of 16 calls and 35 puts in each expiration cycle (the final sample contains 15,588 option observations).

Gross interest rate ($R_{f,t+1}$): It is constructed from (daily) four-week interest rate series (Center for Research in Security Prices) and is scaled to match the number of days in the expiration cycles.

Table Internet Appendix-IV describes the data on realized return, realized return volatility, and interest rate. ■

D. Expressions for conditional return variance

We have shown in equation (5) that $\mathbb{E}_t^{\mathbb{P}}(f[r_{t+1}]) = R_{f,t+1}^{-1} \mathbb{E}_t^{\mathbb{Q}}(g[r_{t+1}])$, where $g[r_{t+1}] \equiv \frac{f[r_{t+1}]}{\Lambda[r_{t+1}]}$, and $r_{t+1} = \log(\frac{S_{t+1}}{S_t})$. Here our interest lies in developing the expressions for the conditional return variance $\text{var}_t^{\mathbb{P}}(r_{t+1}; \Theta) = \mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2) - \{\mathbb{E}_t^{\mathbb{P}}(r_{t+1})\}^2$ to be used in equation (17).

D.1. The SDF takes the form $\Lambda[S_{t+1}] = (\frac{S_{t+1}}{S_t})^{-\gamma}$

Consider the conditional expected (log) return, which implies that

$$f[S_{t+1}] = \log\left(\frac{S_{t+1}}{S_t}\right) \quad \text{and, hence,} \quad g[S_{t+1}] = \log\left(\frac{S_{t+1}}{S_t}\right) / \left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}. \quad (\text{D1})$$

Standard differentiation allows us to present $\mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2)$ and $\mathbb{E}_t^{\mathbb{P}}(r_{t+1})$ in equations (20) and (21), with equation (22) holding.

D.2. The SDF takes the form $\Lambda[S_{t+1}] = (1 + \lambda \log(\frac{S_{t+1}}{S_t}))^{-\gamma}$

Consider

$$g[r_{t+1}] = \{\log(S_{t+1}) - \log(S_t)\}^n (1 + \lambda \{\log(S_{t+1}) - \log(S_t)\})^\gamma. \quad (\text{D2})$$

Omitting some intermediate steps, we can show that $\mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2)$ and $\mathbb{E}_t^{\mathbb{P}}(r_{t+1})$ for this case also satisfy equations (20) and (21), respectively. The positioning in calls and puts are

$$\begin{aligned} a[K] = & \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 \left(\frac{\lambda^2(\gamma-1)\gamma \left(\lambda \log\left(\frac{K}{S_t}\right) + 1\right)^{\gamma-2}}{K^2} - \frac{\lambda\gamma \left(\lambda \log\left(\frac{K}{S_t}\right) + 1\right)^{\gamma-1}}{K^2} \right) \\ & + \frac{4\lambda\gamma \log\left(\frac{K}{S_t}\right) \left(\lambda \log\left(\frac{K}{S_t}\right) + 1\right)^{\gamma-1}}{K^2} + \left(\frac{2}{K^2} - \frac{2\log\left(\frac{K}{S_t}\right)}{K^2} \right) \left(\lambda \log\left(\frac{K}{S_t}\right) + 1\right)^\gamma, \end{aligned} \quad (\text{D3})$$

and

$$\begin{aligned} b[K] = & \log\left(\frac{K}{S_t}\right) \left(\frac{\lambda^2(\gamma-1)\gamma \left(\lambda \log\left(\frac{K}{S_t}\right) + 1\right)^{\gamma-2}}{K^2} - \frac{\lambda\gamma \left(\lambda \log\left(\frac{K}{S_t}\right) + 1\right)^{\gamma-1}}{K^2} \right) \\ & + \frac{2\lambda\gamma \left(\lambda \log\left(\frac{K}{S_t}\right) + 1\right)^{\gamma-1}}{K^2} - \frac{\left(\lambda \log\left(\frac{K}{S_t}\right) + 1\right)^\gamma}{K^2}. \end{aligned} \quad (\text{D4})$$

D.3. The SDF takes the form $\Lambda[S_{t+1}] = \Psi_a (\log(\frac{S_{t+1}}{S_t}))^2 + \Psi_b \log(\frac{S_{t+1}}{S_t}) + \Psi_c$

Via the operation of differentiation, we have

$$\mathbb{E}_t^{\mathbb{P}}(r_{t+1}^2) = \int_{K>S_t} a[K] C_t[K] dK + \int_{K<S_t} a[K] P_t[K] dK, \quad (\text{D5})$$

$$\mathbb{E}_t^{\mathbb{P}}(r_{t+1}) = \frac{1}{\Psi_c} (1 - R_{f,t+1}^{-1}) + \int_{K>S_t} b[K] C_t[K] dK + \int_{K<S_t} b[K] P_t[K] dK. \quad (\text{D6})$$

Then,

$$a[K] = \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 \left(\frac{2 \left(\frac{2\Psi_a \log(\frac{K}{S_t})}{K} + \frac{\Psi_b}{K} \right)^2}{\left(\Psi_a \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 + \Psi_b \log\left(\frac{K}{S_t}\right) + \Psi_c \right)^3} - \frac{-\frac{2\Psi_a \log(\frac{K}{S_t})}{K^2} + \frac{2\Psi_a}{K^2} - \frac{\Psi_b}{K^2}}{\left(\Psi_a \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 + \Psi_b \log\left(\frac{K}{S_t}\right) + \Psi_c \right)^2} \right) \\ + \frac{\frac{2}{K^2} - \frac{2\log(\frac{K}{S_t})}{K^2}}{\Psi_a \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 + \Psi_b \log\left(\frac{K}{S_t}\right) + \Psi_c} - \frac{4 \log\left(\frac{K}{S_t}\right) \left(\frac{2\Psi_a \log(\frac{K}{S_t})}{K} + \frac{\Psi_b}{K} \right)}{K \left(\Psi_a \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 + \Psi_b \log\left(\frac{K}{S_t}\right) + \Psi_c \right)^2} \quad \text{and} \quad (\text{D7})$$

$$b[K] = \log\left(\frac{K}{S_t}\right) \left(\frac{2 \left(\frac{2\Psi_a \log(\frac{K}{S_t})}{K} + \frac{\Psi_b}{K} \right)^2}{\left(\Psi_a \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 + \Psi_b \log\left(\frac{K}{S_t}\right) + \Psi_c \right)^3} - \frac{-\frac{2\Psi_a \log(\frac{K}{S_t})}{K^2} + \frac{2\Psi_a}{K^2} - \frac{\Psi_b}{K^2}}{\left(\Psi_a \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 + \Psi_b \log\left(\frac{K}{S_t}\right) + \Psi_c \right)^2} \right) \\ - \frac{1}{K^2 \left(\Psi_a \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 + \Psi_b \log\left(\frac{K}{S_t}\right) + \Psi_c \right)} - \frac{2 \left(\frac{2\Psi_a \log(\frac{K}{S_t})}{K} + \frac{\Psi_b}{K} \right)}{K \left(\Psi_a \left\{ \log\left(\frac{K}{S_t}\right) \right\}^2 + \Psi_b \log\left(\frac{K}{S_t}\right) + \Psi_c \right)^2}. \quad (\text{D8})$$

We have the desired expressions. ■

E. Do disaster risk premiums predict equicorrelation between stocks?

Anecdotal arguments are often given in support for greater return correlation of stocks during the perceived downside of markets. To formalize our statements, we first construct a measure of affinity between stocks. Let the portfolio return be $r_p = \sum_{i=1}^N \omega_i r_i$, for proportion ω_i invested in stock i with return r_i (suppressing time subscripts).

Under the assumption that the pairwise correlation between stock i and j is a constant, the equicorrelation is

$$\rho_{\{t,t+1\}} \equiv \frac{\text{Var}_t[r_p] - \frac{1}{N} \sum_{i=1}^N \frac{\text{Var}_t[r_i]}{N}}{\frac{1}{N^2} N(N-1) \sum_i \sum_{i \neq j} \frac{\sqrt{\text{Var}_t[r_i] \text{Var}_t[r_j]}}{N(N-1)}} = \frac{\text{Var}_t[r_p] - \frac{1}{N} \text{Average Variance}}{\left(\frac{N-1}{N}\right) \text{Average Pairwise Standard Deviation}}. \quad (\text{E1})$$

Equation (E1) is a consequence of Elton, Gruber, Brown, and Goetzmann (2010, page 56), as also studied in Engle and Kelly (2012), and CBOE (2014, equation (3)), when $\omega_i = 1/N$.

Implementation of equation (E1) requires daily returns (over the options expiration cycles), and we focus on (i) the 30 stocks in the DJIA, and (ii) the MSCI country indexes (dollar-denominated) on 23 developed markets (e.g., Bekaert, Hodrick, and Zhang (2009, Table I)), and 20 emerging markets (e.g., Harvey (1995, Table 1, plus China)). We consider the predictive regression:

$$\rho_{\{t \rightarrow t+1\}} = \Pi_0 + \Pi \text{drp}_t[r^*] + \varepsilon_{t+1}, \quad (\text{E2})$$

where $\rho_{\{t \rightarrow t+1\}}$ is constructed from daily returns over options expiration cycles, and $\text{drp}_t[r^*]$ is inferred from option prices consistent with equations (11)–(13) and is known at time t . For the analysis presented here, we set $r^* = -0.05129$ for brevity.

The null hypothesis is $\Pi = 0$ versus $\Pi \neq 0$. Our test design and question asked is different from Longin and Solnik (2001, Section II), who develop methods to examine the correlation of extreme returns. Finally, we emphasize that the dependent variable $\rho_{\{t \rightarrow t+1\}}$ is not options-market-based.

Table Internet Appendix-III presents the predictive regression results and shows the estimates of the slope coefficient Π . The Π estimates are negative and statistically significant, implying that a one-standard deviation worsening of the disaster risk premiums (i.e., 0.05; see Figure 2) tends to increase $\rho_{\{t \rightarrow t+1\}}$ by 0.05 among international equity indexes. The negative association

between $\rho_{\{t \rightarrow t+1\}}$ and drp_t is robust (across subsamples). Moreover, the disaster risk premiums can explain 13% of the subsequent variation in the equicorrelations (over the entire sample). ■

F. Conditional expectation of functions involving the upside of returns

For functions representing the upside of returns, suppose $x = S_{t+1} - K$ for threshold K . Then

$$\theta[S_{t+1} - K] = -\frac{d}{dK} \max(S_{t+1} - K, 0), \quad (\text{F1})$$

$$\delta[S_{t+1} - K] = -\frac{d}{dK} \theta[S_{t+1} - K] = \frac{d^2}{dK^2} \{\max(S_{t+1} - K, 0)\}. \quad (\text{F2})$$

Thus, $-\delta[S_{t+1} - K]dK = d\theta[S_{t+1} - K]$. For our purposes, we consider threshold K^u such that

$$\frac{S_t}{K^u} - 1 < 0. \quad (\text{F3})$$

By the sifting property in equation (A4):

$$g[S_{t+1}] = \underbrace{\int_0^{K^u} g[K] \delta[K - S_{t+1}] dK}_{=0, \text{ for claims on the upside}} + \int_{K^u}^{+\infty} g[K] \delta[S_{t+1} - K] dK, \quad (\text{F4})$$

$$= \int_{K^u}^{\infty} g[K] \{-d\theta[S_{t+1} - K^u]\} \quad (\text{since } \delta[S_{t+1} - K]dK = -d\theta[S_{t+1} - K]). \quad (\text{F5})$$

Setting $u = g[K]$ and $v = -\theta[S_{t+1} - K]$, integration by parts (i.e., $\int u dv = uv - \int v du$) implies that

$$g[S_{t+1}] = \int_{K^u}^{\infty} g[K] \{-d\theta[K - S_{t+1}]\}, \quad (\text{F6})$$

$$= -g[K] \theta[S_{t+1} - K] \Big|_{K^u}^{\infty} + \int_{K^u}^{\infty} \theta[S_{t+1} - K] g'[K] dK. \quad (\text{F7})$$

Substituting $u = g'[K]$ and $v = -\max(S_{t+1} - K, 0)$, which implies $dv = -\frac{d}{dK} \max(S_{t+1} - K, 0)$, a repeated application of integration by parts to $\int_0^{K^*} \theta[K - S_{t+1}] g'[K] dK$, implies that

$$\int_0^{K^*} \underbrace{g'[K]}_u \underbrace{\theta[S_{t+1} - K]}_{dv} dK = -g'[K] \max(S_{t+1} - K, 0) \Big|_{K^u}^{\infty} + \int_{K^u}^{\infty} \max(S_{t+1} - K, 0) g''[K] dK. \quad (\text{F8})$$

Hence, we can write $g[S_{t+1}] = \int_{K^u}^{\infty} g[K] \delta[S_{t+1} - K] dK$ as

$$g[S_{t+1}] = -g[K] \theta[S_{t+1} - K] \Big|_{K^u}^{\infty} - g'[K] \max(S_{t+1} - K, 0) \Big|_{K^u}^{\infty} + \int_{K^u}^{\infty} \max(S_{t+1} - K, 0) g''[K] dK. \quad (\text{F9})$$

We focus on a representative calculation for the Heaviside function $\theta[S_{t+1} - K^u]$ and suppose

$\Lambda[S_{t+1}] = \left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}$. Then,

$$g[S_{t+1}] = \frac{\theta[S_{t+1} - K^u]}{\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}} = \left(\frac{K^u}{S_t}\right)^{\gamma} \theta[S_{t+1} - K^u] + \frac{\gamma}{S_t} \left(\frac{K^u}{S_t}\right)^{\gamma-1} \max(S_{t+1} - K^u, 0) + \frac{\gamma(\gamma-1)}{S_t^2} \int_{K > K^u} \left(\frac{K}{S_t}\right)^{\gamma-2} \max(S_{t+1} - K, 0) dK. \quad (\text{F10})$$

The remainder follows. ■

Table Internet Appendix-I

Derived values when $\Lambda[S_{t+1}] = (1 + \lambda \log(\frac{S_{t+1}}{S_t}))^{-\gamma}$

The estimation procedure adopted in equation (17) is highly nonlinear in $\Theta = (\lambda, \gamma)$, as λ enters multiplicatively with returns in $\Lambda[S_{t+1}] = (1 + \lambda \log(\frac{S_{t+1}}{S_t}))^{-\gamma}$ and may not be distinctively identified. Hence, we fix λ and estimate γ as

$$\hat{\gamma} \equiv \arg \min_{\gamma} \sum_{t=1}^T \left(\text{var}_t^{\mathbb{P}}(r_{t+1}; \gamma) - \text{rv}_{\{t \rightarrow t+1\}} \right)^2.$$

We verify the minimum by establishing a grid of values of λ and γ . $\text{rv}_{\{t \rightarrow t+1\}}$ is the realized variance (as in equation (18)), while the conditional return variance, $\text{var}_t^{\mathbb{P}}(r_{t+1}; \Theta)$, is computed using the positioning $a[K]$ and $b[K]$ in equations (D3)–(D4) of Internet Appendix D. To obtain the bootstrap distribution of the model parameters, we perform an i.i.d. bootstrap on daily returns and generate the time-series of realized volatility $\text{rv}_{\{t \rightarrow t+1\}}^b$, $b = 1, \dots, 1000$. In each bootstrap iteration, we estimate γ for a fixed λ , yielding the reported mean, standard deviation, lower and upper confidence intervals, and percentiles. We report the disaster probability and the disaster risk premium for a 5% disaster. Reported also are the conditional expected (log) return, the conditional volatility, and the variance risk premium. In our illustrations, we set $r^* = -0.05129$. Results with other illustrative values of $\lambda \in (0, 1)$ are presented in Table Internet Appendix-B.

		Across the 1,000 bootstraps								
		Mean	Std.	95% CI [Lower Upper]	5th	25th	50th	75th	95th	
Panel A: Derived values, when γ is estimated with λ fixed at 1.0										
	Estimated									
γ	2.40	2.37	0.66	[1.10 3.70]	1.20	1.92	2.39	2.84	3.41	
	Average									
$\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$	0.095	0.096	0.007	[0.083 0.109]	0.086	0.091	0.095	0.100	0.108	
$\text{drp}_t[r^*]$	-0.252	-0.248	0.065	[-0.376 -0.120]	-0.348	-0.294	-0.250	-0.204	-0.131	
$\mathbb{E}_t^{\mathbb{P}}(r_{t+1})$	0.125	0.123	0.029	[0.068 0.181]	0.070	0.104	0.124	0.143	0.166	
$\sqrt{\text{var}_t^{\mathbb{P}}}$	0.183	0.183	0.006	[0.172 0.193]	0.175	0.179	0.183	0.187	0.193	
vrp_t	-0.221	-0.217	0.054	[-0.327 -0.115]	-0.299	-0.256	-0.220	-0.182	-0.119	
Panel B: Derived values, when γ is estimated with λ fixed at 0.8										
	Estimated									
γ	3.13	3.10	0.83	[1.50 4.76]	1.64	2.53	3.12	3.69	4.40	
	Average									
$\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$	0.094	0.095	0.006	[0.082 0.108]	0.086	0.090	0.095	0.099	0.107	
$\text{drp}_t[r^*]$	-0.258	-0.255	0.064	[-0.381 -0.129]	-0.354	-0.301	-0.257	-0.211	-0.140	
$\mathbb{E}_t^{\mathbb{P}}(r_{t+1})$	0.128	0.126	0.029	[0.072 0.184]	0.075	0.107	0.128	0.147	0.170	
$\sqrt{\text{var}_t^{\mathbb{P}}}$	0.182	0.183	0.005	[0.172 0.193]	0.175	0.179	0.182	0.186	0.193	
vrp_t	-0.224	-0.220	0.053	[-0.327 -0.120]	-0.301	-0.258	-0.223	-0.185	-0.125	

Table Internet Appendix-II

Derived values when $\Lambda[S_{t+1}] = \psi_a (\log(\frac{S_{t+1}}{S_t}))^2 + \psi_b \log(\frac{S_{t+1}}{S_t}) + \psi_c$

Let $\Theta = (\psi_a, \psi_b, \psi_c)$. Imposing $\psi_a > 0$ and $\frac{\psi_b^2}{4\psi_a} < \psi_c$ (to ensure the positivity of the SDF), and also $\psi_b < 0$, we estimate

$$\hat{\Theta} \equiv \arg \min_{\Theta} \sum_{t=1}^T \left(\text{var}_t^{\mathbb{P}}(r_{t+1}; \Theta) - \text{rv}_{\{t \rightarrow t+1\}} \right)^2,$$

where $\text{rv}_{\{t \rightarrow t+1\}}$ is the realized variance (as in equation (18)). The conditional return variance, $\text{var}_t^{\mathbb{P}}(\Theta)$, is computed using equations (D5)–(D6), in conjunction with the positioning $a[K]$ and $b[K]$ in equations (D7)–(D8) of Appendix D. To obtain the bootstrap distribution of the model parameters, we perform an i.i.d. bootstrap on daily returns and generate the time-series of realized volatility $\text{rv}_{\{t \rightarrow t+1\}}^b$, $b = 1, \dots, 1000$. In each bootstrap iteration, we estimate (ψ_a, ψ_b, ψ_c) , yielding the reported mean, standard deviation, lower and upper confidence intervals, and percentiles. We report the disaster probability and the disaster risk premium for a 5% disaster. Reported also are the conditional expected (log) return, the conditional volatility, and the variance risk premium. In our illustrations, we set $r^* = -0.05129$.

		Across the 1,000 bootstraps								
		Mean	Std.	95% CI		5th	25th	50th	75th	95th
		[Lower Upper]								
	Estimated									
ψ_a	3.10	4.11	2.08	[0.03	8.20]	1.73	2.73	3.49	5.04	8.32
ψ_b	-3.60	-3.72	0.59	[-4.89	-2.56]	-4.57	-4.12	-3.78	-3.38	-2.66
ψ_c	1.08	1.06	0.06	[0.94	1.18]	0.96	1.02	1.06	1.11	1.16
	Average									
$\mathbb{E}_t^{\mathbb{P}}(\theta[r^* - r_{t+1}])$	0.084	0.085	0.006	[0.073	0.097]	0.077	0.081	0.084	0.088	0.096
$\text{drp}_t[r^*]$	-0.375	-0.367	0.069	[-0.503	-0.232]	-0.462	-0.413	-0.375	-0.326	-0.249
$\mathbb{E}_t^{\mathbb{P}}(r_{t+1})$	0.149	0.156	0.023	[0.110	0.202]	0.117	0.143	0.156	0.170	0.200
$\sqrt{\text{var}_t^{\mathbb{P}}}$	0.176	0.177	0.005	[0.166	0.187]	0.170	0.173	0.176	0.180	0.187
vrp_t	-0.235	-0.292	0.060	[-0.409	-0.176]	-0.376	-0.332	-0.302	-0.253	-0.188

Table Internet Appendix-III

Disaster risk premiums and subsequent equicorrelation

We report the results from the predictive regressions with $r^* = -0.05129$:

$$\rho_{\{t \rightarrow t+1\}} = \Pi_0 + \Pi \text{drp}_t[r^*] + \varepsilon_{t+1},$$

where the equicorrelation, $\rho_{\{t \rightarrow t+1\}}$, is computed according to equation (E1), using daily returns over the options expiration cycles. We estimate the equicorrelation using both the DJIA 30 stocks and the MSCI international equity indexes (dollar-denominated) across 23 developed and 20 emerging markets. To correct for autocorrelation and heteroscedasticity, we use the Newey and West (1987) estimator with automatically selected lag, denoted by ℓ^* , as in Newey and West (1994), and the reported two-sided p -values are denoted by $\text{NW}[p]$. Shown also are Hodrick (1992) two-sided p -values, denoted by $\text{H}[p]$. The adjusted R^2 is reported as \bar{R}^2 . The computation of disaster risk premiums are based on the expression for disaster probabilities in equation (13). The intercept Π_0 is not reported.

	DJIA 30 stocks					23 Developed plus 20 emerging markets				
	Π	$\text{NW}[p]$	$\text{H}[p]$	\bar{R}^2 (%)	ℓ^*	Π	$\text{NW}[p]$	$\text{H}[p]$	\bar{R}^2 (%)	ℓ^*
1/22/1990–11/23/2015	-1.20	0.00	0.00	13	8	-1.07	0.00	0.00	13	10
1/22/1990–12/23/2002	-0.73	0.00	0.00	9	2	-0.72	0.00	0.00	18	4
1/21/2003–11/23/2015	-1.81	0.00	0.00	28	7	-1.50	0.00	0.00	26	2
1/22/1996–11/23/2015	-1.23	0.00	0.00	12	2	-0.95	0.00	0.00	9	8

Table Internet Appendix-IV

Features of realized returns and realized volatility over the options expiration cycles

We report the summary statistics for realized returns and realized volatility for the S&P 500 index over the option expiration cycles. There are 311 options expiration cycles over the sample period of January 1990 to December 2015. We calculate the returns over the expiration cycles as $r_{t+1} \equiv \log(\frac{S_{t+1}}{S_t})$, and the realized variance $rv_{\{t \rightarrow t+1\}}$ is calculated from daily (log) returns as

$$rv_{\{t \rightarrow t+1\}} = r_{t+\Delta, \Delta}^2 + r_{t+2\Delta, \Delta}^2 + \dots + r_{t+h\Delta, \Delta}^2, \quad \text{where } r_{t+h\Delta, \Delta} \equiv \log(S_{t+h\Delta}/S_{t+(h-1)\Delta}).$$

In our calculations, h is the number of trading days over an options expiration cycle, and Δ is one day. Reported realized volatility is $\sqrt{rv_{\{t \rightarrow t+1\}}}$. The lower and upper 95% bootstrap confidence intervals on the realized returns are computed using an i.i.d. bootstrap (10,000 trials) using daily returns. The reported lower and upper 95% bootstrap confidence intervals on the realized volatility are based on an AR-1 specification for $\log(rv_{\{t \rightarrow t+1\}})$. The p -value for the Ljung-Box statistic for 20 lags is shown in [.]. We also report the sample properties of the interest rate ($R_{t,t+1} - 1$, annualized).

	Annualized		Std.	Skew.	Kurt.	Autocorrelation, lag j				LB(20) [p -val.]
	Average	Bootstrap 95% CI [Lower Upper]				1	2	5	10	
Realized return	0.094	[0.023 0.164]	0.157	-1.56	9.08	-0.04	0.00	0.07	0.06	[0.79]
Realized volatility	0.161	[0.141 0.182]	0.095	3.04	17.9	0.72	0.56	0.36	0.23	[0.00]
Interest rate	0.033	-	0.026	0.07	1.68	0.98	0.97	0.91	0.77	[0.00]

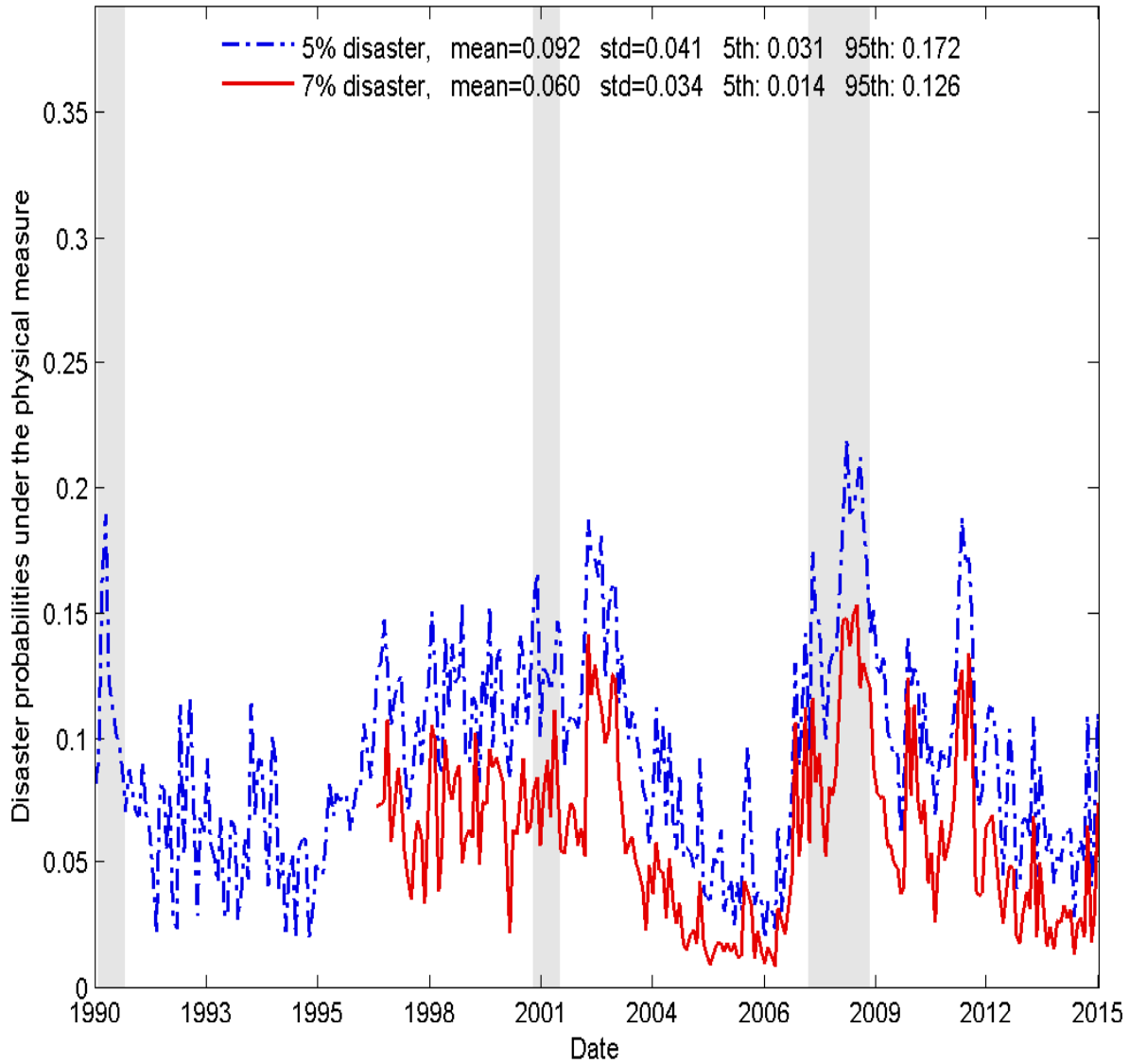


Fig. 1. Expectation of a disaster under the physical probability measure

Plotted are the expectation of disasters according to equation (13), when $\Lambda[S_{t+1}] = (S_{t+1}/S_t)^{-\gamma}$. Our calculations are based on the estimated $\gamma = 2.93$. The first point in the sample is 1/22/1990, and the final point is 11/23/2015. The grey shaded regions represent recessions, as dated by the NBER. The expectation of a 5% disaster over an options expiration cycle is depicted by the dashed-curve (in blue), and the expectation of the 7% disaster is depicted by the solid-curve (in red).

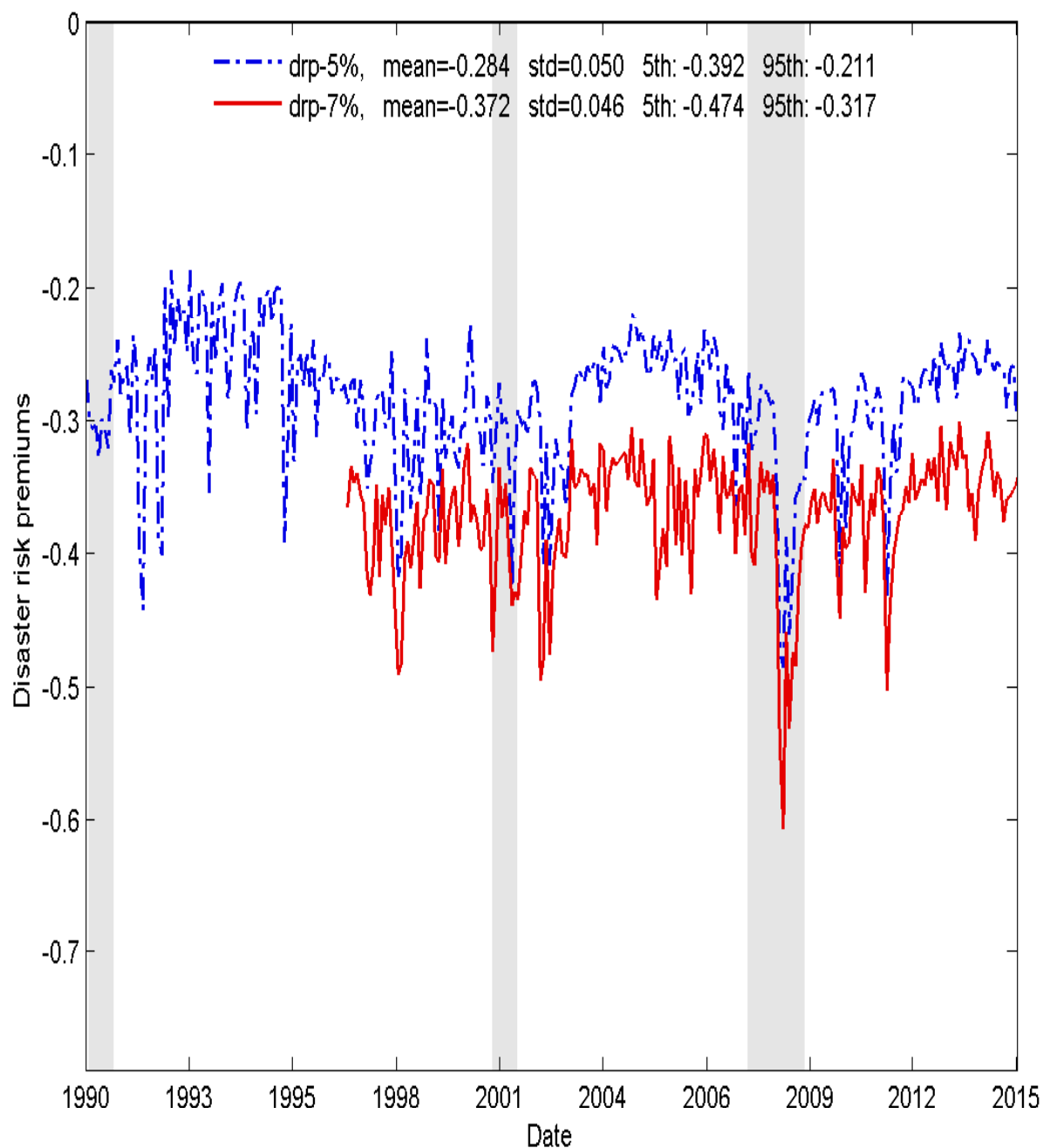


Fig. 2. Disaster risk premiums

Plotted are the disaster risk premiums computed according to equation (12), when $\Lambda[S_{t+1}] = (S_{t+1}/S_t)^{-\gamma}$. Our calculations are based on the estimated $\gamma = 2.93$. The first point in the sample is 1/22/1990, and the final point is 11/23/2015. The grey shaded regions represent recessions, as dated by the NBER. The risk premium underlying a 5% disaster is depicted by the dashed-curve (in blue), and the risk premium underlying a 7% disaster is depicted by the solid-curve (in red).

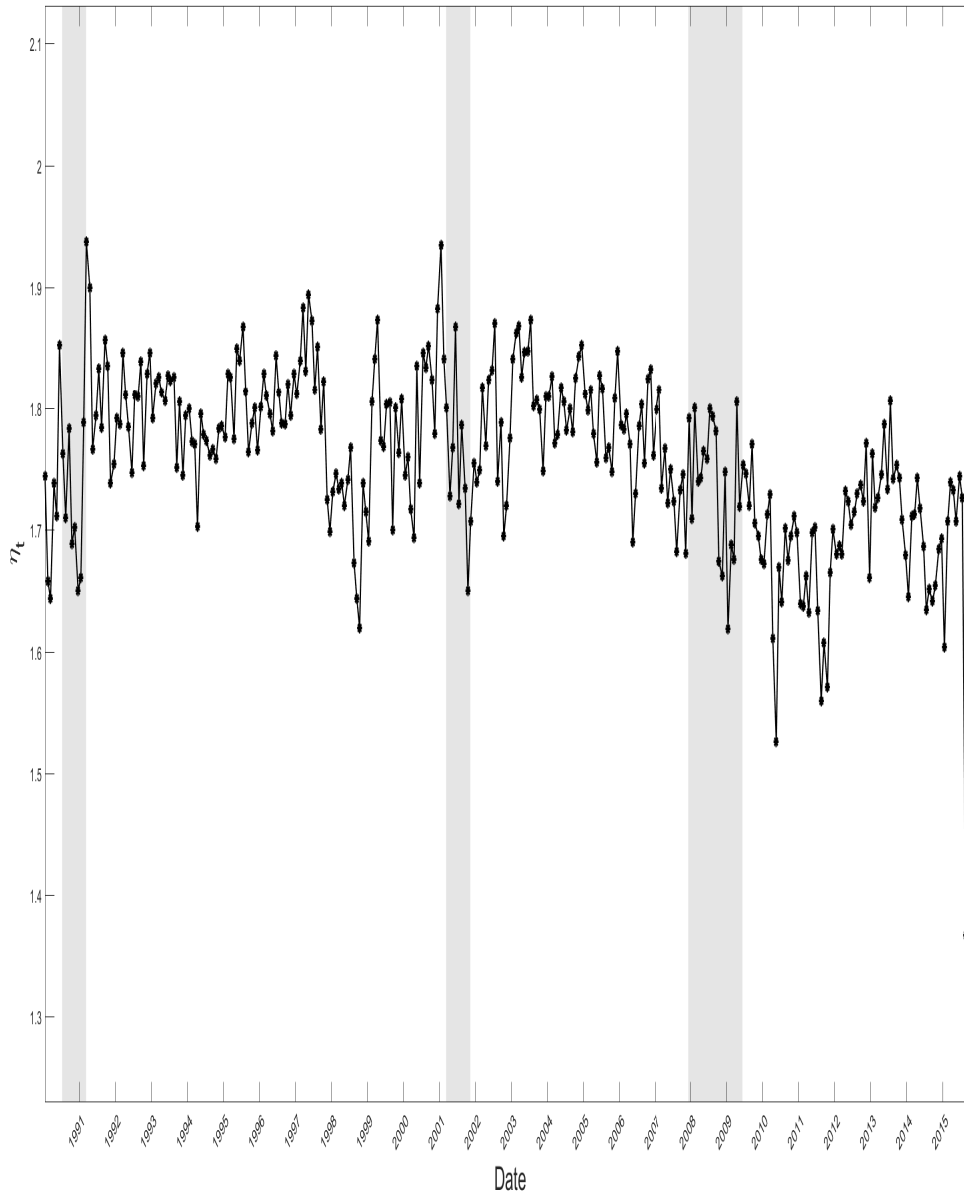


Fig. 3. Time variation in η_t

Plotted is the time-series of η_t estimates, computed as $\eta_t = \frac{\mathbb{E}_t^{\mathbb{P}}(R_{t+1}-1) - (R_{f,t}-1)}{R_{f,t}^{-1} \text{var}_t^{\mathbb{Q}}(R_{t+1})} - 1$. We compute the expected return as $\mathbb{E}_t^{\mathbb{P}}(\frac{S_{t+1}}{S_t} - 1) = 1 - R_{f,t}^{-1} + \int_{K < S_t} w[K] P_t[K] dK + \int_{K > S_t} w[K] C_t[K] dK$, where $w[K] = \gamma K^{-3} (\frac{K}{S_t})^{\gamma+1} \{\gamma K + K - \gamma S_t + S_t\}$. Whereas our null hypothesis is that $\eta_t = 0$ (point-by-point), the minimum value of η_t is 1.37, which coincides with the flash crash on 08/24/2015.