New Entropy Restrictions and the Quest for Better Specified Asset Pricing Models*

Gurdip Bakshi[a†] Fousseni Chabi-Yo[b‡]

[a]Smith School of Business, University of Maryland, College Park, MD 20742, USA

[b]Isenberg School of Management, University of Massachusetts, Amherst, MA 01002, USA

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Abstract

This paper proposes the entropy of m^2 (m is the stochastic discount factor) as a metric to evaluate asset pricing models. We develop a bound on the entropy of m^2 when m correctly prices a finite number of returns and consider models that pass the lower bound on m, yet fail the lower bound on m^2. Interpreting our results, we elaborate on the distinction between the entropy of m^2 versus the entropy of m. We further show that the entropy of m^2 represents an upper bound on the expected excess (log) return of the security with the payoff of m.

KEY WORDS: Stochastic discount factors, lower entropy bounds, individual asset pricing models

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†Tel.: +1-301-405-2261. *E-mail address:* gbakshi@rhsmith.umd.edu

‡Tel.: +1-614-292-8477. *E-mail address:* fchabiyo@isenberg.umass.edu

1. Introduction

The lower bound statistics constructed from observed asset returns have found their applicability in evaluating individual asset pricing models. Taking a cue from the theory underlying the volatility bound of Hansen and Jagannathan (1991), a wave of research has emphasized entropy bounds on the stochastic discount factor and its correlated permanent and transitory components. This line of inquiry includes, among others, Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Bakshi and Chabi-Yo (2012), and Backus, Chernov, and Zin (2014).

Our idea is to focus on a new moment – the entropy of the *square* of the stochastic discount factor (hereby SDF, or simply m) – and we develop the corresponding lower entropy bound.

The novelty of our paper is to show that the entropy of m^2 can be employed as a statistic to reject some models that pass the lower bound on the entropy of m. The defining attribute of the entropy of m^2, from an asset pricing perspective, is that it represents an upper bound on the expected excess (log) gross return of the security with a payoff of m, which is a fundamental risk premium. Additionally, we show that the lower bound on the entropy of m^2 can be derived from a vector of traded asset returns. The proposed entropy bounds (on both the SDF and its permanent component) are distinct from others with no analytical analogs.

The developments here are relevant, as the quest for well-specified SDFs has dominated the agenda in asset pricing. Despite substantial progress, identifying the desirable properties of the SDFs and the embedded permanent component, in addition to their link to economic fundamentals, remains a tall order.

In our framework, we consider models that meet the minimum entropy criterion on the entropy of m, and then ask the question: Is the same model poised for acceptance based on the bound on the entropy of m^2? Our empirical analysis results in the finding that there exist a number of models that respect the lower bound on the entropy of m, but fail to satisfy the lower bound on the entropy of m^2. In this sense, our niche is to show that the lower bound on the entropy of m^2 (constructed from observed asset returns) can offer a way to evaluate individual asset pricing models.[1]

Our approach also clarifies how certain variables can help to improve the workings of a model. For example, a baseline model with market, size, book-to-market, operating profitability, and investment portfolios is rejected (respectively, is not rejected) according to the lower bound on the entropy of m^2 (respectively, m), with eight out of nine bootstrap p-values below 0.1. To the contrary, adding momentum variables to the baseline model is associated with a three-fold increase in the bootstrap p-variables, with five out of nine p-values above 0.1.

2. Entropy of m^2

We employ a result in Alvarez and Jermann (2005, Proposition 1) and Hansen and Scheinkman (2009, page 200), who establish that the SDF (interchangeably, $m_{t,t+1}$) admits the following multiplicative decomposition:

$$m_{t,t+1} = m^P_{t,t+1} m^T_{t,t+1} \quad \text{with} \quad E_t[m^P_{t,t+1}] = 1 \quad \text{and} \quad m^T_{t,t+1} = (R_{t,t+1,\infty})^{-1}, \tag{1}$$

where $m^P_{t,t+1}$ ($m^T_{t,t+1}$) is the permanent (transitory) component of $m_{t,t+1}$, $R_{t,t+1,\infty}$ is the gross return of an infinite-maturity discount bond, and $E_t[.]$ ($E[.]$) represents conditional (unconditional)

[1]Our characterizations (see equations (5) and (9)) show that, compared with the entropy of m, the entropy of m^2 assigns heavier weights to the higher moments (beyond the mean) of m. Intuitively, a candidate model is amiss if it does not generate higher-moment effects consistent with the entropy bound on m^2.

expectation. The $m^P_{t,t+1}$ component of the SDF is unique when $m^T_{t,t+1} = \frac{1}{R_{t,t+1,\infty}}$.[2]

We denote the gross return of the risk-free bond as $R_{t+1,f} \equiv 1/E_t[m_{t,t+1}]$. Our results are not affected by the length of the period for $m_{t,t+1}$, which can be arbitrary. Hence, we use the notations m and $m_{t,t+1}$ interchangeably. The unconditional variance of $m_{t,t+1}$ is denoted as var$[m]$.

2.1. The role of entropy $L[m]$ in testing asset pricing models

In this subsection, we consider the entropy of m, with the understanding that the results could be specialized to the entropy of m^P (with $\log(E[m^P]) = 0$, since $E[m^P] = 1$). This analysis also serves as a bridge for highlighting our new results on the entropy of m^2 and the entropy of $(m^P)^2$.

The unconditional entropy of m, denoted by $L[m]$, is defined as[3]

$$L[m] \;\equiv\; \log(E[m]) \;-\; E[\log(m)]. \tag{2}$$

The entropy measure $L[m]$ is related to *Jensen's gap*, $J\{m\}$, defined as

$$J\{m\} \;\equiv\; E[f\{m\}] \;-\; f\{E[m]\} \;\geq 0, \text{ applied to the } \textit{convex function } f\{m\} = -\log(m). \tag{3}$$

In contrast, the variance measure used in Hansen and Jagannathan (1991) is related to Jensen's gap applied to the convex function $f\{m\} = m^2$. An expansion-based interpretation of entropy

[2]The components $m^P_{t,t+1}$ and $m^T_{t,t+1}$ can be correlated, and, if they exist, can be obtained by solving the eigenfunction problem of Hansen and Scheinkman (2009, Corollary 6.1). In the context of parameterized asset pricing models, Hansen (2012), Borovička, Hansen, and Scheinkman (2016), and Christensen (2017) show that an appropriately solved eigenfunction problem will ensure a unique $m^P_{t,t+1}$.

[3]The asset pricing implications of entropy are explored in the studies of Stutzer (1995), Bansal and Lehmann (1997), Alvarez and Jermann (2005), Bakshi and Chabi-Yo (2012), Backus, Chernov, and Zin (2014), Borovička, Hansen, and Scheinkman (2016), Almeida and Garcia (2017), Qin, Linetsky, and Nie (2016), Orlowski, Sali, and Trojani (2016), Bakshi, Chabi-Yo, and Gao (2017), Christensen (2017), and Ghosh, Julliard, and Taylor (2017).

obtains by expressing $L[m] = \log(E[e^{\log(m)}]) - E[\log(m)]$. To outline this depiction, we take a Taylor expansion of $\exp(n\log(m))$ around $E[n\log(m)]$, for a positive integer n, which implies that

$$\begin{aligned}\exp(\log(m^n)) &= e^{nE[\log(m)]}\left(1 + n\left(\log(m) - E\left[\log(m)\right]\right) + \frac{n^2}{2!}\left(\log(m) - E\left[\log(m)\right]\right)^2\right. \\ &\quad \left. + \frac{n^3}{3!}\left(\log(m) - E\left[\log(m)\right]\right)^3 + \frac{n^4}{4!}\left(\log(m) - E\left[\log(m)\right]\right)^4 + \ldots\right). \qquad (4)\end{aligned}$$

Applying expectation and taking logs on both sides in equation (4), we get $L[m^n] = \log(E[m^n]) - E[\log(m^n)]$ (upon rearranging). Hence, with $n = 1$, it holds that

$$L[m] = \log\left(1 + \sum_{j=2}^{\infty} \frac{\mu_{\log(m)}^{[j]}}{j!}\right), \quad \text{where } \mu_{\log(m)}^{[j]} \equiv E[(\log(m) - E[\log(m)])^j]. \qquad (5)$$

$L[m]$ can be viewed as encapsulating the central moments of $\log(m)$, and distributions of $\log(m)$ that incorporate fatter tails tend to support a higher $L[m]$. When $\log(m)$ follows a normal distribution (i.e., m is distributed lognormal), the correspondence between entropy and the central moments of $\log(m)$ is exact[4] and is given by $L[m] = \frac{1}{2}\text{var}[\log(m)]$.

How is the entropy measure $L[m]$ used in the tests of asset pricing models? Alvarez and Jermann (2005, page 2008, equation (A.1)) and Backus, Chernov, and Zin (2014, page 57, equation (5)) propose the following lower entropy bound:

$$\underbrace{L[m]}_{\text{Entropy from model}} \geq \underbrace{E[\log(R_{t+1}^m) - \log(R_{t+1,f})]}_{\text{Based on observed returns}}. \qquad (6)$$

The bound on $L[m]$ in equation (6) is denominated in units of expected log gross return of a

[4] This follows from employing the expression of the moment generating function of the normal distribution applied to $\log(m)$, i.e., $E[e^{\log(m)}] = \exp(E[\log(m)] + \frac{1}{2}\text{var}[\log(m)])$, using the operation of log and rearranging to use the definition of entropy in equation (2).

generic portfolio (*any* arbitrary portfolio with gross return, R^m_{t+1}, that satisfies correct pricing) in excess of log gross return of a risk-free bond. In contrast, Almeida and Garcia (2017) offer a result in which the entropy bound can be determined as $\inf_m \, (-E[\log(m))]$, where m is extracted from an optimization problem while ensuring that m correctly prices a vector of asset returns.

2.2. Rationale for studying the entropy $L[m^2]$ and bounds on $L[m^2]$

This subsection provides (i) the rationale for studying $L[m^2]$ (and $L[(m^P)^2]$) and (ii) lower bounds on $L[m^2]$ (and $L[(m^P)^2]$) as a metric for evaluating asset pricing models.

The unconditional entropy of m^2 is defined as

$$L[m^2] \;\; \equiv \;\; \log(E[m^2]) \; - \; E[\log(m^2)], \tag{7}$$

which can be expressed in terms of *Jensen's gap* as

$$J\{m\} \; = \; E[f\{m^2\}] \; - \; f\{E[m^2]\}, \text{ applied to the } \textit{convex function } f\{m\} = -\log(m). \tag{8}$$

Additionally, based on equation (4), setting $n = 2$, and comparing with equation (7), we note that

$$L\left[m^2\right] \; = \; \log\left(1 + \frac{2^2}{2!}\mu^{[2]}_{\log(m)} + \frac{2^3}{3!}\mu^{[3]}_{\log(m)} + \frac{2^4}{4!}\mu^{[4]}_{\log(m)} + \ldots\right). \tag{9}$$

Viewed through the prism of the central moments of $\log(m)$, equation (9) shows that $L[m^2]$ assigns a bigger weight to each central moment of $\log(m)$ than $L[m]$. The relative merits of $L[m]$ and $\text{var}[m]$ are addressed in Alvarez and Jermann (2005, page 1985) on the grounds that additional higher moments affect $L[m]$ besides $\text{var}[\log(m)]$.

The assumption of lognormality of m implies that $\mu^{[3]}_{\log(m)} = 0$, $\mu^{[4]}_{\log(m)} = 3(\mu^{[2]}_{\log(m)})^2$, $\mu^{[5]}_{\log(m)} = 0$, $\mu^{[6]}_{\log(m)} = 15(\mu^{[2]}_{\log(m)})^3$, and so on. Then $L[m^2] = 4(\frac{1}{2}\text{var}[\log(m)])$ and, hence, $L[m^2] = 4L[m]$.

Finally, to establish the link among the entropy of m^2, the entropy of m, and variance of m, we subtract twice of $L[m]$ in equation (3) from $L[m^2]$ in equation (7), and note that

$$L[m^2] \;\; = \;\; 2\,L[m] \; + \; \log\left(1 + \frac{\text{var}[m]}{(E[m])^2}\right). \tag{10}$$

Equation (10) elicits three observations. First, Hansen and Jagannathan (1991) derive the lower bound on $\text{var}[m]$, when the SDF correctly prices finitely many assets. Second, the lower bound on $L[m]$ is not known when the SDF correctly prices finitely many assets (and is presented here in equation (23)). Third, when the bounds are not unique, lower bounding the parts in a sum might not be a proper way to lower bound the sum in the right-hand side of equation (10).[5]

We now present a theoretical lower bound on $L[m^2]$ and $L[(m^P)^2]$ that can be inferred from a vector of traded asset returns, when the SDF is required to correctly price finitely many returns. To do so, consider the following set $\mathbb{S}$ of SDFs:

$$\mathbb{S} = \{m_{t,t+1} > 0 : E_t\left[m_{t,t+1}\right] = q_t,\;\; E\left[m_{t,t+1}R_{t,t+1,\infty}\right] = 1,\;\; \text{and}\;\; E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = \mathbf{1}\}\,, \tag{11}$$

where $\mathbf{1}$ is a vector column of ones, and $\mathbf{R}_{t,t+1}$ is an $N \times 1$ vector of gross returns that excludes the risk-free bond and the infinite-maturity discount bond. We further postulate that some SDFs

[5] Suppose you have a function $K[m] = H[m] + G[m]$. If one could bound the function as $H[m] > h^*$ and $G[m] > g^*$, then $h^* + g^*$ can only be a unique lower bound for $K[m]$, if h^* and g^* are *unique* lower bounds.

that belong to the set $\mathbb{S}$ can be uniquely decomposed into permanent and transitory components:

$$\mathbb{S}_P = \left\{ m_{t,t+1} \in \mathbb{S} : m_{t,t+1} = m^P_{t,t+1} m^T_{t,t+1}, \text{ and } m^T_{t,t+1} = (R_{t,t+1,\infty})^{-1} \right\}. \tag{12}$$

Equation (11) requires the SDF to correctly price *each* of the $N+2$ assets. Therefore, due to the pricing of additional risky assets, set $\mathbb{S}$ is smaller than its counterparts based on pricing three assets, for example, a risk-free bond, a long-term bond, and a generic portfolio of risky assets.

Result 1 *The following theoretical bounds are germane to asset pricing models:*

(a) The entropy of m^2 satisfies

$$\begin{aligned} L[m^2] \quad \geq \quad \mathbb{LB}_{m^2} &\equiv 2\left(E\left[\log\left(\frac{(\mathbf{1} - E[q_t] E[\mathbf{R}_{t,t+1}])' \Sigma^{-1}}{\mathbf{1}'\Sigma^{-1}(\mathbf{1} - E[q_t] E[\mathbf{R}_{t,t+1}])} \mathbf{R}_{t,t+1} \right) \right] \; - \; \log\left((E[q_t])^{-1} \right) \right) \\ &+ \; \log\left(1 + (\mathbf{1} - E[q_t] E[\mathbf{R}_{t,t+1}])' \Sigma^{-1} (\mathbf{1} - E[q_t] E[\mathbf{R}_{t,t+1}]) / (E[q_t])^2 \right), \end{aligned} \tag{13}$$

where Σ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}$.

(b) The entropy of $(m^P)^2$ satisfies

$$\begin{aligned} L[(m^P)^2] \quad \geq \quad \mathbb{LB}_{(m^P)^2} &\equiv 2\left(E\left[\log\left(\frac{(\mathbf{1} - E[q_t] E[\mathbf{R}_{t,t+1}])' \Sigma^{-1}}{\mathbf{1}'\Sigma^{-1}(\mathbf{1} - E[q_t] E[\mathbf{R}_{t,t+1}])} \mathbf{R}_{t,t+1} \right) \right] - E\left[\log\left(R_{t,t+1,\infty} \right) \right] \right) \\ &+ \; \log\left(1 + \left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}} \right] \right)' \Sigma_P^{-1} \left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}} \right] \right) \right), \end{aligned} \tag{14}$$

where Σ_P is the variance-covariance matrix of $\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}$.

Proof: See Appendix A. ■

The lower entropy bounds in equations (13) and (14) summarize properties of the distribution of m and m^P. Define, for brevity, the $N \times 1$ vector of constants

$$\mathbf{a} \equiv \frac{\Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right)}{\mathbf{1}'\underbrace{\Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right)}_{\equiv \mathbf{y}}}, \tag{15}$$

which allows us to write the first term of the lower bound in equation (13) as

$$\log(\mathbf{a}'\mathbf{R}_{t+1}) = \log(E[\mathbf{a}'\mathbf{R}_{t+1}]) + \log(1+\tilde{u}) \quad \text{with} \quad \tilde{u} \equiv \frac{\mathbf{a}'\mathbf{R}_{t+1} - E[\mathbf{a}'\mathbf{R}_{t+1}]}{E[\mathbf{a}'\mathbf{R}_{t+1}]}. \tag{16}$$

Taking a Taylor expansion $\log(1+\tilde{u}) = \sum_{i=1}^{\infty} \frac{(-1)^{1+i}}{i}\tilde{u}^i$, we observe that

$$E[\log(\mathbf{a}'\mathbf{R}_{t+1})] = \log(E[\mathbf{a}'\mathbf{R}_{t+1}]) + \sum_{i=1}^{\infty} \frac{(-1)^{1+i}}{i} \frac{E[(\mathbf{a}'\mathbf{R}_{t+1} - E[\mathbf{a}'\mathbf{R}_{t+1}])^i]}{(E[\mathbf{a}'\mathbf{R}_{t+1}])^i}. \tag{17}$$

Hence, the lower bound in equation (13) can be alternatively expressed as

$$\begin{aligned} \mathbb{LB}_{m^2} \;=\; & 2(\log(E[\mathbf{a}'\mathbf{R}_{t+1}]) - \log((E[q_t])^{-1})) \;+\; \log(1 + \mathbf{y}'\Sigma\mathbf{y}/\left(E\left[q_t\right]\right)^2) \\ & +\; 2\sum_{i=1}^{\infty} \frac{(-1)^{1+i}}{i} \frac{E[(\mathbf{a}'\mathbf{R}_{t+1} - E[\mathbf{a}'\mathbf{R}_{t+1}])^i]}{(E[\mathbf{a}'\mathbf{R}_{t+1}])^i}, \end{aligned} \tag{18}$$

and is computable from the time-series of asset returns. Instrumental is the observation that the lower bound on $L[m^2]$ in equation (13) amplifies the effect of higher-order return moments (relative to the lower bound on the entropy of m).

Ghosh, Julliard, and Taylor (2017, Section II.1) construct entropy bounds when the SDF can be factorized into observable and model-specific unobservable components. Our entropy bound on the SDF is distinct from theirs and allows correlated multiplicative components. The lower bound on $L[(m^P)^2]$ in equation (14) is also distinct from the lower bound on $\text{var}[m^P]$ in Bakshi and Chabi-Yo (2012, equation (6)). Analogously, the bound on $\text{var}[m]$, that is, the Hansen and Jagannathan (1991, equation (12)) bound, and our bound on $L[m^2]$ in equation (13), constitute distinctly relevant metrics for evaluating individual asset pricing models.

Liu (2014, Proposition 1 and Collorary 1) derives an upper bound on $E[m^\delta]$ when $\delta \in [0,1]$, and a lower bound on $E[m^\delta]$ when $\delta < 0$, where δ is expressed in terms of the risk aversion parameter $\gamma \equiv \frac{1}{1-\delta}$. One may view the result in Liu (2014) as a particular case of the power divergence bounds in Almeida and Garcia (2017), which optimizes the expected power utility bound over portfolios of traded assets. Finally, our bound on $L[m^2]$ offers a distinction to the noncentral moment bounds considered in Snow (1991, equations (7) and (12)).

2.3. *Economic interpretations of* $L[m^2]$

Focusing on theoretical and economic rationale, we next show that $L[m^2]$ encodes information about the expected excess (log) return of a fundamental asset, namely, the security that entitles the investor a payoff of $m_{t,t+1}$. The security with SDF payoff has return

$$r^{\text{SDF}}_{t,t+1} \equiv \frac{m_{t,t+1}}{E_t[m^2_{t,t+1}]} - 1 \tag{19}$$

and is a hedging asset with gross return lower than the gross return of the risk-free bond. We establish the following result.

Result 2 *The expected excess (log) return of a security with SDF payoff is related to* $L[m^2]$ *as* *follows:*

$$L[m^2] \;\geq\; \underbrace{E[\log(R_{t+1,f}) \;-\; \log(1+r^{\text{SDF}}_{t,t+1})]}_{\text{Expected excess (log) return of SDF security}} \;\geq 0. \tag{20}$$

Proof: See Appendix B. ■

The expected excess (log) return of a security that pays m is limited by the entropy of m^2, much like how the Sharpe ratio is upper bounded by the volatility of m (e.g., Cochrane (2005, page 20))).[6] Our result can be traced to equation (B3) of Appendix B, which shows that the conditional expected excess (log) return of the SDF security reflects the difference between the conditional entropy of m^2 and the conditional entropy of m.

Contemplating our Result 2, there is one additional issue to be addressed: When does the bound in equation (20) bind? The next result formalizes the special restrictions on the SDF, denoted as $m^\bullet$, under which the bound on $L[m^2]$ becomes tight.

Problem 1 *Identify an SDF, denoted by $m^\bullet$, satisfying $L[(m^\bullet)^2] = \mathbb{LB}_{m^2}$ in equation (13) subject to correct pricing (i.e., $E[m^\bullet] = E[q_t]$ and $E[m^\bullet\,\mathbf{R}] = \mathbf{1}$).* ■

Appendix C shows that a solution to Problem 1 is of the form

$$m^\bullet_{t,t+1} = \frac{\psi}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{\nu}}, \tag{21}$$

[6] We can interpret $L[(m^P_{t,t+1})^2]$. The security with the payoff $m^P_{t,t+1}/m^T_{t,t+1}$ has a time t price of $E_t[(m^P_{t,t+1})^2]$ and a well-defined return: $r^{\text{PSDF}}_{t,t+1} \;\equiv\; \frac{m^P_{t,t+1}/m^T_{t,t+1}}{E_t[(m^P_{t,t+1})^2]} - 1$. Adapting the proof of Result 2, we can show that $L[(m^P_{t,t+1})^2] \geq E[\log(R_{t,t+1,\infty})] - E[\log(1+r^{\text{PSDF}}_{t,t+1})]$. This can be distinguished from Alvarez and Jermann (2005, Proposition 3, equation (4)), who establish that $L[m^P_{t,t+1}]$ lower bounds the expected (log) return of a generic portfolio in excess of expected (log) return of a long-term discount bond.

where ν and ψ are sequentially determined via equations (C13) and (C14), and and vector $\mathbf{a}$ is defined in equation (15). Equation (21) facilitates the construction of the SDF security that uses moments based on observed asset returns. While our result is in terms of entropy, it is analogous, in terms of variance, to the approach of Hansen and Jagannathan (1991, page 235, equation (3)).

Pushing interpretations, one remaining question is why not choose any arbitrary $L[m^n]$, for $n > 2$, as a dispersion measure to study asset pricing models? To answer this question, we present the following baseline lower bound result when $n = 3$ (see Appendix D for the steps of the proof):

$$L[m^3] \ \geq \ \log\left(E[(1+r_{t,t+1}^{\text{SDF}})^3]\right). \tag{22}$$

It is seen that the $L[m^3]$ statistic is denominated in terms of return asymmetries of the SDF security, which requires an estimate of the third return moment. With this said, we turn to a further elaboration on the relative merits of $L[m^2]$ versus $L[m]$ as a dispersion measure.

3. Revealing the value added of $L[m^2]$ over $L[m]$

We study the value added of $L[m^2]$ over $L[m]$ from different angles.

3.1. *Models that pass the bound on* $L[m]$ *but not* $L[m^2]$

Appendix E shows that the lower bound on $L[m]$ and $L[m^P]$ is given by (where the vector $\mathbf{a}$ is internally determined from the properties of $\mathbf{R}_{t,t+1}$ via the calculation in equation (15))

$$\left.\begin{aligned} L[m] &\geq \mathbb{LB}_m \equiv E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})] \ - \ \log((E[q_t])^{-1}), \\ L[m^P] &\geq \mathbb{LB}_{m^P} \equiv E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})] \ - \ E[\log(R_{t,t+1,\infty})] \end{aligned}\right\} \text{Entropy bounds.} \tag{23}$$

In this light, we explore a framework in which an individual asset pricing model passes the lower entropy bound on m (i.e., $L[m] \geq \mathbb{LB}_m$), but the $L[m^2]$ statistic potentially serves as a way of rejecting it (i.e., $L[m^2] < \mathbb{LB}_{m^2}$). We formalize the following problem:[7]

Problem 2 *Identify an SDF, denoted by m^G, satisfying $L[m^G] = \mathbb{LB}_m$ in equation (23) subject to correct pricing (i.e., $E[m^G] = E[q_t]$ and $E[m^G \mathbf{R}] = \mathbf{1}$).* ■

The relevance of Problem 2 stems from the feature that m^{G}, by construction, respects the lower bound on $L[m]$ but leaves open the possibility that m^{G} can be rejected using the lower bound on $L[m^2]$. A solution to Problem 2 is (see Appendix F)

$$m^{\mathrm{G}}_{t,t+1} = \frac{\beta}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{\alpha}} \quad \text{for some parameters } \alpha \text{ and } \beta, \tag{24}$$

where α and β sequentially solve the equations

$$\begin{aligned} 0 &= E[q_t] - \exp((\alpha-1)E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})]) \times E[(\mathbf{a}'\mathbf{R}_{t,t+1})^{-\alpha}] \quad \text{and} & (25) \\ 0 &= \beta - \exp((\alpha-1)\, E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})]). & (26) \end{aligned}$$

The class of SDFs proposed in equation (24) is a power function of the portfolio return $\mathbf{a}'\mathbf{R}_{t,t+1}$ with exponent α, and absent the cross-product terms between asset returns in $\mathbf{R}_{t,t+1}$. The solution depends on $\mathbf{R}_{t,t+1}$ and the set of $m^{\mathrm{G}}_{t,t+1}$ satisfying Problem 2 is potentially large.

Turning to our empirical exercises and implementations, Table 1 considers a variety of gross return vectors $\mathbf{R}_{t,t+1}$, which gives rise to different individual pricing models. Here, we take a

[7] We can develop the consequences of imposing $L[m^P] = \mathbb{LB}_{m^P}$, but refrain from doing so to focus on studying the implications of Problem 2. The solution is available from the authors.

collection of return time-series that capture various dimensions of the equity market universe, for example, those highlighted in, among others, Fama and French (2017). These observable characteristics, in addition to the market portfolio, include 25 portfolios formed on the basis of size, in conjunction with (1) book-to-market, (2) momentum, (3) accrual, (4) investment, (5) long-term reversal, (6) net issues, (7) operating profitability, (8) variance, and (9) residual variance.

For each set of returns (there are nine different sets, where $\mathbf{R}_{t,t+1}$ is of dimension 26×1), we solve equations (25) and (26), which guarantees that $L[m^{\mathrm{G}}]$ matches the lower bound, $\mathbb{LB}_m$. In so doing, we circumvent moving parts in our comparisons whereby the considered set of models satisfy the lower bound on $L[m]$ by construction. With this protocol for a comparison, we investigate whether the considered set of models also pass the lower bound on $L[m^2]$.

Table 1 shows, for example, that exponent α is between 2.074 to 3.826, whereas the centering coefficient, β, ranges between 1.032 and 1.122. To gauge statistical significance, we formulate the null hypothesis as

$$L[(m^G)^2] \, - \, \mathbb{LB}_{m^2} \geq 0, \tag{27}$$

which we test using the following bootstrap procedure:

- Generate, with replacement, the return time-series of $\mathbf{R}_{t,t+1}$, matching the respective sample periods;
- In each bootstrap draw, compute (α, β) by solving equations (25) and (26), and construct the time-series of $m^G_{t,t+1}$;
- Using this time-series of $m^G_{t,t+1}$, compute the model-based entropy $L[(m^G)^2]$ in each bootstrap draw and compare it to the corresponding regenerated value of $\mathbb{LB}_{m^2}$. If the criterion

in equation (27) is satisfied, we assign a value of one.

The bootstrap procedure is repeated 100,000 times, and we count the proportion of bootstrap draws that satisfy $L[(m^G)^2] - \mathbb{LB}_{m^2} \geq 0$. We record this proportion as the empirical p-value in Table 1, and low p-values imply *rejection* of the null hypothesis.

The results reveal the value added of $L[m^2]$ over $L[m]$ as a tool to evaluate an individual asset pricing model. Focusing on the results in Table 1, the null hypothesis $L[(m^G)^2] - \mathbb{LB}_{m^2} \geq 0$ is rejected for seven out of nine models with a p-value lower than 0.1.

Consider the model based on market, and 25 portfolios sorted on size and investment, which generates an $L[m^2]$ of 0.1136 and is lower than $\mathbb{LB}_{m^2}$ of 0.2589. The model is rejected with a bootstrap p-value of 0.001. On the other hand, the model based on market and 25 portfolios sorted on size and momentum, generates an $L[m^2]$ of 0.2193, which is higher than $\mathbb{LB}_{m^2}$ of 0.1891. This model furnishes a p-value of 0.383 and is not rejected.[8] The message from our findings is that it is feasible for a model to pass the lower bound on $L[m]$ and at the same time fail to pass the lower bound on $L[m^2]$. The calculations using different $\mathbf{R}_{t,t+1}$ put our results on a firmer footing.

3.2. *Building a further case for considering $L[m^2]$*

This section deviates from the preceding analysis by *not* restricting the model-based entropy to be equal to the lower bound on $L[m]$. Here we alter the playing field and reinforce the idea that models can pass the lower entropy bound on m yet fail the lower entropy bound on m^2.

[8] One could opt in favor of parsimony of the number of assets in $\mathbf{R}_{t,t+1}$ and consider a candidate subset of returns in the flavor of a factor pricing model. Since our procedure involves gross returns, *and not excess returns*, we consider, in addition to the market, the low and high portfolios sorted based on size, book-to-market, operating profitability, and investment (over the sample period of 1963:07–2017:06). We obtain $L[m] = \mathbb{LB}_m = 0.0151$, $L[m^2] = 0.0612$, $\mathbb{LB}_{m^2} = 0.1021$, and an associated bootstrap p-value of 0.026, implying model rejection.

Example 1 [*Linear SDF with returns of market and returns of options to the downside and upside*]: The model is

$$m_{t,t+1} = \boldsymbol{\eta}'[R_{t+1,f} \ \ \mathbf{Q}_{t,t+1}], \qquad \text{with} \qquad \mathbf{Q}_{t,t+1} = \begin{pmatrix} \frac{S_{t+1}}{S_t} \\ \frac{\max(S_t e^{-0.03} - S_{t+1}, 0)}{P_t[S_t e^{-0.03}]} \\ \frac{\max(S_t e^{-0.01} - S_{t+1}, 0)}{P_t[S_t e^{-0.01}]} \\ \frac{\max(S_{t+1} - S_t e^{0.01}, 0)}{C_t[S_t e^{0.01}]} \\ \frac{\max(S_{t+1} - S_t e^{0.03}, 0)}{C_t[S_t e^{0.03}]} \end{pmatrix}, \tag{28}$$

where S_t is the price of the equity market portfolio (proxied here by the S&P 500 index) and $P_t[S_t e^{-d}]$ ($C_t[S_t e^{d}]$) is the price of a put (call) on the market portfolio with strike that is $d\%$ out-of-the-money (OTM), for $d = 0.03$ or $d = 0.01$.

Following Cochrane (2005, pages 65 and 66), we estimate the projection coefficients in equation (28) as $\boldsymbol{\eta} = [-5.72 \ \ 6.63 \ \ 0.21 \ \ 0.07 \ \ -0.14 \ \ 0.04]$. ♣

Table 2 shows that the model in equation (28) generates positive entries of $m_{t,t+1}$ (the minimum is 0.65) with an annualized volatility of 108% ($0.31 \times \sqrt{12}$). The distribution of $m_{t,t+1}$ is positively skewed and heavy tailed. This SDF manifests a decreasing region as well as an increasing region in the return of the market portfolio. The increasing region is a consequence of the positive projection coefficient on the 3% OTM call.[9]

[9] How do we reconcile the use of projected SDFs that include option returns? For one, there is some precedence in considering such models in the derivatives literature in which long and short equity positions exhibit exposures to the downside and the upside (e.g., Bakshi, Madan, and Panayotov (2010), Chabi-Yo (2012), and Christoffersen, Heston, and Jacobs (2013)). In addition, Vanden (2004) builds a model in which agent faces nonnegative wealth constraints and shows that the SDF is a linear function of payoffs on calls and puts.

Taking a stand on the construction of the lower bound, we compute $\mathbb{LB}_m$ and $\mathbb{LB}_{m^2}$ using each of the nine sets of return portfolios comprising 26 assets as in our Table 1 (but matched over the 1990:01 to 2015:12 sample period with 311 observations). The question is: If the model in equation (28) is deemed acceptable by the $\mathbb{LB}_m$ metric, is it rejected by $\mathbb{LB}_{m^2}$?

The bootstrap p-values reported in Table 2 indicate that one *cannot reject* that this model passes $\mathbb{LB}_m$ (the minimum bootstrap p-value is 0.374), but is rejected in seven out of nine instances according to $\mathbb{LB}_{m^2}$. In our bootstrap procedure, we randomly select, with replacement, raw asset returns $\mathbf{R}_{t,t+1}$ and $\mathbf{Q}_{t,t+1}$ together, and recompute the projection coefficients $\boldsymbol{\eta}$. Then we generate the time-series of $m_{t,t+1}$ according to equation (28) and also compute the corresponding lower bounds according to equations (13) and (23). Here we perform 100,000 bootstrap trials.

Example 2 [*SDF is exponentially linear in returns based on Fama and French (2015)*]: Let $\mathbf{er}_{t,t+1} \equiv \mathbf{Q}_{t,t+1} - \mathbf{1}R_{t+1,f}$ be a 9×1 vector of excess returns, where $\mathbf{Q}_{t,t+1}$ is based on market, plus the low and high extreme portfolios featured in Fama and French (2015). Our model is

$$m_{t,t+1} = \exp(\lambda_0 - 1 + \boldsymbol{\lambda}'\mathbf{er}_{t,t+1}), \tag{29}$$

where the constants $(\lambda_0, \boldsymbol{\lambda}) \in \mathbb{R}^{10}$ solve the minimization problem $\inf_{(\lambda_0,\boldsymbol{\lambda})} -E[q_t]\lambda_0 + E[\exp(\lambda_0 - 1 + \boldsymbol{\lambda}'\mathbf{er}_{t,t+1})]$. This form of SDF can be traced to a minimum discrepancy problem described in Almeida and Garcia (2017), Borovička, Hansen, and Scheinkman (2016), and Bakshi, Chabi-Yo, and Gao (2017), where the objective is to minimize $E[m\log(m)]$ subject to $E[m\,\mathbf{er}] = \mathbf{0}$ and $E[m - E[q_t]] = 0$. The Lagrange multipliers on the constraints are $\boldsymbol{\lambda}$ and λ_0, respectively. ♣

Table 3 strengthens our arguments by showing that the point estimate of $L[m] = 0.0304$ exceeds five out of the nine portfolios consisting of 26 returns. Importantly, the null hypothesis of $L[m] \geq \mathbb{LB}_m$ is not rejected with a minimum bootstrap p-value of 0.289. In contrast, based on the reported bootstrap p-values, we can reject that $L[m^2] = 0.1207$ is greater than $\mathbb{LB}_{m^2}$ in eight out of nine instances. This model generates a positive skewness of 1.0 and an excess kurtosis of 6.8.

Example 3 [*SDF is exponentially linear in the baseline returns in Fama and French (2015), augmented with momentum portfolios*]: Let $\mathbf{er}_{t,t+1}^{\text{aug}} \equiv \mathbf{Q}_{t,t+1} - \mathbf{1}R_{t+1,f}$ be a 11×1 vector of excess returns. Here $\mathbf{Q}_{t,t+1}$ is based on market, the low and high extreme portfolios featured in Fama and French (2015), and the portfolio of extreme past losers and winners. More concretely,

$$m_{t,t+1} = \exp(\lambda_0 - 1 + \boldsymbol{\lambda}'\mathbf{er}_{t,t+1}^{\text{aug}}), \tag{30}$$

where $(\lambda_0, \boldsymbol{\lambda}) \in \mathbb{R}^{12}$ also solve the minimization problem outlined in Example 2. ♣

Table 4 is telling about the pricing implications of incorporating the momentum portfolios. First, this model produces sufficient $L[m]$: the point estimate of 0.0569 always exceeds the lower bounds anchored to the nine portfolios. Observe further that the bootstrap p-values rise across the board versus the Table 3 counterparts and supports a minimum value of 0.704. Moreover, the addition of momentum portfolios almost doubles the model-based entropy of m^2 to 0.2428, consistent with a time-series of $m_{t,t+1}$, which manifests pronounced deviations from lognormality.

The takeaway is that the model in equation (30) is associated with reduced likelihood of rejecting the null hypothesis $L[m^2] - \mathbb{LB}_{m^2} \geq 0$, with four out of nine p-values now *less* than 0.1 (with size and momentum, size and net issue, size and variance, and size and residual variance).

From another perspective, the p-values are about three times larger than those in Table 3.

3.3. Additional clarifications

While we show that models that pass the bound on $L[m]$ can often fail to pass the bound on $L[m^2]$, is the reverse true? That is, whether a model that passes the lower bound on $L[m^2]$ fail to pass the lower bound on $L[m]$. To pursue this line of inquiry, we return to Problem 1 and compute the properties of the SDF $m^{\bullet}_{t,t+1}$ at which $L[m^2] = \mathbb{LB}_{m^2}$. Based on a minimum p-value of 0.731 reported in Table 5, all models are seen passing the lower bound on $L[m]$. These models generate SDF volatility (monthly) between 0.25 and 0.53.

Closing, the criterion for model assessment proposed in our paper is adept at flagging models when the m underlying a model is not lognormally distributed. Theoretically, when $\log(m)$ is normally distributed, $L[m^2] = 4L[m]$. In such an economic environment, models that fail the lower bound on $L[m]$ can be expected to fail the lower bound on $L[m^2]$. The Internet Appendix (Section B, Table Internet Appendix-II) highlights this feature in the context of some models, showing that the setting of normally distributed shocks does not offer a proper playing field to understand the distinctions between $L[m^2]$ and $L[m]$.[10] This prompted our empirical approach described in Tables 1 through 4.

[10]Colacito, Ghysels, Meng, and Siwasarit (2016) propose a model of log SDF that depends on innovation in consumption growth by modeling the growth rate of consumption as a skew-normal variable with time-varying parameters. They show that the introduction of time-varying skewness substantially increases equity risk premiums and produces sizable variation in conditional risk premiums. The SDF in their model is conditionally lognormal, but the unconditional distribution of the SDF is not lognormal.

4. Conclusions

A central problem in finance is the specification of the SDF (denoted by $m_{t,t+1}$). We study this problem by providing new asset pricing restrictions that are based on the entropy of $m^2_{t,t+1}$.

In our analysis, we address the conceptual differences between the unconditional entropy of $m^2_{t,t+1}$ versus the unconditional entropy of $m_{t,t+1}$. In particular, we establish that the unconditional entropy of $m^2_{t,t+1}$ is the upper bound on the unconditional expected excess (log) return of a security that pays $m_{t,t+1}$. The entropy restrictions we develop are based on the ability of the SDF to *jointly* price the risk-free bond, the long-term discount bond, and a set of risky assets.

Our focus is on showing that a model can meet the lower bound on the entropy of $m_{t,t+1}$ and can be rejected based on the lower bound on the entropy of $m^2_{t,t+1}$. Thus, the unconditional entropy restrictions on $m^2_{t,t+1}$ proposed in this paper allows one to flag asset pricing models that are not flagged according to the entropy of $m_{t,t+1}$. Our approach can be extended to incorporate conditioning information.

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Appendix A: Proof of Result 1

We adopt the following notation to streamline equation presentation and the steps of the proof:

$$\mathbf{y} \equiv \Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right] E\left[\mathbf{R}_{t,t+1}\right]\right), \quad \mathbf{y}_P \equiv \Sigma_P^{-1}\left(\mathbf{1} - E\left[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}\right]\right), \quad \text{and} \quad \mathbf{a} \equiv \frac{\mathbf{y}}{\mathbf{1}'\mathbf{y}}, \tag{A1}$$

where $q_t = E_t\left[m_{t,t+1}\right]$, Σ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}$, and Σ_P is the variance-covariance matrix of $\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}$. We assume that $\mathbf{a}'\mathbf{R}_{t,t+1}$ is strictly positive. We define

$$\begin{aligned}
\mathrm{er}_R &\equiv E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] - \log((E[q_t])^{-1}) \text{ and} && \text{(A2)}\\
\mathrm{er}_\infty &\equiv E[\log(R_{t,t+1,\infty})] - \log((E[q_t])^{-1}). && \text{(A3)}
\end{aligned}$$

Proof of the entropy bound on $m_{t,t+1}^2$ in equation (13). By the definition of entropy: $L[m^2] = \log(E[m^2]) - E[\log(m^2)]$. Then

$$\begin{aligned}
L[m_{t,t+1}^2] &= \log\left(E[m_{t,t+1}^2]\right) - 2\log\left(E\left[q_t\right]\right) + 2L\left[m_{t,t+1}\right],\\
&= \log\left(1 + \frac{E[m_{t,t+1}^2] - \left(E\left[q_t\right]\right)^2}{\left(E\left[q_t\right]\right)^2}\right) + 2L\left[m_{t,t+1}\right],\\
&= \log\left(1 + \frac{\mathrm{var}\left[m_{t,t+1}\right]}{\left(E[q_t]\right)^2}\right) + 2L\left[m_{t,t+1}\right],\\
&\geq \log\left(1 + \frac{\mathrm{var}[m_{t,t+1}]}{\left(E\left[q_t\right]\right)^2}\right) + 2\,\mathrm{er}_R. \quad (\text{as } L[m] \geq \mathrm{er}_R;\ \text{see } (A2) \text{ and } (E5)) && \text{(A4)}
\end{aligned}$$

Because $E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = \mathbf{1}$ and setting $q_t = E_t\left[m_{t,t+1}\right]$, it follows that

$$E\left[m_{t,t+1}\left(\mathbf{R}_{t,t+1} - E\left(\mathbf{R}_{t,t+1}\right)\right)\right] = \left(\mathbf{1} - E\left[q_t\right] E\left[\mathbf{R}_{t,t+1}\right]\right). \tag{A5}$$

Multiplying equation (A5) by $\left(\mathbf{1}-(E[q_t])\,E[\mathbf{R}_{t,t+1}]\right)'\Sigma^{-1}$ yields

$$\left(\mathbf{1}-(E[q_t])\,E[\mathbf{R}_{t,t+1}]\right)'\Sigma^{-1}\left(\mathbf{1}-E[q_t]\,E[\mathbf{R}_{t,t+1}]\right)=E\left[m_{t,t+1}\left(\mathbf{1}-(E[q_t])\,E[\mathbf{R}_{t,t+1}]\right)'\Sigma^{-1}\left(\mathbf{R}_{t,t+1}-E[\mathbf{R}_{t,t+1}]\right)\right]. \tag{A6}$$

Applying the Cauchy-Schwarz to the right-hand side of equation (A6), it follows that

$$\begin{aligned}
\operatorname{var}[m_{t,t+1}] &\geq \left(\mathbf{1}-(E[q_t])\,E[\mathbf{R}_{t,t+1}]\right)'\Sigma^{-1}\left(\mathbf{1}-(E[q_t])\,E[\mathbf{R}_{t,t+1}]\right), \\
&\geq \left(\mathbf{1}-(E[q_t])\,E[\mathbf{R}_{t,t+1}]\right)'\Sigma^{-1}\Sigma\Sigma^{-1}\left(\mathbf{1}-(E[q_t])\,E[\mathbf{R}_{t,t+1}]\right), \\
&\geq \mathbf{y}'\Sigma\mathbf{y}.
\end{aligned} \tag{A7}$$

Combining the expressions in equations (A4) and (A7), we obtain the bound on $L[m^2_{t,t+1}]$ presented in equation (13) of Result 1. ■

Proof of the entropy bound on $(m^P_{t,t+1})^2$ in equation (14) of Result 1. We write

$$\begin{aligned}
L[(m^P_{t,t+1})^2] &= \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)], \\
&= \log(E[(m^P_{t,t+1})^2]) - 2E[\log(m^P_{t,t+1})], \\
&= \log(E[(m^P_{t,t+1})^2]) + 2L\left[m^P_{t,t+1}\right], \\
&= \log\left(1+\operatorname{var}[m^P_{t,t+1}]\right) + 2L[m^P_{t,t+1}].
\end{aligned} \tag{A8}$$

We show in equation (E9) that $L[m^P_{t,t+1}] \geq \text{er}_R - \text{er}_\infty$ (the complete expressions for er_R and er_∞ are in equations (A2) and (A3), respectively). Therefore, we deduce that

$$L[\left(m^P_{t,t+1}\right)^2] \geq \log\left(1+\operatorname{var}\left[m^P_{t,t+1}\right]\right) + 2\left(\text{er}_R - \text{er}_\infty\right). \tag{A9}$$

Since $E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = E\left[m^P_{t,t+1}\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right] = \mathbf{1}$, we then obtain

$$E\left[m^P_{t,t+1}\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right)\right] = \mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]. \tag{A10}$$

Multiplying each side of equation (A10) by $\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \Sigma_P^{-1}$, we get

$$\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \Sigma_P^{-1}\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right) = E\left[m^P_{t,t+1}\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \Sigma_P^{-1}\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right)\right]. \tag{A11}$$

Applying the Cauchy-Schwarz inequality to the right-hand side of equation (A11), we note that

$$\begin{aligned}
\operatorname{var}\left[m^P_{t,t+1}\right] &\geq \left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \Sigma_P^{-1}\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right), \\
&\geq \left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \left(\Sigma_P^{-1}\right)' \Sigma_P \Sigma_P^{-1}\left(\mathbf{1} - E\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right)\right), \\
&\geq \mathbf{y}_P' \Sigma_P \mathbf{y}_P. \qquad \text{(where noting } \mathbf{y}_P \equiv \Sigma_P^{-1}\left(\mathbf{1} - E\left[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}\right]\right)\text{)}
\end{aligned} \tag{A12}$$

Inserting the bound derived in equation (A12) into equation (A9) leads to the bound in equation (14) of Result 1. ■

Appendix B: Proof of equation (20) of Result 2

The gross return of the security with SDF payoff is $1 + r^{\text{SDF}}_{t,t+1} = \frac{m_{t,t+1}}{E_t[m^2_{t,t+1}]}$, and it satisfies the Euler equation with $E_t[m_{t,t+1}(1 + r^{\text{SDF}}_{t,t+1})] = 1$.

There are two parts of the proof of equation (20). First, we establish that $L[m^2] \geq E[\log(R_{t+1,f})] - E[\log(1 + r^{\text{SDF}}_{t,t+1})]$. Next, we show that the security that pays the SDF is a hedging asset with expected return lower than the risk-free return. Let $L_t[m]$ denote the conditional entropy.

Taking logs of the expression for $1+r^{\text{SDF}}_{t,t+1}$, and adding and subtracting $\log(m^2_{t,t+1})$, we obtain

$$\log(1+r^{\text{SDF}}_{t,t+1}) = \log(m_{t,t+1}) - \log(E_t[m^2_{t,t+1}]) + \log(m^2_{t,t+1}) - 2\log(m_{t,t+1}). \tag{B1}$$

Netting out $\log(m_{t,t+1})$ and taking expectations on both sides of equation (B1), we have

$$E_t[\log(1+r^{\text{SDF}}_{t,t+1})] + E_t[\log(m_{t,t+1})] \;=\; \overbrace{E_t[\log(m^2_{t,t+1})] - \log(E_t[m^2_{t,t+1}])}^{-L_t[m^2_{t,t+1}] \text{ from eq. (7)}}. \tag{B2}$$

Subtracting $\log(E_t[m_{t,t+1}])$ from both sides of equation (B2) and rearranging, it follows that

$$\begin{aligned} L_t[m^2_{t,t+1}] + \log(E_t[m_{t,t+1}]) \;&=\; -E_t[\log(1+r^{\text{SDF}}_{t,t+1})] + \underbrace{\log(E_t[m_{t,t+1}]) - E_t[\log(m_{t,t+1})]}_{L_t[m_{t,t+1}]\geq 0} && \text{(B3)} \\ &\geq\; -E_t[\log(1+r^{\text{SDF}}_{t,t+1})]. && \text{(B4)} \end{aligned}$$

Rearranging, we obtain the following expression for the conditional entropy of $m^2_{t,t+1}$:

$$L_t[m^2_{t,t+1}] \;\geq\; -E_t[\log(1+r^{\text{SDF}}_{t,t+1})] \;-\; \log(E_t[m_{t,t+1}]). \tag{B5}$$

Since the gross return of the risk-free bond satisfies $E_t\left[m_{t,t+1}\right] = 1/R_{t+1,f}$, we obtain

$$L_t[m^2_{t,t+1}] \;\geq\; \log(R_{t+1,f}) \;-\; E_t[\log(1+r^{\text{SDF}}_{t,t+1})]. \tag{B6}$$

Taking unconditional expectations on both sides of equation (B6),

$$E[L_t[m^2_{t,t+1}]] \;\geq\; E[\log(R_{t+1,f})] \;-\; E[\log(1+r^{\text{SDF}}_{t,t+1})]. \tag{B7}$$

Now exploit the following relation

$$E[L_t[m_{t,t+1}^2]] \;\leq\; L[m_{t,t+1}^2] \quad \text{since} \quad L[u^2] = E[L_t[u^2]] \;+\; L[E_t[u^2]] \;\text{ for any random variable } u. \tag{B8}$$

Therefore, $L[m_{t,t+1}^2] \geq E[\log(R_{t+1,f})] - E[\log(1+r_{t,t+1}^{\text{SDF}})]$. Our measure is tied to the maximum expected (log) return on a security that pays the SDF. ♣

Completing the picture, we need to show that the security with SDF payoff is a hedging asset. Observe that $E_t[m_{t,t+1}^2] \geq (E_t\,[m_{t,t+1}])^2$, because $\text{var}\,(m_{t,t+1}) > 0$. Hence,

$$E_t[1+r_{t,t+1}^{\text{SDF}}] = \frac{E_t[m_{t,t+1}]}{E_t[m_{t,t+1}^2]} \;\leq\; \frac{E_t\,[m_{t,t+1}]}{(E_t\,[m_{t,t+1}])^2} = \frac{1}{E_t\,[m_{t,t+1}]} = R_{t+1,f}. \tag{B9}$$

Equation (B9), thus, shows that $E_t[1+r_{t,t+1}^{\text{SDF}}] \leq R_{t+1,f}$, which implies that

$$\log\left(E_t\left[1+r_{t,t+1}^{\text{SDF}}\right]\right) \;\leq\; \overbrace{\log\left(\frac{1}{E_t[m_{t,t+1}]}\right)}^{\log(R_{t+1,f})}. \tag{B10}$$

By an application of Jensen's inequality,

$$E_t[\log(1+r_{t,t+1}^{\text{SDF}})] \;\leq\; \log\left(E_t\left[1+r_{t,t+1}^{\text{SDF}}\right]\right) \;\leq\; \log(R_{t+1,f}). \tag{B11}$$

It then follows that

$$\log(R_{t+1,f}) - E_t[\log(1+r_{t,t+1}^{\text{SDF}})] \geq 0. \quad \text{Therefore,} \quad E[\log(R_{t+1,f})] - E[\log(1+r_{t,t+1}^{\text{SDF}})] \geq 0.$$

The proof is complete. ∎

Appendix C: Proof of equation (21)

Our Result 1 implies that $L[m^2] \geq \mathbb{LB}_{m^2}$, where

$$\mathbb{LB}_{m^2} \equiv 2\left(E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] - \log((E[q_t])^{-1})\right) + \log(1 + \mathbf{y}'\Sigma\mathbf{y}/(E[q_t])^2), \tag{C1}$$

with $\mathbf{a}$ and $\mathbf{y}$ displayed in equation (A1).

To investigate the restrictions under which the bound becomes binding, we need to show that there exists $m^{\bullet}_{t,t+1}$ with

$$1 + r^{\bullet\text{SDF}}_{t,t+1} = \frac{m^{\bullet}_{t,t+1}}{E[(m^{\bullet}_{t,t+1})^2]} \text{ such that } \mathbb{LB}_{m^2} = E[\log(\frac{R_{t+1,f}}{1 + r^{\bullet\text{SDF}}_{t,t+1}})]. \tag{C2}$$

Guided by Result 1, the identity (C2) becomes

$$\mathbb{LB}_{m^2} \equiv E\left[\log\left(((E[q_t])\,\mathbf{a}'\mathbf{R}_{t,t+1})^2(1 + \mathbf{y}'\Sigma\mathbf{y}/(E[q_t])^2)\right)\right] = E[\log(\frac{R_{t+1,f}}{1 + r^{\bullet\text{SDF}}_{t,t+1}})]. \tag{C3}$$

Upon simplification,

$$\frac{R_{t+1,f}}{1 + r^{\bullet\text{SDF}}_{t,t+1}} = (\mathbf{a}'\mathbf{R}_{t,t+1})^2\left((E[q_t])^2 + \mathbf{y}'\Sigma\mathbf{y}\right). \tag{C4}$$

Thus,

$$1 + r^{\bullet\text{SDF}}_{t,t+1} = \frac{R_{t+1,f}\,\mho^{\bullet}}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^2} \quad \text{with} \quad \mho^{\bullet} \equiv \frac{1}{(E[q])^2 + \mathbf{y}'\Sigma\mathbf{y}}. \tag{C5}$$

With the return of the SDF security of the form $\frac{m^{\bullet}_{t,t+1}}{E[(m^{\bullet}_{t,t+1})^2]}$, one needs to find an SDF $m^{\bullet}_{t,t+1}$ that satisfies

$$\frac{\mho^{\bullet}R_{t+1,f}}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^2} = \frac{m^{\bullet}_{t,t+1}}{E[(m^{\bullet}_{t,t+1})^2]}. \tag{C6}$$

We conjecture and then verify that a solution is as given in equation (21). Direct substitution implies that

$$\frac{m^{\bullet}_{t,t+1}}{E[(m^{\bullet}_{t+1})^2]} = \frac{\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\nu}}}{E\left[\frac{\psi}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{2\nu}}\right]}. \tag{C7}$$

With this step, we replace the SDF in identity (C6) and obtain

$$\frac{\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\nu}}}{E\left[\frac{\beta}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{2\nu}}\right]} = \frac{\mho^{\bullet} R_{t+1,f}}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{2}}. \tag{C8}$$

Proceeding we take logs and then apply the expectations operator. Thus, it follows that

$$E\left[\log\left(\frac{\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\nu}}}{E\left[\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{2\nu}}\right]}\right)\right] - \log(\psi) = E\left[\log\left(\frac{\mho^{\bullet} R_{t+1,f}}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{2}}\right)\right]. \tag{C9}$$

To enforce the correct pricing of the risk-free return, it must hold that

$$E[q_t] = \psi E\left[\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\nu}}\right]. \text{ Hence, } \log(\psi) = \log\left(E\left[q_t\right]\right) - \log\left(E\left[\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\nu}}\right]\right). \tag{C10}$$

With the aid of equations (C10) and (C9), we determine that

$$E\left[\log\left(\frac{\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\nu}}}{E\left[\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{2\nu}}\right]}\right)\right] - \log\left(E\left[q_t\right]\right) + \log\left(E\left[\frac{1}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\nu}}\right]\right) = E\left[\log\left(\frac{\mho^{\bullet} R_{t+1,f}}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{2}}\right)\right]. \tag{C11}$$

This expression simplifies to

$$E\left[\log\left(\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{\nu}}\right)\right]-\log\left(E\left[\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{2\nu}}\right]\right)-\log\left(E\left[q_t\right]\right)$$
$$+\log\left(E\left[\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{\nu}}\right]\right)-E\left[\log\left(\frac{\mho^{\bullet}R_{t+1,f}}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{2}}\right)\right]=0. \tag{C12}$$

Finally, we determine ν and ψ as solutions to the equations

$$E\left[\log\left(\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{\nu}}\right)\right]-\log\left(E\left[\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{2\nu}}\right]\right)-\log\left(E\left[q_t\right]\right)$$
$$+\log\left(E\left[\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{\nu}}\right]\right)-E\left[\log\left(\frac{\mho^{\bullet}R_{t+1,f}}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{2}}\right)\right]=0, \tag{C13}$$

and

$$E\left[\log\left(\frac{\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{\nu}}}{E\left[\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{2\nu}}\right]}\right)\right]-\log\left(E\left[q_t\right]\right)+\log\left(E\left[\frac{1}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{\nu}}\right]\right)-E\left[\log\left(\frac{\mho^{\bullet}R_{t+1,f}}{\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)^{2}}\right)\right]=0. \tag{C14}$$

In this case, the lower bound on $L[m^2]$ becomes binding. ■

Appendix D: Proof of bound on $L[m^3]$ in equation (22)

The gross return of the security with SDF payoff is $1+r_{t,t+1}^{\text{SDF}}=\frac{m_{t,t+1}}{E_t[m_{t,t+1}^2]}$. Hence,

$$\left(1+r_{t,t+1}^{\text{SDF}}\right)^3=\frac{m_{t,t+1}^3}{(E_t[m_{t,t+1}^2])^3} \qquad \text{and} \qquad E_t[(1+r_{t,t+1}^{\text{SDF}})^3]=\frac{E_t[m_{t,t+1}^3]}{(E_t[m_{t,t+1}^2])^3}. \tag{D1}$$

The log of the above expression is

$$
\begin{aligned}
\log\left(E_t[(1+r^{\text{SDF}}_{t,t+1})^3]\right) &= \log(E_t\left[m^3_{t,t+1}\right]) - 3\log(E_t\left[m^2_{t,t+1}\right]), && \text{(D2)}\\
&= L_t\left[m^3_{t,t+1}\right] + E_t\left[\log(m^3_{t,t+1})\right] - 3\log\left(E_t\left[m^2_{t,t+1}\right]\right), && \text{(D3)}\\
&= L_t\left[m^3_{t,t+1}\right] + E_t\left[\log(m^3_{t,t+1})\right] - 3\left(L_t[m^2_{t,t+1}] + E_t\left[\log(m^2_{t,t+1})\right]\right), &&\\
&= L_t[m^3_{t,t+1}] - 3L_t[m^2_{t,t+1}] - 3E_t[\log(m_{t,t+1})], && \text{(D4)}\\
&= L_t[m^3_{t,t+1}] - 3L_t[m^2_{t,t+1}] + 3\left(\log\left(E_t[m_{t,t+1}]\right) - E_t[\log(m_{t,t+1})]\right) &&\\
&\quad -3\log\left(E_t[m_{t,t+1}]\right), && \text{(D5)}\\
&= L_t[m^3_{t,t+1}] - 3L_t[m^2_{t,t+1}] + 3\left(L_t[m_{t,t+1}] - \log\left(E_t[m_{t,t+1}]\right)\right), &&\\
&= L_t[m^3_{t,t+1}] - 3L_t\left[m^2_{t,t+1}\right] + 3L_t[m_{t,t+1}] - 3\log\left(E_t[m_{t,t+1}]\right). && \text{(D6)}
\end{aligned}
$$

It is then true that

$$\log(E_t[(1+r^{\text{SDF}}_{t,t+1})^3]) + 3\log(E_t[m_{t,t+1}]) + 3L_t[m^2_{t,t+1}] - 3L_t[m_{t,t+1}] = L_t[m^3_{t,t+1}]. \tag{D7}$$

Relying on the fact that

$$L_t[m^2_{t,t+1}] + \log\left(E_t[m_{t,t+1}]\right) = -E_t[\log(1+r^{\text{SDF}}_{t,t+1})] + L_t[m_{t,t+1}]. \tag{D8}$$

Since $-E_t[\log(1+r^{\text{SDF}}_{t,t+1})] \geq 0$, it follows that

$$L_t[m^2_{t,t+1}] + \log\left(E_t[m_{t,t+1}]\right) \geq L_t[m_{t,t+1}]. \tag{D9}$$

We can therefore express equation (D7) as

$$L_t[m_{t,t+1}^3] = \log\left(E_t[(1+r_{t,t+1}^{\text{SDF}})^3]\right) + 3\left(\underbrace{\left(L_t[m_{t,t+1}^2] + \log(E_t[m_{t,t+1}])\right) - L_t[m_{t,t+1}]}_{\geq 0}\right). \quad \text{(D10)}$$

As a result,

$$L_t[m_{t,t+1}^3] \geq \log(E_t\left[(1+r_{t,t+1}^{\text{SDF}})^3\right]). \quad \text{(D11)}$$

We now exploit the relation

$$L[m_{t,t+1}^3] \geq E\left[L_t[m_{t,t+1}^3]\right]. \quad \text{(D12)}$$

Therefore,

$$L[m_{t,t+1}^3] \geq L_t\left[m_{t,t+1}^3\right] \geq \log\left(E[(1+r_{t,t+1}^{\text{SDF}})^3]\right). \quad \text{(D13)}$$

Finally,

$$\frac{1}{2}(L[m_{t,t+1}^3] + E[\log R_{t+1,f}]) \geq \frac{1}{2}\left(\log(E[(1+r_{t,t+1}^{\text{SDF}})^3]) + E[\log R_{t+1,f}]\right). \quad \text{(D14)}$$

We have the complete proof. ■

Appendix E: Lower bound on $L[m]$ and $L[m^P]$

Here we present a lower bound on the entropy of $m_{t,t+1}$ and a lower bound on the entropy of $m_{t,t+1}^P$, when $m_{t,t+1}$ correctly price $N+2$ assets. Internet Appendix (Section A, Table Internet Appendix-I) illustrate the sharpness of these bounds.

We maintain the following notation (as in equation (A1)):

$$\mathbf{y} \equiv \Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right), \qquad \text{and} \qquad \mathbf{a} \equiv \frac{\mathbf{y}}{\mathbf{1}'\mathbf{y}}, \tag{E1}$$

where Σ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}$.

Proof of the bound on the entropy of $m_{t,t+1}$**:** Consider the following return:

$$\begin{aligned} E[\log(m_{t,t+1}\mathbf{a}'\mathbf{R}_{t,t+1})] &\leq \log(E[m_{t,t+1}\mathbf{a}'\mathbf{R}_{t,t+1}]), \\ &\leq \log(\mathbf{a}'E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right]), \\ &\leq \log(\mathbf{a}'\mathbf{1}) = \log(1), \\ &\leq 0. \end{aligned} \tag{E2}$$

From equation (E2) and noting that $\log(m_{t,t+1}\mathbf{a}'\mathbf{R}_{t,t+1}) = \log(m_{t,t+1}) + \log(\mathbf{a}'\mathbf{R}_{t,t+1})$, we deduce that

$$E\left[\log\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right)\right] \leq -E\left[\log\left(m_{t,t+1}\right)\right]. \tag{E3}$$

Adding $\log\left(E\left[m_{t,t+1}\right]\right)$ to both sides of equation (E3) yields

$$\log\left(E\left[m_{t,t+1}\right]\right) + E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] \;\leq\; \log(E[m_{t,t+1}]) - E\left[\log\left(m_{t,t+1}\right)\right] = L\left[m_{t,t+1}\right]. \tag{E4}$$

Since $q_t = E_t\left[m_{t,t+1}\right]$, equation (E4) simplifies to

$$L\left[m_{t,t+1}\right] \;\geq\; E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})] \;-\; \log\left(1/E\left[q_t\right]\right), \text{ where } \mathbf{a}' \text{ is as defined in equation } (E1). \tag{E5}$$

The result depends on the variance-covariance matrix of returns and the mean return vector. ■

Proof of the bound on the entropy of $m^P_{t,t+1}$: We note that

$$E[\log(m_{t,t+1}\mathbf{a}^{'}\mathbf{R}_{t,t+1})] = E[\log(m^P_{t,t+1}\frac{\mathbf{a}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}})]. \tag{E6}$$

Using Jensen's inequality, we have

$$\begin{aligned} E[\log(m^P_{t,t+1}\frac{\mathbf{a}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}})] &= E[\log(m_{t,t+1}\mathbf{a}^{'}\mathbf{R}_{t,t+1})], \\ &\leq \log(\mathbf{a}^{'}E[m_{t,t+1}\mathbf{R}_{t,t+1}]), \\ &\leq \log(\mathbf{a}^{'}\mathbf{1}) = 0. \end{aligned} \tag{E7}$$

From equation (E7), it is true that

$$E[\log(\frac{\mathbf{a}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}})] \leq -E[\log\left(m^P_{t,t+1}\right)] = L[m^P_{t,t+1}]. \tag{E8}$$

Hence,

$$L[m^P_{t,t+1}] \;\geq\; E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})] \;-\; E[\log(R_{t,t+1,\infty})]. \tag{E9}$$

This bound incorporates the information from $N+2$ assets. ■

Appendix F: Proof of solution to Problem 2

Observe that

$$L[m^{\mathrm{G}}_{t,t+1}] = E[\log(\frac{E[q_t]}{m^{\mathrm{G}}_{t,t+1}})], \tag{F1}$$

and it holds that $L[m^{\mathrm{G}}_{t,t+1}] = E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})] - \log((E[q_t])^{-1})$. This implies that

$$E\left[\log\left(\frac{\frac{\mathbf{a}^{'}\mathbf{R}_{t,t+1}}{(E[q_t])^{-1}}}{\frac{E[q_t]}{m^{\mathrm{G}}_{t,t+1}}}\right)\right] = 0. \tag{F2}$$

Hence,

$$E\left[\log\left(m^{\mathrm{G}}_{t,t+1}\mathbf{a}^{'}\mathbf{R}_{t,t+1}\right)\right] = 0. \tag{F3}$$

Now conjecture that $m^{\mathrm{G}}_{t,t+1} = \frac{\beta}{\left(\mathbf{a}^{'}\mathbf{R}_{t,t+1}\right)^{\alpha}}$. To ensure the SDF prices correctly the risk-free return

$$E[q_t] = E[m^{\mathrm{G}}_{t,t+1}] = \beta E[(\mathbf{a}^{'}\mathbf{R}_{t,t+1})^{-\alpha}]. \tag{F4}$$

Next, to ensure that equation (F3) holds,

$$E[\log(\frac{\beta}{\left(\mathbf{a}^{'}\mathbf{R}_{t,t+1}\right)^{\alpha}}\mathbf{a}^{'}\mathbf{R}_{t,t+1})] = 0. \tag{F5}$$

This implies that

$$\log(\beta) - (\alpha - 1)E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})] = 0. \text{ Rearranging, } \beta = \exp((\alpha-1)E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})]). \tag{F6}$$

Replace β in equation (F4):

$$E\left[q_t\right] = \exp\left((\alpha-1)E[\log(\mathbf{a}^{'}\mathbf{R}_{t,t+1})]\right)E\left(\frac{1}{\left(\mathbf{a}^{'}\mathbf{R}_{t,t+1}\right)^{\alpha}}\right). \tag{F7}$$

We have the corroborated the form of the solution to Problem 2. ■

Table 1
Models that pass the lower bound on $L[m]$, but may or may not pass the lower bound on $L[m^2]$

The lower bound on $L[m^2]$, denoted by $\mathbb{LB}_{m^2}$, is based on equation (13), while the lower bound on $L[m]$, denoted by $\mathbb{LB}_m$, is based on equation (23). Both bounds rely on the ability of the SDF to correctly price $N+2$ assets (the risk-free bond, the long-term discount bond, and N risky assets). The criterion is

$$\text{Model passes} = \begin{cases} \text{If } L[m] \geq \mathbb{LB}_m \\ \text{fails otherwise} \end{cases} \quad \text{and/or,} \;\; \text{Model passes} = \begin{cases} \text{If } L[m^2] \geq \mathbb{LB}_{m^2} \\ \text{fails otherwise.} \end{cases}$$

The set of models we consider in this table are of the type (see Problem 2)

$$m^{\mathrm{G}}_{t,t+1} = \frac{\beta}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\alpha}} \;\; \text{for some parameters } \alpha \text{ and } \beta,$$

where α and β solve $0 = E[q_t] - \exp((\alpha-1)E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})])E[(\mathbf{a}'\mathbf{R}_{t,t+1})^{-\alpha}]$, and $0 = \beta - \exp((\alpha - 1)E[\log(\mathbf{a}'\mathbf{R}_{t,t+1})])$. We consider nine choices of the portfolio return $\mathbf{a}'\mathbf{R}_{t,t+1}$: (1) Market plus 25 portfolios based on size and book-to-market (1931:07–2017:06); (2) Market plus 25 portfolios based on size and momentum (1931:07–2017:06); (3) Market plus 25 portfolios based on size and accrual (1963:07-2017:06); (4) Market plus 25 portfolios based on size and investment (1963:07-2017:06); (5) Market plus 25 portfolios based on size and long-term reversal (1931:07–2017:06); (6) Market plus 25 portfolios based on size and net issues (1963:07-2017:06); (7) Market plus 25 portfolios based on size and operating profitability (1963:07-2017:06); (8) Market plus 25 portfolios based on size and variance (1963:07-2017:06); (9) Market plus 25 portfolios based on size and residual variance (1963:07-2017:06). The reported bootstrap p-value is the proportion of bootstrap draws (out of 100,000) and tests the null hypothesis that $L[m^2] \geq \mathbb{LB}_{m^2}$ (low p-values imply rejection). $\mathbf{R}_{t,t+1}$ is of dimension 26×1.

	Market plus			$L[m] = \mathbb{LB}_m$ Entropy of $L[m]$			Entropy of $L[m^2]$			
	25 portfolios	α	β	$L[m]$	$\mathbb{LB}_m$	Pass	$L[m^2]$	$\mathbb{LB}_{m^2}$	Pass	p-val. (bootstrap)
1	Size×B/M	2.484	1.038	0.0221	0.0221	Yes	0.1115	0.1415	Yes	0.123
2	Size×Momentum	2.074	1.038	0.0321	0.0321	Yes	0.2193	0.1891	Yes	0.383
3	Size×Accrual	3.493	1.072	0.0238	0.0238	Yes	0.0968	0.2011	No	0.008
4	Size×Investment	3.826	1.094	0.0279	0.0279	Yes	0.1136	0.2589	No	0.001
5	Size×Long-Term Reversal	2.547	1.032	0.0176	0.0176	Yes	0.0678	0.0944	No	0.091
6	Size×Net Issues	3.619	1.122	0.0402	0.0402	Yes	0.1668	0.3413	No	0.006
7	Size×Operating Profitability	2.823	1.043	0.0191	0.0191	Yes	0.0774	0.1281	No	0.025
8	Size×Variance	3.657	1.104	0.0332	0.0332	Yes	0.1474	0.3115	No	0.067
9	Size×Residual Variance	3.601	1.113	0.0374	0.0374	Yes	0.1658	0.3399	No	0.070

Table 2
Model of SDF with returns of risk-free bond, equity market portfolio, and out-of-the-money puts and calls

The sample period for this exercise is 1990:01 to 2015:12 (311 observations). The model is

$$m_{t,t+1} = \boldsymbol{\eta}'[R_{t+1,f} \ \ \mathbf{Q}_{t,t+1}].$$

The 5×1 vector of gross returns $\mathbf{Q}_{t,t+1}$ contains

- gross return of the market (S&P 500 index);
- gross return of a 3% out-of-the-money put on the S&P 500 index;
- gross return of a 1% out-of-the-money put on the S&P 500 index;
- gross return of a 1% out-of-the-money call on the S&P 500 index;
- gross return of a 3% out-of-the-money call on the S&P 500 index.

The projection coefficients are estimated to be (i.e., Cochrane (2005, pages 65 and 66))

$$\boldsymbol{\eta} = [-5.72 \ \ 6.63 \ \ 0.21 \ \ 0.07 \ \ -0.14 \ \ 0.04].$$

We present the monthly properties of m and then $L[m]$ and $L[m^2]$. The lower bounds on $\mathbb{LB}_m$ and $\mathbb{LB}_{m^2}$ are extracted over the sample period 1990:01 to 2015:12. The reported p-values are based on a bootstrap and tests the null hypothesis that the model based entropy is greater than the lower bound. In each bootstrap draw, we randomly select, with replacement, raw asset returns $\mathbf{R}_{t,t+1}$ together with $\mathbf{Q}_{t,t+1}$, and recompute the projection coefficients $\boldsymbol{\eta}$. Then we generate the time-series of $m_{t,t+1}$ according to equation (28) and also compute the corresponding lower bounds according to equations (13) and (23). Here we perform 100,000 bootstrap trials.

Model	$L[m]$	$L[m^2]$	Properties of the SDF (monthly) Mean	Std.	Skewness	Kurtosis (excess)	Min.	Max.
Returns of risk-free bond, equity, puts, and calls	0.0314	0.1567	0.9976	0.31	3.7	18.4	0.65	2.85

	1990:01–2015:12 $\mathbf{R}_{t,t+1}$ is of dimension 26×1			
	$\mathbb{LB}_m$	Bootstrap p-value	$\mathbb{LB}_{m^2}$	Bootstrap p-value
Size×B/M	0.0298	0.572	0.3975	0.018
Size×Momentum	0.0266	0.570	0.2668	0.057
Size×Accrual	0.0196	0.652	0.1779	0.203
Size×Investment	0.0368	0.395	0.3509	0.029
Size×Long-Term Reversal	0.0352	0.374	0.2692	0.095
Size×Operating Profitability	0.0226	0.633	0.2012	0.195
Size×Net Issues	0.0348	0.433	0.4030	0.011
Size×Variance	0.0204	0.767	0.3138	0.038
Size×Residual Variance	0.0226	0.666	0.2762	0.050

Table 3
Model where log SDF is linear in a set of baseline excess returns

The sample period for this exercise is 1963:07 to 2017:06 (648 observations). The specific model we consider takes the form

$$m_{t,t+1} = \exp(\lambda_0 - 1 + \boldsymbol{\lambda}' \mathbf{er}_{t,t+1}),$$

where $\mathbf{er}_{t,t+1} \equiv \mathbf{Q}_{t,t+1} - \mathbf{1}R_{t+1,f}$ is a 9×1 vector of excess returns. The excess return $\mathbf{er}_{t,t+1}$ are based on $\mathbf{Q}_{t,t+1}$ from the five-factor model of Fama and French (2015):

- gross return of the market;
- gross returns of the two extreme low and high size portfolios;
- gross returns of the two extreme low and high book-to-market portfolios;
- gross returns of the two extreme low and high operating profitability portfolios;
- gross returns of the two extreme low and high investment portfolios.

The constants $(\lambda_0, \boldsymbol{\lambda})$ are a solution to the minimization problem $\inf_{(\lambda_0,\boldsymbol{\lambda})} -E[q_t]\lambda_0 + E[\exp(\lambda_0 - 1 + \boldsymbol{\lambda}'\mathbf{er}_{t,t+1})]$. This form of SDF obtains by minimizing the objective $E[m \log(m)]$ subject to $E[m\,\mathbf{er}] = \mathbf{0}$ and $E[m - E[q_t]] = 0$. The Lagrange multipliers on the equality constraints are $\boldsymbol{\lambda}$ and λ_0, respectively. Our procedure implies that

$$\lambda_0 = 1.01, \text{ and } \quad \boldsymbol{\lambda} = [36.3\ 0.0\ -0.2\ -13.8\ -10.0\ -0.1\ -0.2\ -10.1\ -4.2].$$

We present the monthly properties of m, and the model-based $L[m]$, and $L[m^2]$. The lower bounds on $\mathbb{LB}_m$ and $\mathbb{LB}_{m^2}$ are over the sample period 1963:07 to 2017:06. In each bootstrap draw, we randomly select, with replacement, raw asset returns $\mathbf{R}_{t,t+1}$ together with $\mathbf{Q}_{t,t+1}$, and recompute $(\lambda_0, \boldsymbol{\lambda})$. Then we generate the time-series of $m_{t,t+1}$, according to equation (30), and also compute the corresponding lower bounds according to equations (13) and (23). Here we perform 100,000 bootstrap trials.

			Properties of the SDF (monthly)					
	$L[m]$	$L[m^2]$	Mean	Std.	Skewness	Kurtosis (excess)	Min.	Max.
$m_{t,t+1} = \exp(\lambda_0 - 1 + \boldsymbol{\lambda}' \mathbf{er}_{t,t+1})$	0.0304	0.1207	0.988	0.25	1.1	6.8	0.33	2.31

	1963:07–2017:06 $\mathbf{R}_{t,t+1}$ is of dimension 26×1			
	$\mathbb{LB}_m$	Bootstrap p-value	$\mathbb{LB}_{m^2}$	Bootstrap p-value
Size×B/M	0.0243	0.692	0.2311	0.039
Size×Momentum	0.0352	0.337	0.2795	0.021
Size×Accrual	0.0238	0.644	0.2011	0.052
Size×Investment	0.0279	0.569	0.2589	0.028
Size×Long-Term Reversal	0.0176	0.745	0.0944	0.086
Size×Operating Profitability	0.0402	0.753	0.3413	0.201
Size×Net Issues	0.0191	0.289	0.1281	0.013
Size×Variance	0.0332	0.435	0.3115	0.017
Size×Residual Variance	0.0374	0.357	0.3399	0.015

Table 4

Model where log SDF is linear in a set of baseline excess returns, augmented with momentum portfolios

The sample period for this exercise is 1963:07 to 2017:06 (648 observations). The specific model we consider takes the form

$$m_{t,t+1} = \exp(\lambda_0 - 1 + \boldsymbol{\lambda}' \mathbf{er}^{\text{aug}}_{t,t+1}),$$

where $\mathbf{er}^{\text{aug}}_{t,t+1} \equiv \mathbf{Q}_{t,t+1} - \mathbf{1}R_{t+1,f}$ is a 11×1 vector of excess returns. The excess return $\mathbf{er}^{\text{aug}}_{t,t+1}$ are based on $\mathbf{Q}_{t,t+1}$ from the baseline five-factor model of Fama and French (2015) augmented with two momentum portfolios (reflecting the gross returns of past losers and winner):

- gross return of the market;
- gross returns of the two extreme low and high size portfolios;
- gross returns of the two extreme low and high book-to-market portfolios;
- gross returns of the two extreme low and high operating profitability portfolios;
- gross returns of the two extreme low and high investment portfolios;
- gross returns of the two extreme low and high momentum portfolios.

The constants $(\lambda_0, \boldsymbol{\lambda})$ are a solution to the minimization problem $\inf_{(\lambda_0,\boldsymbol{\lambda})} -E[q_t]\lambda_0 + E[\exp(\lambda_0 - 1 + \boldsymbol{\lambda}'\mathbf{er}^{\text{aug}}_{t,t+1})]$. This form of SDF obtains by minimizing the objective $E[m\log(m)]$ subject to $E[m\,\mathbf{er}^{\text{aug}}] = \mathbf{0}$ and $E[m - E[q_t]] = 0$. The Lagrange multipliers on the equality constraints are $\boldsymbol{\lambda}$ and λ_0, respectively. Our procedure implies that

$$\lambda_0 = 1.01, \text{ and } \quad \boldsymbol{\lambda} = [53.8 \;\; 4.2 \;\; 0.1 \;\; -20.4 \;\; -15.0 \;\; -3.3 \;\; 0.3 \;\; -12.0 \;\; -2.9 \;\; 0.1 \;\; -5.9].$$

We present the monthly properties of m, and the model-based $L[m]$, and $L[m^2]$. The lower bounds on $\mathbb{LB}_m$ and $\mathbb{LB}_{m^2}$ are over the sample period 1963:07 to 2017:06. In each bootstrap draw, we randomly select, with replacement, raw asset returns $\mathbf{R}_{t,t+1}$ together with $\mathbf{Q}_{t,t+1}$, and recompute $(\lambda_0, \boldsymbol{\lambda})$. Then we generate the time-series of $m_{t,t+1}$, according to equation (30), and also compute the corresponding lower bounds according to equations (13) and (23). Here we perform 100,000 bootstrap trials.

	$L[m]$	$L[m^2]$	Properties of the SDF (monthly) Mean	Std.	Skewness	Kurtosis (excess)	Min.	Max.
$m_{t,t+1} = \exp(\lambda_0 - 1 + \boldsymbol{\lambda}' \mathbf{er}^{\text{aug}}_{t,t+1})$	0.0569	0.2428	0.981	0.36	3.0	24.2	0.16	4.66

	1963:07–2017:06 $\mathbf{R}_{t,t+1}$ is of dimension 26×1 $\mathbb{LB}_m$	Bootstrap p-value	$\mathbb{LB}_{m^2}$	Bootstrap p-value
Size×B/M	0.0243	0.978	0.2311	0.275
Size×Momentum	0.0352	0.802	0.2795	0.064
Size×Accrual	0.0238	0.946	0.2011	0.396
Size×Investment	0.0279	0.927	0.2589	0.147
Size×Long-Term Reversal	0.0176	0.970	0.0944	0.638
Size×Operating Profitability	0.0402	0.975	0.3413	0.810
Size×Net Issues	0.0191	0.704	0.1281	0.019
Size×Variance	0.0332	0.795	0.3115	0.052
Size×Residual Variance	0.0374	0.735	0.3399	0.040

Table 5

Models that pass the entropy bound on $L[m^2]$ also pass the entropy bound on $L[m]$

The lower bound on $L[m^2]$, denoted by $\mathbb{LB}_{m^2}$, is based on equation (13), while the lower bound on $L[m]$, denoted by $\mathbb{LB}_m$, is based on equation (23). Both bounds rely on the ability of the SDF to correctly price $N+2$ assets (the risk-free bond, the long-term discount bond, and N risky assets). The criterion is

$$\text{Model passes} = \begin{cases} \text{If } L[m] \geq \mathbb{LB}_m \\ \text{fails otherwise} \end{cases} \quad \text{and/or,} \quad \text{Model passes} = \begin{cases} \text{If } L[m^2] \geq \mathbb{LB}_{m^2} \\ \text{fails otherwise.} \end{cases}$$

The set of models that pass the bound on $L[m^2]$ are of the type (see Problem 1)

$$m^{\bullet}_{t,t+1} = \frac{\psi}{(\mathbf{a}'\mathbf{R}_{t,t+1})^{\nu}} \quad \text{for some parameters } \nu \text{ and } \psi,$$

where ν and ψ, respectively, solve equations (C13) and (C14). We consider nine choices of the portfolio return $\mathbf{a}'\mathbf{R}_{t,t+1}$: (1) Market plus 25 portfolios based on size and book-to-market (1931:07–2017:06); (2) Market plus 25 portfolios based on size and momentum (1931:07–2017:06); (3) Market plus 25 portfolios based on size and accrual (1963:07-2017:06); (4) Market plus 25 portfolios based on size and investment (1963:07-2017:06); (5) Market plus 25 portfolios based on size and long-term reversal (1931:07–2017:06); (6) Market plus 25 portfolios based on size and net issues (1963:07-2017:06); (7) Market plus 25 portfolios based on size and operating profitability (1963:07-2017:06); (8) Market plus 25 portfolios based on size and variance (1963:07-2017:06); (9) Market plus 25 portfolios based on size and residual variance (1963:07-2017:06). The reported bootstrap p-value is the proportion of bootstrap draws (out of 100,000) and tests the null hypothesis that $L[m] \geq \mathbb{LB}_m$ (low p-values imply rejection). We report the (monthly) standard deviation and mean of the SDFs. $\mathbf{R}_{t,t+1}$ is of dimension 26×1.

Market plus			Entropy of $L[m]$				$L[m^2] = \mathbb{LB}_{m^2}$ Entropy of $L[m^2]$		SDF	
25 portfolios	ν	ψ	$L[m]$	$\mathbb{LB}_m$	Pass	p-val. (bootstrap)	$L[m^2]$	$\mathbb{LB}_{m^2}$	Std.	Mean
Size×B/M	2.9	1.039	0.031	0.022	Yes	0.932	0.142	0.142	0.34	0.997
Size×Momentum	2.1	1.038	0.032	0.032	Yes	0.731	0.189	0.189	0.41	0.997
Size×Accrual	5.7	1.094	0.064	0.024	Yes	0.996	0.201	0.201	0.38	0.996
Size×Investment	6.5	1.129	0.080	0.028	Yes	1.000	0.259	0.259	0.44	0.996
Size×Long-Term Reversal	3.5	1.037	0.033	0.018	Yes	0.980	0.094	0.094	0.25	0.997
Size×Net Issues	5.7	1.157	0.102	0.040	Yes	0.998	0.341	0.341	0.52	0.996
Size×Operating Profitability	4.1	1.051	0.041	0.019	Yes	0.989	0.128	0.128	0.30	0.996
Size×Variance	5.5	1.140	0.090	0.037	Yes	0.967	0.340	0.340	0.53	0.996
Size×Residual Variance	5.6	1.130	0.082	0.033	Yes	0.970	0.312	0.312	0.51	0.996

New Entropy Restrictions and the Quest for Better Specified Asset Pricing Models

Internet Appendix: Not for Publication

Abstract

The Internet Appendix provides detailed steps and expressions for some results presented in the paper. Section A sheds light on the sharpness of the lower bounds on $L[m]$ and $L[m^P]$. Section B considers additional applications and presents three asset pricing models (i) difference habit, (ii) recursive utility with stochastic variance, and (iii) recursive utility with constant jump intensity. Section B.3 is devoted to empirical assessment, showing that each model is rejected based on the lower bounds on both $L[m]$ and $L[m^2]$. We show that for some models $L[m^2]$ is close to $4L[m]$. Finally, Section B.4 (Section B.5) provides the solution of the eigenfunction problem for the difference habit formation (recursive utility) model.

I. Internet Appendix

A. Sharpness of the entropy bounds

How sharp is our bound on $L[m]$ compared to the bound constructed from a generic portfolio return in Backus, Chernov, and Zin (2014, Column 2 of Table I) and the corresponding bound on $L[m^P]$ in Alvarez and Jermann (2005) (based on pricing the risk-free bond return, the long-term bond return, and a generic portfolio return)?

Table Internet Appendix-I reports our lower bounds on $L[m]$ and $L[m^P]$ and the associated bootstrap p-values. We consider several N (the dimensionality of $\mathbf{R}_{t,t+1}$) and draw two conclusions. First, our bounds on $L[m]$ and $L[m^P]$ are quantitatively sharper with $N > 1$, implying greater hurdles on pricing models (e.g., compare bounds in Panel V versus those in Panels I through IV). Second, the bounds obtained with a portfolio are far less stringent than the corresponding bounds that rely on the SDFs correctly pricing each of the assets composing the portfolio. This can be seen by comparing the bound displayed in row (c) versus (i) and between row (d) versus (j). ■

B. Other example asset pricing models

Our goal is to learn about the properties of $m_{t,t+1}$ and $m^P_{t,t+1}$, and their consistency with bound restrictions. Additionally, we compare $L[m^2]$ to $4L[m]$. We focus on three models:

(i) Difference habit,

(ii) Recursive utility with stochastic variance, and

(iii) Recursive utility with constant jump intensity.

Some of the model solutions require loglinearization, whose effects are explored and elaborated in the study of Pohl, Schmedders, and Wilms (2015).

B.1. Difference habit model

The shocks in the difference habit model model are normally distributed, and the SDF is (Campbell and Cochrane (1999))

$$m_{t,t+1} \ = \ \beta g_{t+1}^{\rho-1} \left(s_{t+1}/s_t\right)^{\rho-1}, \tag{IA-1}$$

where g_{t+1} is consumption growth, β is the time discount parameter, and $1-\rho$ is the coefficient of relative risk aversion. Define $s_t \equiv 1-\exp(z_t)$ and $z_t \equiv \log(h_t) - \log(c_t)$, where s_t is the surplus ratio corresponding to z_t, and the habit h_{t+1} is known at t. The laws of motion for h_t and g_t are

$$\log(h_{t+1}) = \log(h) + \eta\,[B]\,\log(c_t) \quad \text{ and } \quad \log(g_{t+1}) = \log(g) + \gamma[B]\,\upsilon^{\frac{1}{2}}\,\omega_{gt+1}, \tag{IA-2}$$

where B is the lag operator, such that $B\{s_{t+1}\} = s_t$, with backshift operators $\gamma[B] = \sum_{j=0}^{\infty} \gamma_j B^j$ and $\eta\,[B] = \sum_{j=0}^{\infty} \eta_j B^j$. Moreover, υ denotes the constant variance of $\log(g_t)$, and ω_{gt+1} is i.i.d. standard normal variable.

Loglinear approximation of $\log(s_t)$, in conjunction with equation (IA-2), leads to the following dynamics:

$$\log(s_{t+1}) - \log(s_t) = \left(\frac{s-1}{s}\right)(\eta\,[B]\,B-1)\log(g_{t+1}). \tag{IA-3}$$

Completing the model description, we define the state variable $x_t = (\gamma[B]-\gamma_0)\,\upsilon^{\frac{1}{2}}\,\omega_{gt+1}$, which governs the following dynamics of the log consumption growth:

$$x_t = \gamma_1 \upsilon^{\frac{1}{2}} \omega_{gt} + \varphi_g x_{t-1} \qquad \text{with} \qquad \varphi_g = \frac{\gamma_2}{\gamma_1}. \tag{IA-4}$$

We solve the eigenfunction problem to recover $m^P_{t,t+1}$: $E_t\left[m_{t,t+1}\, e_{t+1}\right] = \zeta e_t$, where ζ is the eigenvalue and e_{t+1} is the eigenfunction (Hansen and Scheinkman (2009, Definition 6.1)). For the SDF of the habit model specified in equation (IA-1), the permanent component is

$$m^P_{t,t+1} \;=\; \exp\left(-D_1 + D_2\, x_{t-1} + D_3\, x_t + D_4\, x_{t+1}\right), \tag{IA-5}$$

where D_1 through D_4 are in (IA-25) through (IA-28) of the Internet Appendix (Section B.4). We employ equation (IA-5) to compute the left-hand side of equation (14) of Result 1.[1]

B.2. Recursive utility models

The two following recursive utility models are adopted from Backus, Chernov, and Zin (2014):

$$U_t = \left[(1-\beta)\, c_t^{\rho} \;+\; \beta\left(\mu_t\left[U_{t+1}\right]\right)^{\rho}\right]^{\frac{1}{\rho}}, \tag{IA-6}$$

with certainty equivalent function $\mu_t\left[U_{t+1}\right] = \left(E_t\left[U^{\alpha}_{t+1}\right]\right)^{\frac{1}{\alpha}}$. Moreover, β is the time preference parameter, $1/(1-\rho)$ is the intertemporal elasticity of substitution, and $1-\alpha$ is the coefficient of relative risk aversion.

[1]Models that accommodate habit have shown promise in matching salient attributes of the asset market data, including the equity premium, procyclicality of stock prices, counter-cyclicality of stock volatility, and return predictability at long horizons (e.g., see, among others, Bekaert and Engstrom (2017), Chapman (1998), Chan and Kogan (2002), and Santos and Veronesi (2010)).

The shocks ω_{gt}, z_{gt}, and ω_{ht} are standard normal random variables, independent of each other and across time. The jump component z_{gt} is a Poisson mixture of normals: conditional on the number of jumps j, z_{gt} is normal, with mean $j\theta$ and variance $j\delta^2$. The probability of $j \geq 0$ jumps at date t is $e^{h_{t-1}} h_{t-1}^j / j!$, and the jump intensity, h_{t-1}, is the mean of j.

With backshift operators characterized by $\nu[B] = \sum_{j=0}^{\infty} \nu_j B^j$ and $\psi[B] = \sum_{j=0}^{\infty} \psi_j B^j$, the state-variables in this model obey the following dynamics:

$$\log(g_t) = \log(g) + \gamma[B]\, \upsilon_{t-1}^{1/2} \omega_{gt} + \psi[B]\, z_{gt} - \psi[1]\, h\theta, \qquad h_t = h + \eta[B]\, \omega_{ht}, \tag{IA-7}$$

$$\upsilon_t = \upsilon \; + \; \nu[B]\, \omega_{\upsilon t}, \quad z_{gt} | j \sim \mathcal{N}\left(j\theta, j\delta^2\right), \quad P[j] = \exp(-h_{t-1}) \frac{(h_{t-1})^j}{j!}. \tag{IA-8}$$

A. Recursive utility model with stochastic variance. Set $h = 0$, $\eta[B] = 0$, $\psi[B] = 0$ in equations (IA-7) and (IA-8). For tractability, we consider the evolution of the transformed variable:

$$x_t \;=\; \varphi_g\, x_{t-1} \;+\; \gamma_1\, \upsilon_{t-1}^{1/2}\, \omega_{gt}. \tag{IA-9}$$

Then the $m_{t,t+1}^P$ component of the SDF of the recursive utility model with stochastic variance is

$$m_{t,t+1}^P = \exp\left(H_6 + (H_2 - \tau_0)\, x_t \;+\; (H_3 + \tau_0)\, x_{t+1} + (H_4 - \tau_1)\, \upsilon_t + (H_5 + \tau_1)\, \upsilon_{t+1}\right), \tag{IA-10}$$

where H_2 through H_6, τ_0, and τ_1 are presented in the Internet Appendix (Section B.5).

B. Recursive utility model with constant jump intensity: In equations (IA-7) and (IA-8), set $\nu[B] = 0$. Then the permanent component of the SDF of the recursive utility model with constant jump intensity is:

$$m^P_{t,t+1} = \exp\left(G_9 - G_8\,h_t + (G_5+\varsigma_1)\,z_{gt+1} + (G_6+\varsigma_2\gamma_1)\,\upsilon^{\frac{1}{2}}\omega_{gt+1} + (G_7+\varsigma_0\eta_0)\,\omega_{ht+1}\right), \quad \text{(IA-11)}$$

where G_5 through G_9, ς_0 through ς_3, and η_0 are presented in the Internet Appendix (Section B.5).[2]

B.3. Empirical evidence and connection to our findings

How do the models under consideration fare when viewed from the perspective of data-based lower bounds on the entropy of m, entropy of m^2, and the volatility of m?

Our implementation of the models with difference habit (hereby DH), recursive utility with stochastic variance (hereby RU-SV), and recursive utility with constant jump intensity (hereby RU-CJI) follows the calibration procedure in Backus, Chernov, and Zin (2014, respectively, Model (4) in Table 2, Model (1) in Table 3, and Model (4) in Table 4). The corresponding model parameterizations are displayed in our Table Internet Appendix-III, which indicates that each model reasonably calibrates to consumption growth data.

Aided by the analytical representations of $m^P_{t,t+1}$ (e.g., as derived in equation (IA-5)), we generate the paths for $m^P_{t,t+1}$, along with those of $m_{t,t+1}$. The paths are based on the model parameters in Table Internet Appendix-III and shocks driving the fundamentals (e.g., $\omega_{\upsilon t}$ and ω_{gt} for the RU-SV). Then we obtain the sample averages of the series $\{m^2_{t,t+1}, m_{t,t+1}, (m^P_{t,t+1})^2, m^P_{t,t+1}:$

[2]Models that incorporate recursive preferences in conjunction with stochastic variance or jumps in the consumption growth dynamics have proved successful in explaining asset pricing quantities. Notable applications include, among others, Epstein and Zin (1991), Bansal and Yaron (2004), Campbell and Vuolteenaho (2004), Hansen, Heaton, and Li (2008), Wachter (2013), and Zhou and Zhu (2009). Wachter (2013) emphasizes that her model can reconcile the size of the equity premium, the behavior of equity volatility, and the return predictability of Treasury bonds, pointing to a possible link between seemingly disparate phenomena from equity and bond markets.

$t = 1, \ldots T\}$, and accordingly compute the entropies $L[m^2_{t,t+1}]$, $L[m_{t,t+1}]$, $L[(m^P_{t,t+1})^2]$, $L[m^P_{t,t+1}]$, and the volatilities of $m_{t,t+1}$ and $m^P_{t,t+1}$.

Next, we draw 50,000 paths for the shocks driving a model and, hence, obtain 50,000 paths for $m_{t,t+1}$ and $m^P_{t,t+1}$. Panels A, B, and C of Table Internet Appendix-II report the entropies and volatilities across the models, obtained by averaging the entropies over the replications. The p-values, shown in square brackets, represent the proportion of replications for which the model-based entropy and volatility measures exceeds the corresponding lower bound obtained from the returns data in 50,000 replications of a simulation over 966 months.

How successful are the three models in generating $L[m]$ that is consistent with the data? Panel A of Table Internet Appendix-II reveals an $L[m]$ of 0.0196, 0.0217, and 0.0190, respectively, for the DH, RU-SV, and RU-CJI models. Based on our data-based performance measure, computed based on SET B, all the models are rejected at the 5% level (as seen by the bootstrap p-values).

Such an implication from our bound, calculated using the return properties of the risk-free bond, the long-term discount bond, the equity market, and the 25 portfolios sorted by size and momentum, differ from a finding in Backus, Chernov, and Zin (2014). Specifically, the data-based lower bound in Backus, Chernov, and Zin (2014, Table 1) are generally of an order lower than the average conditional entropy $E[L_t[m]]$ obtained from asset pricing models. In particular, all of the 11 $E[L_t[m]]$ in Backus, Chernov, and Zin (2014, Tables II through IV) exceed the lower bound inferred from the returns on a generic portfolio taken to be the S&P 500 index.

How does one explain this discrepancy? We note that the magnitude of the lower bound on $L[m]$ in the calculations of Backus, Chernov, and Zin (2014, Table 1, row S&P 500) is 0.0040, whereas it is 0.0367, based on our lower bound and SET B. It bears emphasizing that the lower bound on $L[m_{t,t+1}]$ constructed from the returns of a (single) generic portfolio may provide an insufficient hurdle in evaluating the merits of an asset pricing model. The bounds on both $L[m]$ and $L[m^P]$ agree in suggesting that the models are misspecified.

Elaborating further, we now argue that considering the entropy $L[m^2]$ (or $L[(m^P)^2]$) in the model assessment can provide an important contrast to our findings based on the entropy $L[m]$ (or $L[m^P]$). One noteworthy result is that the entropy $L[m^2]$ of the RU-CJI model is about 15-fold higher than the other two models that do not incorporate the random jump feature in the dynamics of the consumption growth. For example, the DH, RU-SV, and RU-CJI models generate $L[m_{t,t+1}^2]$ of 0.0785, 0.0869, and 1.4331, respectively (see the entries in Panel B of Table Internet Appendix-II). We further note that since the lower bound restriction implied from asset prices is 0.1956, the DH and RU-SV models are rejected at the 5% level. However, the RU-CJI model with constant jump intensity cannot be rejected at the 5% level, which is a point of departure based on the entropy $L[m]$.

Accordingly, one question emerges: Why does the RU-CJI fail to explain features of m (and likewise m^P), as reflected in asset prices when $L[m]$-based performance measure is used, while the model is successful in explaining features of m, as reflected in asset prices when $L[m^2]$-based performance measure is used? To investigate a source of model performance, we note that the entropy measure $L[m^2]$ is substantially more sensitive to tail asymmetries and tail size of the m distribution as opposed to the entropy measure $L[m]$.

Taking such a trait of entropies into consideration, we report the moments of $m_{t,t+1}$ and $m_{t,t+1}^P$ for each of the models in Panel D and Panel E of Table Internet Appendix-II. The unexpected finding is that the RU-CJI model embeds excessive levels of skewness and kurtosis of $m_{t,t+1}$, while generating variance that is almost 90 times its DH and RU-SV model counterparts. Our contention is that the inordinate levels of the higher-order moments of $m_{t,t+1}$ ($m^P_{t,t+1}$) give rise to the reported $L[m^2_{t,t+1}]$ ($L[(m^P_{t,t+1})^2]$) of 1.4331 (1.4858) for the RU-CJI model.

How should one interpret a model, such as the RU-CJI, that calibrates well to the first-moment, the second-moment, and the autocorrelation of consumption growth, but does not produce finite central moments for the distribution of both $m_{t,t+1}$ and $m^P_{t,t+1}$? This result arises because a convex transform of a random variable, which is here Poisson-distributed, increases the skewness to the right (see van Zwet (1966, page 10, Theorem 2.2.1)). To see this analytically, we can use the density of the Poisson random variable to show that $E_t[(m_{t,t+1})^k] = E_t\left[e^{k\log\left(m_{t,t+1}\right)}\right] = E_t\left[E_t\left[e^{k\log\left(m_{t,t+1}\right)}|j\right]\right] = e^{G[k]}E_t\left[e^{H[k]j}\right]$, for constants $G[k]$ and $H[k]$. Note that $e^{H[k]j}$ is a convex transformation of the Poisson variable J, and, for certain parameterizations, does not admit finite higher-moments of $m_{t,t+1}$. The inordinate amounts of skewness and kurtosis do not appear to be a reasonable depiction of valuation operators, which are likely to be characterized by exponential, rather than power, tails.

Finally, consider the volatility bound on m using the Hansen and Jagannathan (1991, equation (12)) and the volatility bound on m^P using Bakshi and Chabi-Yo (2012, equation (6)). As seen from Panel C of Table Internet Appendix-II, the DH and RU-SV models are rejected, but the RU-CJI model is not rejected for reasons discussed, namely, that the RU-CJI model embeds an unreasonable volatility, skewness, and kurtosis of m.

B.4. Details: Permanent component of the SDF of the difference habit model

Using a loglinear approximation of $\log(s_t)$,

$$\log(m_{t,t+1}) = D_0 + (\rho - 1)\frac{1}{s}\left(1 - (1-s)\eta[B]B\right)\gamma[B]\upsilon^{\frac{1}{2}}\omega_{gt+1}, \tag{IA-12}$$

$$\text{where } D_0 = \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\frac{\eta_0}{1-\varphi_h} - 1\right)\log(g).$$

Given the approximation $\log(s_t) \approx 1 + \frac{(s-1)}{s}z_t$, the dynamics of the surplus consumption ratio are

$$\log(s_{t+1}) - \log(s_t) = \frac{(s-1)}{s}\left(\eta[B]B - 1\right)\log(g_{t+1}). \tag{IA-13}$$

Therefore, we may write the log SDF as

$$\begin{aligned} \log(m_{t,t+1}) &= \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\eta[B]B - 1\right)\log(g) \\ &\quad + (\rho-1)\frac{1}{s}\left(1-(1-s)\eta[B]B\right)\gamma[B]\upsilon^{\frac{1}{2}}\omega_{gt+1}. \end{aligned} \tag{IA-14}$$

To solve for the permanent and transitory components of the SDF, we write the log SDF as

$$\begin{aligned} \log(m_{t,t+1}) &= \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\eta[B]B - 1\right)\log(g) + (\rho-1)\frac{1}{s}x_t \\ &\quad - (\rho-1)\frac{1}{s}(1-s)\eta[B]Bx_t - (\rho-1)\frac{1}{s}(1-s)\eta[B]B\upsilon^{\frac{1}{2}}\omega_{gt+1} \\ &\quad + (\rho-1)\frac{1}{s}\upsilon^{\frac{1}{2}}\omega_{gt+1}, \end{aligned} \tag{IA-15}$$

$$\text{where } x_t = (\gamma[B] - \gamma_0)\upsilon^{\frac{1}{2}}\omega_{gt+1}, \qquad \text{implying} \qquad x_{t+1} - \varphi_g x_t = \gamma_1\upsilon^{1/2}\omega_{gt+1}. \tag{IA-16}$$

We simplify the log SDF as

$$\begin{aligned}\log(m_{t,t+1}) &= D_0+(\rho-1)\frac{1}{s}\left(1-\frac{1}{\gamma_1}\varphi_g\right)x_t+(\rho-1)\frac{1}{s\gamma_1}x_{t+1} \\ &+ (\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g-1\right)\eta[B]x_{t-1}-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B]x_t.\end{aligned}$$

Consider the following eigenfunction problem:

$$E_t[m_{t,t+1}e_{t+1}]=\zeta e_t, \text{ where } \zeta \text{ is the eigenvalue and } e_{t+1} \text{ is the eigenfunction.} \tag{IA-17}$$

Accordingly, the permanent and transitory components of the SDF are

$$m^P_{t,t+1}=m_{t,t+1}\left(\frac{e_{t+1}}{\zeta e_t}\right) \qquad \text{and} \qquad m^T_{t,t+1}=\frac{\zeta e_t}{e_{t+1}}. \tag{IA-18}$$

We conjecture that e_{t+1} in equation (IA-17) is of the form

$$\log(e_{t+1})=\delta[B]x_{t+1}, \qquad \text{where} \qquad \delta[B]=\sum_{j=0}^{\infty}\delta_j B^j \qquad \text{with} \qquad \delta_0=1. \tag{IA-19}$$

To verify the solution, we expand to the following:

$$\begin{aligned}\log(m_{t,t+1})+\log\left(\frac{e_{t+1}}{e_t}\right) &= D_0+(\rho-1)\frac{1}{s}\left(1-\frac{1}{\gamma_1}\varphi_g\right)x_t+(\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g-1\right)\eta[B]x_{t-1} \\ &- \left((\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B]+\delta[B]\right)x_t+(\delta[B]-\delta_0)x_{t+1} \\ &+ \left((\rho-1)\frac{1}{s}\frac{1}{\gamma_1}+\delta_0\right)x_{t+1}.\end{aligned} \tag{IA-20}$$

Upon simplifying the expectation involving the eigenfunction problem, we derive ζ as

$$
\begin{aligned}
\log(\zeta) \quad = \quad & \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}(\eta[B]B-1)\log(g) + \frac{1}{2}\left((\rho-1)\frac{1}{s}\frac{1}{\gamma_1}+\delta_0\right)^2 \gamma_1^2 \upsilon \\
& + \left((\rho-1)\frac{1}{s}\left(1-\frac{1}{\gamma_1}\varphi_g\right) + \left((\rho-1)\frac{1}{s}\frac{1}{\gamma_1}+\delta_0\right)\varphi_g\right)x_t + (\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta[B]x_{t-1} \\
& + \left(-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B] - \delta[B]\right)x_t + (\delta[B]-\delta_0)x_{t+1}. \qquad \text{(IA-21)}
\end{aligned}
$$

Using the identification approach, we deduce

$$
\begin{aligned}
\log(\zeta) &= D_0 + \frac{1}{2}\left((\rho-1)(s\gamma_1)^{-1}+\delta_0\right)^2 \gamma_1^2 \upsilon, \qquad \text{(IA-22)} \\
\delta_1 &= -\left((\rho-1)\frac{1}{s}+\delta_0\varphi_g\right) - \left(-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta_0 - \delta_0\right), \\
\delta_{j+1} &= -(\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta_{j-1} - \left(-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta_j - \delta_j\right) \text{ for } j \geq 1.
\end{aligned}
$$

Then the transitory component of the SDF is

$$
m_{t,t+1}^T \quad = \quad \exp\left(D_0 + D_1 + D_5\,(x_t - x_{t+1})\right). \qquad \text{(IA-23)}
$$

Equation (IA-23) implies the permanent component in equation (IA-5), where

$$D_0 = \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\frac{\eta_0}{1-\varphi_h} - 1\right)\log(g), \quad \text{(IA-24)}$$

$$D_1 = \frac{1}{2}\left((\rho-1)(s\gamma_1)^{-1} + \delta_0\right)^2 \gamma_1^2 \upsilon, \quad \text{(IA-25)}$$

$$D_2 = (\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta[B], \quad \text{(IA-26)}$$

$$D_3 = -\delta[B] - (\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B] + (\rho-1)\frac{1}{s}\left(1 - \frac{1}{\gamma_1}\varphi_g\right), \quad \text{(IA-27)}$$

$$D_4 = (\rho-1)(s\gamma_1)^{-1} + \delta[B], \quad \text{and} \quad \text{(IA-28)}$$

$$D_5 = \delta[B]. \quad \text{(IA-29)}$$

This ends the proof. ■

B.5. Details: Permanent component of the SDF of the recursive utility models

We derive the expressions of the permanent component of the SDFs, as presented in equations (IA-10) and (IA-11).

Based on equations (IA-6) and (IA-8), we note that ω_{gt}, z_{gt}, and ω_{ht} are standard normal random variables, independent of each other and across time. The jump component z_{gt} is a Poisson mixture of normals: conditional on the number of jumps j, z_{gt} is normal with mean $j\theta$ and variance $j\delta^2$. The probability of $j \geq 0$ jumps at date $t+1$ is $e^{h_t} h_t^j / j!$ expands to

$$m_{t,t+1} = \exp\left(\chi_0 + a_g[B]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} + a_z[B]z_{gt+1} + a_\upsilon[B]\omega_{\upsilon t+1} + a_h[B]\omega_{ht+1}\right), \quad \text{(IA-30)}$$

$$\chi_0 = \log(\beta) + (\rho-1)\log(g)$$
$$- (\alpha-\rho)(D\upsilon - Jh) - (\alpha-\rho)(\alpha/2)\left((Db_1\nu[b_1])^2 + (Jb_1\eta[b_1])^2\right), \quad \text{(IA-31)}$$

where $a_g[B]$, $a_z[B]$, $a_\upsilon[B]$, and $a_h[B]$ are backshift operators defined as follows:

$$a_g[B] = (\rho-1)\gamma[B] + (\alpha-\rho)\gamma[b_1], \qquad a_z[B] = (\rho-1)\psi[B] + (\alpha-\rho)\psi[b_1], \tag{IA-32}$$

$$a_\upsilon[B] = (\alpha-\rho)D(b_1\nu[b_1] - \nu[B]B), \quad a_h[B] = (\alpha-\rho)J(b_1\eta[b_1] - \eta[B]B), \tag{IA-33}$$

$$D = (\alpha/2)(\gamma[b_1])^2, \quad \text{and} \qquad J = \left(\frac{e^{\alpha\psi[b_1]\theta + (\alpha\psi[b_1]\delta)^2} - 1}{\alpha}\right). \tag{IA-34}$$

The functions $\eta[b_1]$, $\nu[b_1]$, and $\gamma[b_1]$ are polynomial functions of b_1:

$$\eta[b_1] = \sum_{j=0}^{\infty} b_1^j \eta_j, \quad \gamma[b_1] = \sum_{j=0}^{\infty} b_1^j \gamma_j, \quad \nu[b_1] = \sum_{j=0}^{\infty} b_1^j \nu_j, \quad \psi[b_1] = \sum_{j=0}^{\infty} b_1^j \psi_j, \tag{IA-35}$$

with $\gamma_0 = 1$, where

$$\sum_{j=1}^{\infty} \gamma_j < \infty, \quad \sum_{j=1}^{\infty} \eta_j < \infty, \quad \sum_{j=1}^{\infty} \nu_j < \infty, \quad \sum_{j=1}^{\infty} \psi_j < \infty, \tag{IA-36}$$

and

$$\nu[B] = \sum_{j=0}^{\infty} \nu_j B^j \qquad \text{and} \qquad \psi[B] = \sum_{j=0}^{\infty} \psi_j B^j. \tag{IA-37}$$

A. Recursive utility with stochastic variance: The SDF is a special case of (IA-30) with $h = 0$, $\eta[B] = 0$, $J = 0$. The SDF takes the form

$$m_{t,t+1} = \exp\left(\begin{array}{c} H_0 + (\rho-1)\gamma[B]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} + (\alpha-\rho)\gamma[b_1]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} \\ +(\alpha-\rho)Db_1\nu[b_1]\omega_{\upsilon t+1} - (\alpha-\rho)D\nu[B]B\omega_{\upsilon t+1} \end{array}\right),$$

with

$$H_0 = \log(\beta) + (\rho - 1)\log g - (\alpha - \rho)(D\upsilon) - (\alpha - \rho)(\alpha/2)\left((Db_1\nu[b_1])^2\right). \tag{IA-38}$$

Now, define

$$x_t = (\gamma[B] - \gamma_0)\,\upsilon_t^{\frac{1}{2}}\omega_{gt+1}. \tag{IA-39}$$

The state variable x_t dynamics is

$$x_t = \varphi_g x_{t-1} + \gamma_1 \upsilon_{t-1}^{\frac{1}{2}}\omega_{gt}, \quad \text{with} \quad \gamma_j = \varphi_g \gamma_{j-1} \text{ for } j \geq 2 \quad \text{and} \quad \varphi_g = \frac{\gamma_2}{\gamma_1}. \tag{IA-40}$$

It can be shown that the dynamics of the state variable υ_t is

$$\upsilon_t - \upsilon = \varphi_\upsilon(\upsilon_{t-1} - \upsilon) + \nu_0\omega_{\upsilon t}, \qquad \text{for } j \geq 2 \qquad \text{and} \qquad \varphi_\upsilon = \frac{\nu_1}{\nu_0}. \tag{IA-41}$$

The SDF can be expressed as

$$m_{t,t+1} = \exp\left(H_1 + H_2 x_t + H_3 x_{t+1} + H_4 \upsilon_t + H_5 \upsilon_{t+1}\right), \tag{IA-42}$$

where

$$H_1 = H_0 + (\alpha - \rho) D\upsilon + (\alpha - \rho) D b_1 \nu [b_1] \frac{(\varphi_\upsilon - 1)}{\nu_0} \upsilon, \tag{IA-43}$$

$$H_2 = (\rho - 1) - ((\alpha - \rho)\gamma[b_1] + (\rho - 1)) \frac{\varphi_g}{\gamma_1}, \tag{IA-44}$$

$$H_3 = \frac{(\rho - 1)}{\gamma_1} + \frac{(\alpha - \rho)\gamma[b_1]}{\gamma_1}, \tag{IA-45}$$

$$H_4 = (\alpha - \rho) D \left(-b_1 \nu [b_1] \frac{\varphi_\upsilon}{\nu_0} - 1 \right), \text{ and} \tag{IA-46}$$

$$H_5 = (\alpha - \rho) D b_1 \frac{\nu [b_1]}{\nu_0}. \tag{IA-47}$$

Proceeding, we now solve the eigenfunction problem specified in equations (IA-17) and (IA-18). We conjecture that $\log(e_{t+1}) = \tau_0 x_{t+1} + \tau_1 \upsilon_{t+1}$. Hence,

$$\log(m_{t,t+1} e_{t+1}/e_t) = H_1 + (H_2 - \tau_0) x_t + (H_3 + \tau_0) x_{t+1} + (H_4 - \tau_1) \upsilon_t + (H_5 + \tau_1) \upsilon_{t+1} \tag{IA-48}$$

and

$$\begin{aligned} \log(\zeta) &= H_1 + (H_5 + \tau_1) \upsilon (1 - \varphi_\upsilon) + \frac{1}{2} (H_5 + \tau_1)^2 \nu_0^2 + (H_2 - \tau_0 + (H_3 + \tau_0) \varphi_g) x_t \\ &\quad + \left((H_4 - \tau_1) + \frac{1}{2} (H_3 + \tau_0)^2 \gamma_1^2 + (H_5 + \tau_1) \varphi_\upsilon \right) \upsilon_t. \end{aligned} \tag{IA-49}$$

Using the identification approach, we arrive at the following expressions:

$$\log(\zeta) = H_1 + (H_5 + \tau_1) \upsilon (1 - \varphi_\upsilon) + \frac{1}{2} (H_5 + \tau_1)^2 \nu_0^2 \tag{IA-50}$$

and

$$\tau_0 = \frac{H_2 + H_3\varphi_g}{1-\varphi_g} \qquad \text{and} \qquad \tau_1 = \frac{H_4 + \frac{1}{2}(H_3+\tau_0)^2\gamma_1^2 + H_5\varphi_\upsilon}{1-\varphi_\upsilon}. \tag{IA-51}$$

With these results, we are in a position to state the transitory and permanent components as

$$\begin{aligned} m_{t,t+1}^T &= \exp\left(H_1 + (H_5+\tau_1)\upsilon(1-\varphi_\upsilon) + \tfrac{1}{2}(H_5+\tau_1)^2\nu_0^2 + \tau_0(x_t - x_{t+1}) + \tau_1(\upsilon_t - \upsilon_{t+1}) \right), \\ m_{t,t+1}^P &= \exp\left(\begin{array}{c} -(H_5+\tau_1)\upsilon(1-\varphi_\upsilon) - \frac{1}{2}(H_5+\tau_1)^2\nu_0^2 \\ (H_2-\tau_0)x_t + (H_3+\tau_0)x_{t+1} + (H_4-\tau_1)\upsilon_t + (H_5+\tau_1)\upsilon_{t+1} \end{array} \right). \end{aligned} \tag{IA-52}$$

Setting $H_6 \equiv -(H_5+\tau_1)\upsilon(1-\varphi_\upsilon) - (H_5+\tau_1)^2\nu_0^2/2$, we have equation (IA-10). ■

B. Recursive utility model with constant jump intensity: Consider the consumption growth dynamics with $\nu[B] = 0$ (in this case $\upsilon_t = \upsilon$). It can be shown that the SDF reduces to

$$m_{t,t+1} = \exp\left(\begin{array}{c} \chi_0 \\ +(\rho-1)x_t + ((\rho-1)\gamma_0 + (\alpha-\rho)\gamma[b_1])\upsilon^{\frac{1}{2}}\omega_{gt+1} \\ +(\rho-1)(\psi[B]-\psi_0)z_{gt+1} + ((\rho-1)\psi_0 + (\alpha-\rho)\psi[b_1])z_{gt+1} \\ +(\alpha-\rho)Jb_1\eta[b_1]\omega_{ht+1} - (\alpha-\rho)(h_t - h)J \end{array} \right). \tag{IA-53}$$

Now denote

$$\widetilde{x}_t = (\psi[B]-\psi_0)z_{gt+1}. \tag{IA-54}$$

The law of motion of $\widetilde{x}_t$ becomes

$$\widetilde{x}_t = \varphi_z\widetilde{x}_{t-1} + \psi_1 z_{gt}, \quad \text{with} \quad \varphi_z = \frac{\psi_2}{\psi_1} \quad \text{and} \quad \psi_{j+2} = \varphi_z\psi_{j+1} \quad \text{for } j \geq 1. \tag{IA-55}$$

The SDF in equation (IA-53) reduces to

$$m_{t,t+1} = \exp\left(G_0 + G_1 x_t + G_2 \widetilde{x}_{t-1} + G_3 z_{gt} + G_4 h_t + G_5 z_{gt+1} + G_6 \upsilon^{\frac{1}{2}} \omega_{gt+1} + G_7 \omega_{ht+1}\right), \tag{IA-56}$$

with

$$\begin{aligned} G_0 &= \chi_0 + (\alpha - \rho)\, hJ, & G_1 &= (\rho - 1), \\ G_2 &= (\rho - 1)\, \varphi_z, & G_3 &= (\rho - 1)\, \psi_1, \\ G_4 &= -(\alpha - \rho)\, J, & G_5 &= (\rho - 1)\, \psi_0 + (\alpha - \rho)\, \psi[b_1], \\ G_6 &= (\rho - 1)\, \gamma_0 + (\alpha - \rho)\, \gamma[b_1], & G_7 &= (\alpha - \rho)\, Jb_1 \eta[b_1]. \end{aligned}$$

For the eigenfunction problem in equations (IA-17)–(IA-18), i.e., $E_t[m_{t,t+1} e_{t+1}] = \zeta e_t$, we conjecture that the eigenfunction is of the form

$$e_{t+1} = \exp(\varsigma_0 h_{t+1} + \varsigma_1 z_{gt+1} + \varsigma_2 x_{t+1} + \varsigma_3 \widetilde{x}_t). \tag{IA-57}$$

Algebraic manipulation yields the expression

$$\begin{aligned} m_{t,t+1} \frac{e_{t+1}}{e_t} &= \exp\left(\begin{array}{c} G_0 + \varsigma_0 h - \varsigma_0 \varphi_h h + (G_1 - \varsigma_2 + \varsigma_2 \varphi_g)\, x_t + G_2 \widetilde{x}_{t-1} + (G_3 - \varsigma_1 + \varsigma_3 \psi_1)\, z_{gt} \\ + (G_4 - \varsigma_0 + \varsigma_0 \varphi_h)\, h_t + (\varsigma_3 \varphi_z - \varsigma_3)\, \widetilde{x}_{t-1} \end{array} \right) \\ &\quad \times \exp\left((G_5 + \varsigma_1)\, z_{gt+1} + (G_6 + \varsigma_2 \gamma_1)\, \upsilon^{\frac{1}{2}} \omega_{gt+1} + (G_7 + \varsigma_0 \eta_0)\, \omega_{ht+1} \right). \end{aligned} \tag{IA-58}$$

Upon further manipulation of equation (IA-58), we get

$$\zeta = \xi \times \exp\left(\begin{array}{c} G_0 + \varsigma_0 h - \varsigma_0 \varphi_h h + (G_1 - \varsigma_2 + \varsigma_2 \varphi_g)\, x_t + G_2 \widetilde{x}_{t-1} + (G_3 - \varsigma_1 + \varsigma_3 \psi_1)\, z_{gt} \\ + (G_4 - \varsigma_0 + \varsigma_0 \varphi_h)\, h_t + (\varsigma_3 \varphi_z - \varsigma_3)\, \widetilde{x}_{t-1} \end{array} \right) \tag{IA-59}$$

with

$$\xi = E_t\left(\exp\left(\left(G_5+\varsigma_1\right)z_{gt+1}+\left(G_6+\varsigma_2\gamma_1\right)\upsilon^{\frac{1}{2}}\omega_{gt+1}+\left(G_7+\varsigma_0\eta_0\right)\omega_{ht+1}\right)\right). \tag{IA-60}$$

One may observe that

$$\begin{aligned}\xi &= \left(E_t\exp\left(\left(G_5+\varsigma_1\right)z_{gt+1}\right)\right)\left(E_t\exp\left(\left(G_6+\varsigma_2\gamma_1\right)\upsilon^{\frac{1}{2}}\omega_{gt+1}\right)\right)\left(E_t\left(\left(G_7+\varsigma_0\eta_0\right)\omega_{ht+1}\right)\right) \\ &= E_t\left(\exp\left(\left(\left(G_5+\varsigma_1\right)\theta+\frac{1}{2}\left(G_5+\varsigma_1\right)^2\delta^2\right)j\right)\right)\exp\left(\frac{1}{2}\left(G_6+\varsigma_2\gamma_1\right)^2\upsilon+\frac{1}{2}\left(G_7+\varsigma_0\eta_0\right)^2\right)\end{aligned} \tag{IA-61}$$

and

$$E_t\left(\exp\left(\left(\left(G_5+\varsigma_1\right)\theta+\frac{1}{2}\left(G_5+\varsigma_1\right)^2\delta^2\right)j\right)\right)=\exp\left(G_8h_t\right), \tag{IA-62}$$

with

$$G_8 = e^{\left(\left(G_5+\varsigma_1\right)\theta+\frac{1}{2}\left(G_5+\varsigma_1\right)^2\delta^2\right)}-1. \tag{IA-63}$$

As a consequence, equation (IA-61) simplifies to

$$\xi = \exp\left(G_8h_t+\frac{1}{2}\left(G_6+\varsigma_2\gamma_1\right)^2\upsilon+\frac{1}{2}\left(G_7+\varsigma_0\eta_0\right)^2\right). \tag{IA-64}$$

We substitute equation (IA-64) in equation (IA-59) and rearrange to obtain

$$\begin{aligned}\log(\varsigma) &= G_0+\varsigma_0h-\varsigma_0\varphi_hh+\left(G_1-\varsigma_2+\varsigma_2\varphi_g\right)x_t \\ &\quad+\left(G_3-\varsigma_1+\varsigma_3\psi_1\right)z_{gt}+\left(G_4-\varsigma_0+\varsigma_0\varphi_h+G_8\right)h_t \\ &\quad+\left(\varsigma_3\varphi_z-\varsigma_3+G_2\right)\widetilde{x}_{t-1}+\frac{1}{2}\left(G_6+\varsigma_2\gamma_1\right)^2\upsilon+\frac{1}{2}\left(G_7+\varsigma_0\eta_0\right)^2.\end{aligned} \tag{IA-65}$$

Using the identification approach, we then have

$$\log(\varsigma) = G_0 + \varsigma_0 h(1-\varphi_h) + \frac{1}{2}(G_6 + \varsigma_2\gamma_1)^2 \upsilon + \frac{1}{2}(G_7 + \varsigma_0\eta_0)^2 \tag{IA-66}$$

and

$$\begin{aligned} G_1 - \varsigma_2 + \varsigma_2\varphi_g &= 0, & G_4 - \varsigma_0 + \varsigma_0\varphi_h + G_8 &= 0, \\ G_3 - \varsigma_1 + \varsigma_3\psi_1 &= 0, & \varsigma_3\varphi_z - \varsigma_3 + G_2 &= 0. \end{aligned} \tag{IA-67}$$

Finally, we get

$$\varsigma_0 = \tfrac{G_8+G_4}{1-\varphi_h}, \quad \varsigma_1 = G_3 + \varsigma_3\psi_1, \quad \varsigma_2 = \tfrac{G_1}{1-\varphi_g}, \quad \varsigma_3 = \tfrac{G_2}{1-\varphi_z}. \tag{IA-68}$$

The transitory component is, therefore, $m^T_{t,t+1} = \zeta \exp(e_t - e_{t+1})$, and we obtain

$$m^T_{t,t+1} = \zeta \exp(\varsigma_0(h_t - h_{t+1}) + \varsigma_1(z_{gt} - z_{gt+1}) + \varsigma_2(x_t - x_{t+1}) + \varsigma_3(\widetilde{x}_{t-1} - \widetilde{x}_t)). \tag{IA-69}$$

We can establish the relation in equation (IA-11) by setting $G_9 \equiv -\frac{1}{2}(G_6 + \varsigma_2\gamma_1)^2 \upsilon - \frac{1}{2}(G_7 + \varsigma_0\eta_0)^2$.

■

Table Internet Appendix-I

Sharpness of our entropy bounds on $m_{t,t+1}$ and $m^P_{t,t+1}$, when SDFs correctly price each of the $N+2$ assets

Reported are the lower entropy bounds with the one-sided p-values in $\langle.\rangle$. Our lower entropy bounds on $m_{t,t+1}$ and $m^P_{t,t+1}$ are based on equations (23), and rely on the ability of the SDF to correctly price *each of the* $N+2$ assets (the risk-free bond, the long-term discount bond, and N risky assets). The Backus, Chernov, and Zin (2014, equation (5)) lower bound on the entropy of $m_{t,t+1}$ (denoted by BCZ) is based on the expression $E[\log(R^m_{t,t+1})] - \log(R_{t+1,f})$, while the Alvarez and Jermann (2005, equation (4)) lower bound on the entropy of $m^P_{t,t+1}$ (denoted by AJ) is based on the expression $E[\log(R^m_{t,t+1})] - E[\log(R_{t,t+1,\infty})]$, where $R^m_{t,t+1}$ is the return on a single risky asset or a benchmark portfolio (i.e., which we proxy, for instance, by the value-weighted equity market return or equally weighted portfolio of 25 Fama-French size and book-to-market portfolios). Moreover, $R_{t,t+1,\infty}$ is the return on an infinite-maturity bond, which we proxy by the return of a 30-year Treasury bond. $R_{t+1,f}$ is the gross return of the three-month Treasury bond. We employ different assets and N in the construction of the bounds. For example, in Panel I, the N risky assets are based on two data sets: SET A contains the value-weighted market returns, together with the 25 Fama-French size and book-to-market portfolios, while SET B contains the value-weighted market returns together with the 25 Fama-French size and momentum portfolios. The sample period is from July 1931 to December 2011 (966 observations). To compute these p-values, we first use the block bootstrap with a block size of 20 to generate 50,000 samples from the original data. Then we compute the lower bounds in each sample and tabulate the proportion of bootstrap samples for which the lower bound is less than zero.

	Lower bound on $m_{t,t+1}$		Lower bound on $m^P_{t,t+1}$	
	Bound	p-value	Bound	p-value
Panel I. SDF correctly prices each of the $N+2$ assets, and we set $N=26$				
(a) Market, 25 size & B/M	**0.023**	$\langle 0.000 \rangle$	**0.021**	$\langle 0.000 \rangle$
(b) Market, 25 size & momentum	**0.037**	$\langle 0.003 \rangle$	**0.035**	$\langle 0.003 \rangle$
Panel II. SDF correctly prices each of the $N+2$ assets, and we set $N=25$				
(c) 25 size & B/M	**0.022**	$\langle 0.000 \rangle$	**0.020**	$\langle 0.000 \rangle$
(d) 25 size & momentum	**0.029**	$\langle 0.000 \rangle$	**0.027**	$\langle 0.000 \rangle$
Panel III. SDF correctly prices each of the $N+2$ assets, and we set $N=11$				
(e) Market, 10 momentum	**0.020**	$\langle 0.000 \rangle$	**0.018**	$\langle 0.001 \rangle$
Panel IV. SDF correctly prices each of the $N+2$ assets, and we set $N=2$				
(f) Market, Low Momentum	**0.010**	$\langle 0.000 \rangle$	**0.008**	$\langle 0.000 \rangle$
(g) Market, high Momentum	**0.014**	$\langle 0.010 \rangle$	**0.012**	0.011
Panel V. SDF correctly prices each of the $N+2$ assets, and we set $N=1$				
	(BCZ, Eq. 5)		(AJ, Eq. 4)	
(h) Market portfolio only	**0.005**	$\langle 0.005 \rangle$	**0.003**	$\langle 0.066 \rangle$
(i) EWI portfolio of 25 size & B/M	**0.007**	$\langle 0.001 \rangle$	**0.005**	$\langle 0.018 \rangle$
(j) EWI portfolio of 25 size & momentum	**0.007**	$\langle 0.001 \rangle$	**0.005**	$\langle 0.021 \rangle$

Table Internet Appendix-II

Model comparisons using bounds

Reported are the results for bounds on the entropy of m, the entropy of m^2, and the volatility of m, for three models: the difference habit (denoted by DH), the recursive utility with stochastic variance (denoted by RU-SV), and the recursive utility with constant jump intensity (denoted by RU-CJI). The companion results are presented for the permanent component of the SDF. The one-sided p-values shown in square brackets represent the proportion of replications for which the model-based quantity (entropy or volatility) exceeds, in 50,000 replications, the lower bound computed from observed asset prices. Our lower bounds on the entropy of m^2 and $(m^P)^2$ are based on equation (23). and (14) of Result 1, and rely on the ability of the SDF to correctly price $N+2$ assets (the risk-free bond, the long-term discount bond, and N risky assets). The N risky assets are based on SET B, which contains the value-weighted market returns, together with the 25 Fama-French size and momentum portfolios. The sample period is from July 1931 to December 2011. The lower bounds on the entropy of m and m^P are based on equation (23) and also rely on the ability of the SDF to correctly price $N+2$ assets (the risk-free bond, the long-term discount bond, and N risky assets). The lower bound on the volatility of m are based on Hansen and Jagannathan (1991, equation (12)) and the lower bound on the volatility of m^P are based on Bakshi and Chabi-Yo (2012, equation (6)). We focus on SET B, as it corresponds to the maximum lower bound on entropy measures (as in our Table Internet Appendix-I). Panels D and E present the variance, skewness, and kurtosis of m and m^P, which are consistent with model parameterizations in Table Internet Appendix-III. The one-sided p-values $\langle . \rangle$ reported below the lower bounds, represent the proportion of bootstrap samples for which the lower bound is less than zero.

	Habit model	Recursive utility models		
	DH	RU-SV	RU-CJI	Lower bounds (Set B)
Panel A: Entropies of m and m^P, when m correctly prices $N+2$ returns				
$L[m]$	0.0196	0.0217	0.0190	0.0367
	[0.000]	[0.000]	[0.000]	$\langle 0.003 \rangle$
$L[m^P]$	0.0203	0.0237	0.0197	0.0348
	[0.000]	[0.000]	[0.000]	$\langle 0.003 \rangle$
Panel B: Entropies of m^2 and $(m^P)^2$				
$L[m^2]$	0.0785	0.0869	1.4331	0.1956
	[0.000]	[0.000]	[1.000]	$\langle 0.003 \rangle$
$L[(m^P)^2]$	0.0811	0.095	1.4858	0.1851
	[0.000]	[0.000]	[1.000]	$\langle 0.003 \rangle$
Panel C: Volatility bounds				
Hansen and Jagannathan (1991)	0.0415	0.0444	3.344	0.1292
	[0.000]	[0.000]	[1.000]	$\langle 0.000 \rangle$
Bakshi and Chabi-Yo (2012)	0.0403	0.0487	3.248	0.1225
	[0.000]	[0.000]	[1.000]	$\langle 0.000 \rangle$
Panel D: Moments of the $m_{t,t+1}$ distribution				
Variance	0.0403	0.0444	3.3438	
Skewness	0.6041	0.6476	$+\infty$	
Kurtosis	3.6447	3.8061	$+\infty$	
Panel E: Moments of the $m^P_{t,t+1}$ distribution				
Variance	0.0415	0.0487	3.2480	
Skewness	0.6142	0.6778	$+\infty$	
Kurtosis	3.6654	3.8786	$+\infty$	

Table Internet Appendix-III

Parameters employed in model implementation

Displayed in this table are the parameters that govern preferences and the dynamics of consumption growth. These parameters are adopted from Tables 2, 3, and 4 of Backus, Chernov, and Zin (2014), and likewise $\log(g)$ and η_0 are taken from their page 16. Our implementation of the models with difference habit (hereby DH), recursive utility with stochastic variance (hereby RU-SV), and recursive utility with constant jump intensity (hereby RU-CJI) follows Backus, Chernov, and Zin (2014, respectively, Model (4) in Table 2, Model (1) in Table 3, and Model (4) in Table 4). We use US annual real personal consumption expenditures as a proxy for aggregate consumption over the sample period of 1931:07 to 2011:12 (966 observations). To compare model implications with the data, we simulate a finite sample of consumption growth, c_{t+1}/c_t, over 966 months. Following convention, we then compute the annualized consumption growth as $\exp(\sum_{j=1}^{12}\log(c_{t+j}/c_{t+j-1}))$. The reported model mean, standard deviation, and autocorrelation are based on the annualized consumption growth.

Parameter	DH	RU-SV	RU-CJI	Data implied
Panel A: Preferences				
ρ	-9.0000	0.3333	0.3333	
α		-9.0000	-9.0000	
β	0.9980	0.9980	0.9980	
φ_h	0.9000			
s	0.5000			
Panel B: Consumption growth dynamics				
γ_0	1.0000	1.0000	1.0000	
$\log(g)$	0.0015	0.0015	0.0015	
η_0	0.1000			
γ_1	0.0271	0.0271	0.0281	
φ_g	0.9790	0.9790	0.9690	
$\upsilon^{1/2}$	0.0099	0.0099	0.0079	
ν_0		0.23×10^{-5}		
φ_υ		0.9870		
h			0.0008	
θ			-0.1500	
δ			0.1500	
ψ_0			1.0000	
b_1		0.9977	0.9979	
Panel C: Consumption growth				
Mean (annualized)	1.0192	1.0190	1.0189	1.0339
Std. Dev. (annualized)	0.0416	0.0415	0.0369	0.0287
Autocorrelation	0.2424	0.2433	0.1771	0.2386