A Recovery That We Can Trust? Deducing and Testing the Restrictions of the Recovery Theorem*

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Abstract

How reliable is the recovery theorem of Ross (2015)? We explore this question in the context of options on the 30-year Treasury bond futures, allowing us to deduce restrictions that link the physical and risk-neutral return distributions. Our empirical results undermine the implications of the recovery theorem. First, we reject an implicit assumption of the recovery theorem that the martingale component of the stochastic discount factor is identical to unity. Second, we consider the restrictions between the physical and risk-neutral return moments when the recovery theorem holds, and reject them in both forecasting regressions and generalized method of moment estimations.

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1. Introduction

Is it possible to recover both a stochastic discount factor (SDF) and the physical probabilities from option prices? The underlying asset for the option could be an equity index, an individual stock, or the futures of a 30-year Treasury bond, and the answer from Ross (2015, Theorem 1, pages 622 and 646) is that it is feasible, provided (i) that uncertainty is driven by a finite-state, irreducible Markov chain, and (ii) the pricing kernel satisfies the transition independence property. His solution approach and formalization, based on the Perron-Frobenius theorem, appear to have inspired an across-the-board intellectual conversation.

How reliable is the recovery theorem of Ross (2015) in applied work? This paper builds on Borovička, Hansen, and Scheinkman (2016) and provides a framework to assess the reliability of the Ross recovery theorem using data on futures of the 30-year Treasury bond and its options. The motivation is that finance theory has derived much of its analytical power and appeal from a few simplifying assumptions (e.g., the Black-Scholes formula), and the defense of a theory eventually resides in its empirical validity.

What do we do in this paper? First, we use the framework of Alvarez and Jermann (2005), Hansen and Scheinkman (2009), and Qin and Linetsky (2017) that uniquely decomposes SDFs into martingale and transitory components. Second, we feature the long-term bond market and their futures contract, and directly measure the transitory component using the returns of the long-term bond. Third, we examine the implications of imposing the restriction that the martingale component of SDFs is degenerate (identical to unity), following Borovička, Hansen, and Scheinkman (2016). Measuring the transitory component in the setting of the long-term bond market is instrumental to isolating the form of the SDF when the martingale component is degenerate.

The crux of our approach (and testing framework) is to establish a set of theoretical restrictions between the moment-generating function of the physical and risk-neutral return distributions that have to hold when the martingale component is degenerate. The novelty of the link is that it allows us to set up forecasting equations that predict the first (second) moment of futures return of the long-term bond under the physical measure, from the second (third) risk-neutral return moment. Our null hypothesis is that the intercept is zero and the slope coefficient is one. We further show how to operationalize the forecasting equations using options on the long-term bond futures, without invoking distributional assumptions.[1] Then we implement tests related to the adequacy of return and variance forecasts in the long-term bond market. Additionally, we consider the merits of the theory, relying on unconditional tests and generalized method of moments (GMM) estimation. Both sets of results undermine the implications of the recovery theorem.

We also consider a minimum discrepancy problem to infer the dispersion of the martingale component of SDFs in the long-term bond market, and is at the heart of the skepticism in Borovička, Hansen, and Scheinkman (2016) that the martingale component may not be degenerate. This involves posing a convex optimization problem that minimizes the expectation of a convex function of the martingale component subject to the constraints that the martingale component is nonnegative and the SDF correctly prices a set of test assets in the long-term bond market.

Our approach and empirical implementation indicate that the martingale component of the SDF is notably volatile. When the test assets include the returns of the risk-free bond, the long-term bond futures, and a collection of out-of-the-money puts and calls on the 30-year Treasury bond futures, the solution for the martingale component yields a minimum dispersion in the ball-

[1]The long-term bond futures and its options are uniquely important and appropriate objects to empirically assess the recovery theorem. Our approach does not rely on the Markovian assumption, and it obviates the step of solving the Perron-Frobenius problem in contrast to the solution technique in Ross (2015). Thus, we sidestep the task of identifying the relevant state variables.

park of 50% (annualized). Moreover, the extracted martingale component is positively correlated with the transitory component. These solution properties are contrary to the treatment of Ross (2015), and are robust under a bootstrap procedure and to our choices for the convex function.

Related literature. The Ross (2015) and Borovička, Hansen, and Scheinkman (2016) papers have been impactful, and there is a collection of theoretical and empirical studies that address various aspects of the recovery theorem and related ideas. This includes Audrino, Huitema, and Ludwig (2015), Carr and Yu (2012), Dubynskiy and Goldstein (2013), Liu (2015), Martin and Ross (2013), Massacci, Williams, and Zhang (2016), Tran and Xia (2015), and Walden (2017).[2] The study of Jensen, Lando, and Pedersen (2015) extends Ross (2015) to incorporate general probability distributions (without invoking time-homogeneity or the Markov property), whereas the work of Christensen (2017) introduces econometric methods to extract the martingale and transitory components of the SDF.

Qin, Linetsky, and Nie (2016) propose a term structure model driven by a two-factor diffusion with market price of risks that are linear in latent state variables. They estimate the martingale component from the yield curve and show that the implications of the recovery theorem are inconsistent with a class of asset pricing models. Our study is parallel to Qin, Linetsky, and Nie (2016) in that both studies, while adopting different methodologies, corroborate the presence of a substantial martingale component.

Additionally, Huang and Shaliastovich (2014) consider a framework with recursive preferences to recover physical probabilities and risk adjustments from risk-neutral probabilities, finding support for early resolution of uncertainty in the equity index market. Schneider and Trojani

[2] The joint laws of the risk-neutral and physical density have been studied in the context of the equity market under assumptions about SDFs prior to Ross (2015). For a partial list, see Aït-Sahalia and Lo (2000), Jackwerth (2000), Rosenberg and Engle (2002), Bakshi, Kapadia, and Madan (2003), Carr and Wu (2003), Huang and Wu (2004), Chabi-Yo, Garcia, and Renault (2008), Bakshi, Madan, and Panayotov (2010), Bollerslev and Todorov (2011), Kozhan, Neuberger, and Schneider (2013), Christoffersen, Jacobs, and Heston (2013), and Chaudhuri and Schroder (2015).

(2015) derive bounds on the conditional physical return moments and identify the pricing kernel from equity options. The work of Jackwerth and Menner (2016) studies the performance of the recovery theorem in the context of the equity index market and shows that the recovery theorem does not reproduce reasonable SDFs.

The road map of our paper is as follows. Section 2 develops the testable restrictions of the recovery theorem that have to hold when the martingale component is degenerate. Section 3 explores an empirical application of the minimum discrepancy problem to quantitatively characterize the martingale component. In this regard, we emphasize that Almeida and Garcia (2017) study the dispersion of SDFs, while Borovička, Hansen, and Scheinkman (2016) consider a theoretical formulation without developing a solution. Next, we conduct regression tests of the forecasting equations that explicitly link physical and risk-neutral return moments. Finally, complementing the forecasting exercises, we investigate whether the unconditional physical return moments implied by the recovery theorem and option prices are compatible with the data in a GMM setting. Distinguishing our work from others, our approach and the setting of options on the Treasury bond futures allow us to present some new evidence on the controversy surrounding the importance of the martingale component of SDFs.

2. The theoretical framework and testable implications

We center our attention on developing the testable implications of the recovery theorem in a stochastic environment that is somewhat more general than a finite-state Markov chain. Our theoretical framework synthesizes various elements of the analysis in Ross (2015) and Borovička, Hansen, and Scheinkman (2016), and serves as the basis for the empirical investigation.

Let $\mathbb{E}_t(.)$ indicate time t conditional expectation under the physical probability measure. Further, the positive random variable M_{t+1} represents the pricing kernel at date $t+1$. The nominal discount bond price at date t, denoted by $V_t[1_{t+k}]$, is a claim to \$1 at date $t+k$: $V_t[1_{t+k}] = \mathbb{E}_t(\frac{M_{t+k}}{M_t} \times 1)$. Accordingly, a k-period discount bond bought at time t at price $V_t[1_{t+k}]$, and sold at time $t+1$ at price $V_{t+1}[1_{t+k}]$, has gross return $R_{t+1,k}$, given by $R_{t+1,k} \equiv \frac{V_{t+1}[1_{t+k}]}{V_t[1_{t+k}]} = \frac{\mathbb{E}_{t+1}(\frac{M_{t+k}}{M_{t+1}} \times 1)}{\mathbb{E}_t(\frac{M_{t+k}}{M_t} \times 1)}$.

The gross return of a long-term discount bond corresponds to a large k, that is, $\lim_{k \to \infty} R_{t+1,k} \equiv R_{t+1,\infty}$, and the gross return of a risk-free bond corresponds to $k=1$, that is, $R_{t+1,\text{rf}} \equiv \frac{1}{V_t[1_{t+1}]}$. We denote the SDF by $m_{t+1} \equiv \frac{M_{t+1}}{M_t}$.

This paper studies the empirical properties of the decomposition of the SDF, exploiting the work of Alvarez and Jermann (2005), Hansen and Scheinkman (2009), and Qin and Linetsky (2017). In a discrete-time environment, Alvarez and Jermann (2005, Assumptions 1 and 2) show that the SDF can be uniquely decomposed as

$$m_{t+1} = m_{t+1}^P \, m_{t+1}^T, \qquad \text{where} \qquad m_{t+1}^T = \frac{1}{R_{t+1,\infty}}. \tag{1}$$

In equation (1), m_{t+1}^P is the martingale component that satisfies $\mathbb{E}_t(m_{t+1}^P) = 1$ and the transitory component m_{t+1}^T is the inverse of the gross return of a long-term discount bond. Hansen and Scheinkman (2009), Borovička, Hansen, and Scheinkman (2016), and Qin and Linetsky (2017) further show that m_{t+1}^P defines a long-term risk-neutral probability measure.

In a Markov environment, Hansen and Scheinkman (2009, Proposition 6.1) and Borovička, Hansen, and Scheinkman (2016, equation (17)) show that the decomposition (1) can be uniquely obtained by solving the Perron-Frobenius (eigenfunction) problem. To elaborate, one analytically solves $\mathbb{E}_t(m_{t+1} \, e^{-\rho} \frac{\phi[\mathbf{s}_{t+1}]}{\phi[\mathbf{s}_t]}) - 1 = 0$, by the choice of $\phi[\mathbf{s}_t]$ and ρ, for a state vector $\mathbf{s}$, where $\phi[\mathbf{s}_t]$ represents the eigenfunction, and e^{ρ} is the eigenvalue (ρ is typically negative). With $\phi[\mathbf{s}_t]$ and ρ determined (subject to a stability restriction on the long-term risk-neutral probability measure), the transitory component is $m^T_{t+1} = e^{\rho} \frac{\phi[\mathbf{s}_t]}{\phi[\mathbf{s}_{t+1}]}$ and $m^P_{t+1} = m_{t+1}/m^T_{t+1}$.

We recognize that implementation in Ross (2015) uses the Perron-Frobenius theorem to recover the SDF and the transition probabilities (i.e., the data-generating probability measure) from market data in conjunction with option prices. Given the assumptions in Ross (2015), it is established in Borovička, Hansen, and Scheinkman (2016, Section I.B and Section III) that the SDF takes the form $m_{t+1} = e^{\rho} \frac{\phi[\mathbf{s}_t]}{\phi[\mathbf{s}_{t+1}]} = m^T_{t+1}$, which also satisfies the Perron-Frobenius problem. Therefore, for the recovery theorem to hold, the martingale component of the SDF must be equal to unity, more exactly, $m^P_{t+1} = 1$ for all t.

Our approach does not hinge on the Markovian assumption. We exploit the direct measurement of m^T_{t+1}, using the returns of the long-term bond (or its futures), and this approach is model-free. Hence, we circumvent the identification of the state vector $\mathbf{s}$, while inferring $m_{t+1} = e^{\rho} \frac{\phi[\mathbf{s}_t]}{\phi[\mathbf{s}_{t+1}]}$. This is in contrast to the implementation in Ross (2015), in which the direct inference of $\phi[\mathbf{s}_t]$ is considerably involved. Moreover, our identification of m^T_{t+1} is crucial and allows us to extract the minimum dispersion of m^P_{t+1} (in the long-term bond market).

In this light, the issue we address is how to infer the dispersion and characteristics of m^P_{t+1} that capture the wedge between features of the return distribution implied by the recovery theorem and option prices, and features of the physical (data-generating) distribution observed in the return time-series. We ask two questions: Can we test whether $m^P_{t+1} = 1$? What are the implications of imposing $m_{t+1} = e^{\rho} \frac{\phi[\mathbf{s}_t]}{\phi[\mathbf{s}_{t+1}]} = m^T_{t+1}$?

The discount bond with infinite maturity is not traded, and we surrogate $R_{t+1,\infty}$ with the returns of the futures on the 30-year Treasury bond using spot-futures arbitrage. Importantly, we develop testable implications, in terms of the relevant observable quantities, and we rely on, in particular, both the futures on the 30-year Treasury bond and options on the 30-year Treasury bond futures. Our approach and the setting of the futures on the long-term bond and its options enable us to address some not yet fully resolved questions associated with the volatility of the martingale component of SDFs in the long-term bond market.

To impart empirical content to the derived equations, let $F_t^{[k]}$ represent the time-t price of a one-period futures contract on the k-period discount bond. Then, from Cox, Ingersoll, and Ross (1981, equation (46)) and for a marked-to-market futures contract,

$$\begin{aligned} F_t^{[k]} &= \mathbb{E}_t^{\mathbb{Q}}(V_{t+1}[1_{t+k}]), \text{where } \mathbb{E}_t^{\mathbb{Q}}(.) \text{ indicates expectation under the risk-neutral measure,} \\ &= R_{t+1,\mathrm{rf}}\, V_t[1_{t+k}]. \end{aligned} \tag{2}$$

Thus, we obtain the following relationship:

$$e^{z_{t+1,\infty}} \equiv \lim_{k\to\infty} \frac{V_{t+1}[1_{t+k}]}{F_t^{[k]}} = \frac{1}{R_{t+1,\mathrm{rf}}} \lim_{k\to\infty} \frac{V_{t+1}[1_{t+k}]}{V_t[1_{t+k}]} = \frac{R_{t+1,\infty}}{R_{t+1,\mathrm{rf}}}. \tag{3}$$

With the relation in equation (3) and $m_{t+1}^P = 1$, the SDF in equation (1) can be expressed as

$$m_{t+1} = m_{t+1}^T = \frac{1}{R_{t+1,\mathrm{rf}}}\, e^{-z_{t+1,\infty}}. \qquad \text{(from equation (3))} \tag{4}$$

The resulting m_{t+1} is free of any parametrization, and we further note that

$$\mathbb{E}_t(m_{t+1}) = \frac{1}{R_{t+1,\mathrm{rf}}}, \quad \text{which implies that } \mathbb{E}_t(e^{-z_{t+1,\infty}}) = 1. \tag{5}$$

Consider claims written on the futures of the long-term bond, and let $p[z_{t+1,\infty}]$ denote the physical density of $z_{t+1,\infty} \in (-\infty, +\infty)$. The associated risk-neutral pricing density is

$$q[z_{t+1,\infty}] = \frac{\overbrace{R_{t+1,\mathrm{rf}}^{-1} e^{-z_{t+1,\infty}}}^{m_{t+1}} p[z_{t+1,\infty}]}{R_{t+1,\mathrm{rf}}^{-1} \int_{-\infty}^{+\infty} e^{-z_{t+1,\infty}} p[z_{t+1,\infty}] \, dz_{t+1,\infty}} = e^{-z_{t+1,\infty}} p[z_{t+1,\infty}], \quad \text{since } \mathbb{E}_t(e^{-z_{t+1,\infty}}) = 1. \tag{6}$$

The form of $q[z_{t+1,\infty}]$ presented in equation (6) corresponds to the risk-neutral probability in Ross (2015, equation (25)). Guided by this implication, we consider a class of restrictions for an arbitrary parameter n:

$$\mathbb{E}_t^{\mathbb{Q}}\left(e^{(n+1)\,z_{t+1,\infty}}\right) = \int_{-\infty}^{+\infty} e^{(n+1)\,z_{t+1,\infty}} q[z_{t+1,\infty}] \, dz_{t+1,\infty} \tag{7}$$

$$= \int_{-\infty}^{+\infty} e^{(n+1)\,z_{t+1,\infty}} e^{-z_{t+1,\infty}} p[z_{t+1,\infty}] \, dz_{t+1,\infty} \tag{8}$$

$$= \mathbb{E}_t\left(e^{n\,z_{t+1,\infty}}\right). \tag{9}$$

The takeaway is that the Ross recovery theorem implies a restriction on the moment-generating function of the physical and risk-neutral distributions of $z_{t+1,\infty}$, provided $\left|\mathbb{E}_t^{\mathbb{Q}}(e^{(n+1)\,z_{t+1,\infty}})\right| < +\infty$ for suitable choices of n. The restriction (9) is testable if one can compute $\mathbb{E}_t^{\mathbb{Q}}(e^{(n+1)\,z_{t+1,\infty}})$.

We synthesize $\mathbb{E}_t^{\mathbb{Q}}(e^{(n+1)\,z_{t+1,\infty}})$ by pricing the powers of the gross return of the futures of a long-term bond (e.g., adapting the relations in Carr and Madan (2001) or Bakshi, Kapadia, and Madan (2003) to stochastic interest rates). Specifically,

$$\mathbb{E}_t^{\mathbb{Q}}\left(e^{(n+1)\,z_{t+1,\infty}}\right) = 1 + \frac{n(n+1)R_{t+1,\mathrm{rf}}}{(F_t^{[k]})^2} \left(\int_{K>F_t^{[k]}} \left(\frac{K}{F_t^{[k]}}\right)^{n-1} C_t[K] \, dK + \int_{K<F_t^{[k]}} \left(\frac{K}{F_t^{[k]}}\right)^{n-1} P_t[K] \, dK \right) \quad \text{for a large } k, \tag{10}$$

where $C_t[K]$ ($P_t[K]$) is the time t price of the one-period European call (put) option on the futures of a k-period bond with strike price K. The testable restriction when $m^P_{t+1} = 1$, is that, for $n \geq 1$,

$$\mathbb{E}_t \left(\frac{e^{n\, z_{t+1,\infty}}}{x_t^{[n+1]}} - 1 \right) = 0, \qquad \text{where} \qquad x_t^{[n+1]} \equiv \mathbb{E}_t^{\mathbb{Q}} \left(e^{(n+1)\, z_{t+1,\infty}} \right). \tag{11}$$

The restrictions in (11) are the focus of the goodness-of-fit empirical tests in Sections 3.3 and 3.4.

It may potentially be the case that for some markets, the simplification of a degenerate martingale component leads to little pricing impact and is acceptable. At the same time, testing the restrictions of the recovery theorem outside of the long-term bond market can be cumbersome and entails further assumptions about the joint physical density of $z_{t+1,\infty}$ and the returns in that market.

3. Empirical results and interpretation

The characterization in equation (11) shows that one could evaluate whether certain moments of the physical distribution, deduced from the recovery theorem in conjunction with option prices, are aligned with the historical record. In the discussion that follows, we describe the data, motivate a convex optimization problem to study the martingale component of SDFs, and consider empirical exercises that investigate the reliability of the recovery theorem.

3.1. Data on the 30-year Treasury bond futures and options

Our investigation emphasizes a testing framework that exploits data of options on the futures of the 30-year Treasury bond. We focus on the 30-year Treasury bond futures, as they manifest contingent claims with a long tenor. The futures and options on the futures inherit the properties of the underlying 30-year Treasury bond (which carries a fixed nominal coupon), and are compatible with the stochastic evolution of interest rates (e.g., Amin and Jarrow (1991)). There are no exchange-traded options on Treasury bonds.

The master file from the Chicago Mercantile Exchange (CME) has daily data of options on the 30-year Treasury bond futures, and includes options across all expiration cycles. This data includes (i) the strike price, (ii) the remaining maturity, (iii) the option price, (iv) the identifier for a call or put option, and (v) the futures price. The CME does not provide bid and ask prices. The data is available from October 1982 to December 2015, with 1,093,510 daily option records.[3]

We apply the following steps to process the daily master file. First, we retain out-of-the-money options with the *nearest maturity*. We define out-of-the-money calls as having moneyness $\log(F_t^{[360m]}/K) < 0$ and out-of-money puts as having moneyness $\log(K/F_t^{[360m]}) < 0$, where $F_t^{[360m]}$ denotes the time t price of the futures on a 30-year Treasury bond. Our proxy for large k is 360 months. Next, we omit the data prior to January 1985 to maintain a total of at least eight out-of-the-money options every month, which can enable a more accurate valuation of the payoff curvature via options.

Finally, we build a set of option prices and time-series of option returns, all at *the end of the month*.

- **Nearest maturity options at the monthly frequency**: These options expire on the last Friday, at least two business days from the last business day of the next month. The options so constructed have an average maturity of 26 days. The number of out-of-the-money calls (puts) vary from four (four) to 42 (49), with a total of 10,336 option observations.

- **Returns of a 3% and 1% out-of-the-money put**: At the end of each month, we search for a put option that is closest to 3% and 1% out-of-the-money, respectively. For ex-

[3]Treasury bond futures and options are among the most actively traded derivatives, easing concerns of illiquidity. For example, the average open interest (sampled on the last day of the month) for the nearest-maturity options is 225,989 contracts, while the average trading volume on the last day of the month for the nearest-maturity options is 37,191 contracts.

ample, the gross return of a 3% out-of-the-money put is constructed as $R_{t+1,3\% \text{ otm put}} \equiv \frac{\max(K - F_{t+1}^{[360m]}, 0)}{P_t[K]}$, where $K = e^{-0.03} \times F_t^{[360m]}$.

- **Returns of a 1% and 3% out-of-the-money call**: At the end of each month, we search for a call option that is closest to 1% and 3% out-of-the-money, respectively. The gross return of a 1% out-of-the-money call is constructed as $R_{t+1,1\% \text{ otm call}} \equiv \frac{\max(F_{t+1}^{[360m]} - K, 0)}{C_t[K]}$, where K solves $F_t^{[360m]}/K = e^{-0.01}$.

The returns of out-of-the-money puts and calls indexed by strikes on the 30-year Treasury bond futures are employed as test assets in our procedure to gauge variations in the martingale component of SDFs, which is a constant under the treatment of Ross (2015), as established in Borovička, Hansen, and Scheinkman (2016).

There are additional data features that deserve clarification and elaboration. First, Treasury bond futures options are American-style. Since our analysis predominantly uses out-of-the-money puts and calls, the impact of the early exercise premium in this market is understandably small (e.g., Flesaker (1993, footnote 7), and Mueller, Vedolin, and Yen (2017)). Second, participants that are short Treasury bond futures could choose the timing of the physical delivery and which bond to deliver. This attribute of the Treasury futures market bears no consequence in our empirical tests, because when the option expires, the settlement of the futures occur at least one month thereafter.

What about the liquidity of 30-year Treasury bond futures options across moneyness? To answer this question, we compute the dollar open interest (dollar trading volume) as the number of options contracts outstanding (number of options contracts traded) multiplied by the Treasury bond futures price, all observed on the last day of the month. For comparison, we also tabulate the corresponding values for the S&P 500 index options.

	Average dollar open interest ($m)				Average dollar trading volume ($m)			
	3% put	1% put	1% call	3% call	3% put	1% put	1% call	3% call
30-year bond futures options	2,102	1,858	2,191	1,819	464	543	591	350
S&P 500 index options	2,077	1,892	1,874	1,775	280	341	272	277

The takeaway is that the dollar open interest and the dollar trading volume in the Treasury bond futures options are comparable to the counterparts of the S&P 500 index options.

3.2. *The extracted martingale components do not favor the $m^P_{t+1} = 1$ treatment*

Is the extracted martingale component m^P_{t+1} identical to unity in the long-term bond market? Even when the constancy of the martingale component of SDFs can be refuted, what can be said about the correlation between the *extracted* martingale component and the transitory component? In this subsection, our interest lies in isolating the martingale component, and this exercise could shed light on the reliability of the recovery theorem.

The existing literature (e.g., in the manner elaborated in Hansen and Scheinkman (2009), Borovička, Hansen, and Scheinkman (2016), and Qin and Linetsky (2016, 2017)) has shown theoretically that the recovered measure corresponds to a long-term risk-neutral measure. This recovered measure coincides with the physical distribution only when $m^P_{t+1} = 1$. This is a case that is potentially subject to falsification, and it is one of the centerpieces of our investigation.

3.2.1. The framework for the convex optimization problem

In what follows, we write $\mathbb{E}(.)$ to express unconditional expectation. Define

$$\mathbf{R}_{t+1} \equiv (R_{t+1,\infty}, \mathbf{R}'_{t+1,j})' \qquad \text{and} \qquad \mathbf{Z}_{t+1} \equiv \frac{\mathbf{R}_{t+1} - R_{t+1,\mathrm{rf}}\mathbf{1}}{R_{t+1,\infty}}, \tag{12}$$

where $\mathbf{R}_{t+1,j}$, for $j = 1,\ldots,J$, is a $J \times 1$ vector of gross returns of risky assets that excludes the long-term bond return, and $\mathbf{1}$ is a vector of ones. We assume $\mathbb{E}(|\mathbf{Z}|^2) < +\infty$. Consider the set

$$\mathcal{M} \equiv \{m^P \geq 0 \text{ such that } \mathbb{E}\left(m^P \mathbf{Z}\right) = \mathbf{0} \text{ and } \mathbb{E}\left(m^P\right) = 1\}. \tag{13}$$

To address the dispersion of the martingale component, we pose the following convex optimization (minimum discrepancy) problem:

$$\inf_{m^P \in \mathcal{M}} \ \mathbb{E}(\psi[m^P]), \ \text{ for a convex function } \psi[m^P] \text{ satisfying } \mathbb{E}(\psi[m^P]) < +\infty. \tag{14}$$

We emphasize that the pricing and martingale restrictions are imposed unconditionally. The equality constraints $\mathbb{E}\left(m^P \mathbf{Z}\right) = \mathbf{0}$ are a statement about the absence of arbitrage and reflect correct pricing, while the equality constraint $\mathbb{E}(m^P) = 1$ is the martingale condition. Additionally, the condition $m^P \geq 0$ is aimed at enforcing the nonnegativity of the martingale component.[4]

The minimization in equation (14) is over a possibly infinite-dimensional space, but solving the dual enables tractability. Let $\boldsymbol{\lambda}$ be the $(J+1) \times 1$ vector of Lagrange multipliers associated with $\mathbb{E}\left(m^P \mathbf{Z}\right) = \mathbf{0}$ and ν be the Lagrange multiplier associated with $\mathbb{E}(m^P) = 1$.

We explore solutions with two different convex functions to establish robustness. The proofs are in the appendix.

Case 1 *Consider $\psi[m^P] = \frac{1}{2}(m^P)^2$. Since $\mathbb{E}(m^P) = 1$, minimizing $\mathbb{E}((m^P)^2)$ is equivalent to minimizing the variance, $\mathbb{E}((m^P - 1)^2)$. The optimal solution can be characterized as*

$$m^{P*} \ = \ \max\left(\nu^* + \boldsymbol{\lambda}^{*\prime}\mathbf{Z}, 0\right), \tag{15}$$

[4] We repose the constraints on correct pricing differently from Borovička, Hansen, and Scheinkman (2016, equation (29)), as it simplifies analytics, because the inclusion of $R_{t+1,\infty}$ as the first element of $\mathbf{R}_{t+1}$ ensures the correct pricing of the risk-free bond.

where $(\boldsymbol{\lambda}^*, \nu^*)$ *solve*

$$\inf_{(\boldsymbol{\lambda},\nu)} \quad -\nu \,+\, \frac{1}{2}\,\mathbb{E}\left([(\nu \,+\, \mathbf{Z}'\boldsymbol{\lambda}) \times 1_{\{\nu+\mathbf{Z}'\boldsymbol{\lambda}\geq 0\}}]^2\right). \tag{16}$$

In equation (16), $1_{\{a\geq 0\}}$ *is an indicator function for the event* $\{a \geq 0\}$. ♣

Case 2 *Consider* $\psi[m^P] = m^P \log(m^P)$ *for* $m^P > 0$*. The optimal solution can be characterized as*

$$m^{P*} \;=\; \exp\left(-1+\nu^* + \boldsymbol{\lambda}^{*\prime}\mathbf{Z}\right), \tag{17}$$

where $(\boldsymbol{\lambda}^*, \nu^*)$ *solve*

$$\inf_{(\boldsymbol{\lambda},\nu)} \quad -\nu \,+\, \mathbb{E}\left(\exp\left(-1+\nu+\boldsymbol{\lambda}'\mathbf{Z}\right)\right). \tag{18}$$

Since $\mathbb{E}(\psi[m^P]) - \underbrace{\psi[\mathbb{E}(m^P)]}_{=0} = \mathbb{E}(\frac{1}{2}(m^P-1)^2 - \frac{1}{6}(m^P-1)^3 + \frac{1}{12}(m^P-1)^4 + O((m^P-1)^5))$*, the objective weighs (higher) central moments of* m^P. ♣

3.2.2. The extracted m^P_{t+1} *is not a constant and is positively correlated with* m^T_{t+1}

Germane to implementing the solution, either via equation (15) or (17), is the question of which data to feature from the long-term bond market. We consider the $\mathbf{Z}_{t+1}$ below:

$$\mathbf{Z}_{t+1} = \frac{1}{R_{t+1,\mathrm{rf}}\, e^{z_{t+1,\infty}}} \begin{pmatrix} R_{t+1,\mathrm{rf}}\, e^{z_{t+1,\infty}} \;-\; R_{t+1,\mathrm{rf}} \\ R_{t+1,3\%\text{ otm put}} \;-\; R_{t+1,\mathrm{rf}} \\ R_{t+1,1\%\text{ otm put}} \;-\; R_{t+1,\mathrm{rf}} \\ R_{t+1,1\%\text{ otm call}} \;-\; R_{t+1,\mathrm{rf}} \\ R_{t+1,3\%\text{ otm call}} \;-\; R_{t+1,\mathrm{rf}} \end{pmatrix}. \tag{19}$$

This choice of $\mathbf{Z}_{t+1}$ makes the SDFs consistent with returns of the risk-free bond, the long-term bond, and a collection of state contingent returns to the downside and upside of long-term bond futures. Our analysis in Internet Appendix (Section I, Table Internet Appendix-I, and Table Internet Appendix-II) explores the implications of using different sets of test assets. First, a restricted version of $\mathbf{Z}_{t+1}$ (in equation (19)) that omits 3% out-of-the-money puts and calls lowers the variance of the extracted m_{t+1}^P, and this reduction is statistically significant. Second, augmenting $\mathbf{Z}_{t+1}$ with the returns of straddles slightly raises the variance of the extracted m_{t+1}^P, and this increase is statistically insignificant.

We draw on extant approaches and numerically solve the sample analog to equations (16) and (18) by searching over $(\boldsymbol{\lambda}, \nu)$, analogous to, for example, Hansen, Heaton, and Luttmer (1995, Sections 2 and 4) and Gospodinov, Kan, and Robotti (2016). Panels A and B of Table 1 report the properties of the extracted m_{t+1}^P series, according to equations (15) and (17), respectively. In both cases, we consider a block bootstrap procedure to generate $\mathbf{Z}_{t+1}$, and report the 5th, 25th, 50th, 75th, and 95th percentile values of the m_{t+1}^P distribution, across the 25,000 bootstrap trials.

We emphasize that the convex optimization problems could become ill-posed if the objective is unbounded (e.g., Boyd and Vandenberghe (2004, Chapter 4)). Such an outcome is precluded with (i) $m^P \geq 0$, (ii) m^P being centered at unity, and (iii) the imposing of $\mathbb{E}(\psi[m^P]) < +\infty$ and $\mathbb{E}(|\mathbf{Z}|^2) < +\infty$. Throughout our bootstrap implementation, the optimizations in equations (16) and (18) are well-posed with a finite objective and well-defined Lagrange multipliers (see Schennach (2007) for further elaboration on the practical issues that arise when the Lagrange multipliers are unconstrained).[5]

The results are informative from a number of perspectives. For one, the entries for the solution

[5]There is no guarantee that the objective is finite when $(\boldsymbol{\lambda}, \nu)$ are unrestricted and no distributional assumptions are imposed on $\mathbf{Z}_{t+1}$. The objective would exist in implementations even if it does not in population. Our confidence is bolstered by the fact that the analysis with the two convex functions leads to quantitative results that are in agreement.

in Panels A and B of Table 1 are in agreement for a given $\mathbf{Z}_{t+1}$, indicating that our choice of the convex functions does not materially affect the properties of the extracted martingale component. Our results further suggest that the extracted martingale component is not identical to unity. When $\psi[m^P] = \frac{1}{2}(m^P)^2$, the optimal solution generates an annualized minimum $\sqrt{\text{Variance}}$ of 48.7%, with a 50th percentile bootstrap value of 66.4%. When $\psi[m^P] = m^P \log(m^P)$, the $\sqrt{\text{Variance}}$ of the extracted m^P_{t+1} is 49%, with a 50th percentile bootstrap value of 67.3%.[6] The distribution of the extracted m^P_{t+1} is positively skewed and fat-tailed.[7]

Additionally, the extracted martingale component (consistent with the convex optimization problems) is correlated with the transitory component (measured by $1/R_{t+1,\infty}$). This correlation, denoted by $\rho_{P,T}$, is 0.53, whereas the 50th percentile bootstrap value is 0.39 (Panel A of Table 1). The null hypothesis that $\rho_{P,T} \leq 0$ is rejected with the highest bootstrap p-value of 0.027.[8]

The exercises in Internet Appendix (Section I and Table Internet Appendix-I and -II) provide additional findings in conjunction with Table 1. First, the null hypothesis of $\rho_{P,T} \leq 0$ is also rejected when the number of test assets is three or six. Second, we consider the null hypothesis that $\rho_{P,T}$ is increasing with the number of test assets using a bootstrap procedure. In this regard, we find that when the number of test assets is increased from three to five (five to six), the estimated correlation decreases from 0.64 to 0.53 (0.53 to 0.50), and this effect is statistically significant (insignificant). At the same time, we acknowledge that the documented correlation may be a poor estimate of the true correlation if the variance of the extracted m^P_{t+1} constitutes a small part of the

[6]Is the difference in the variance of the extracted m^P_{t+1} from the convex problems in Cases 1 and 2 statistically significant (the optimization problem furnishes the same mean of unity by construction)? To address this question, we construct the time-series of $(m^P_{t+1} - 1)^2$, corresponding to Cases 1 and 2, and regress the difference on a constant. The hypothesis of unequal variance is rejected, with a p-value of 0.000 in our 25,000 bootstrap trials.

[7]We note that our empirical implementation and the proposed solutions for the martingale component of the SDFs (i.e., Cases 1 and 2) differentiate our convex optimization problems from Almeida and Garcia (2012, 2017), and the theoretical formulations in Borovička, Hansen, and Scheinkman (2016).

[8]Our estimates of $\rho_{P,T}$ line up with the positive association between m^P_{t+1} and m^T_{t+1}, shown in the context of Bakshi and Chabi-Yo (2012, Table 5), and with the nonparametric approach of Christensen (2017, Section 6).

variance of the true m^P_{t+1}.

In summary, both pieces of evidence, namely, the pronounced volatility of the extracted martingale component and the positive correlation of the extracted martingale component (from a problem that minimizes $\frac{1}{2}(m^P)^2$ or $m^P \log(m^P)$) with the transitory component, are not supportive of the $m^P_{t+1} = 1$ treatment of Ross (2015).

3.2.3. Further discussion

What are the consequences of our finding that the extracted m^P_{t+1} (from Cases 1 and 2) and m^T_{t+1} are correlated? It turns out that if m^P_{t+1} is not degenerate, recovery is still possible in the long-term bond market under the assumption that the martingale component is independent of the transitory component. To elaborate, we note that $\mathbb{E}^{\mathbb{Q}}_t(e^{(n+1)z_{t+1,\infty}}) = R_{t+1,\text{rf}}\,\mathbb{E}_t(m^P_{t+1} m^T_{t+1} e^{(n+1)z_{t+1,\infty}}) = \mathbb{E}_t(e^{n\,z_{t+1,\infty}})$, where the last equality follows from $m^T_{t+1} = \frac{1}{R_{t+1,\text{rf}}} e^{-z_{t+1,\infty}}$, $\mathbb{E}_t(m^P_{t+1}) = 1$, and the independence of m^P_{t+1} and m^T_{t+1}. More specifically, the pricing restrictions in equation (9) still hold for each n, when the martingale component is stochastic and independent of m^T_{t+1}.

Our exercises with respect to the dispersion of the martingale component are complementary to Qin, Linetsky, and Nie (2016, Sections 2 and 3), who solve for the transitory and martingale components of SDFs through the eigenfunction problem. They parameterize a two-factor diffusion model of the spot interest rate and derive the instantaneous volatility of the martingale component (their equation (27)), allowing them to test for the degeneracy of the martingale component. Our departure is that we provide an alternative computation of the martingale component by considering minimum discrepancy problems via a discrete-time formulation (and by using option returns), and we do so without taking a stand on the specification of the spot interest rate process or the functional form of the factor risk premiums.

Closing, how do we position our findings and results in light of Alvarez and Jermann (2005, Proposition 2, equation (4)), who develop a lower entropy bound on the martingale component of SDFs (when SDFs correctly price a generic portfolio, the long-term bond, and the risk-free bond)? Qin, Linetsky, and Nie (2016, Table 4) apply the Alvarez and Jermann bound to the bond market when the generic portfolio consists of leveraged investments in short-maturity bonds. For the set of assets in our $\mathbf{Z}_{t+1}$, the lower entropy bound can be obtained by solving $\inf_{m^P \in \mathcal{M}} \ \mathbb{E}(-\log(m^P))$. For this case, it is provable that the optimal solution $m^{P*} = \frac{1}{-(\mathbf{Z}'\boldsymbol{\lambda}^* + \nu^*)}$ requires $\mathbf{Z}'\boldsymbol{\lambda}^* + \nu^* < 0$, where $(\boldsymbol{\lambda}^*, \nu^*)$ solve $\inf_{(\boldsymbol{\lambda},\nu)} \ -\nu - \mathbb{E}(\log(-(\mathbf{Z}'\boldsymbol{\lambda} + \nu)))$. Schennach (2007) notes that the objective for such problems cannot be guaranteed to exist, as the function (i.e., $\log(-(\mathbf{Z}'\boldsymbol{\lambda} + \nu))$), that underlies the moment condition, can become undefined. For this reason, we feature the solutions to the minimum discrepancy problems in Cases 1 and 2.

3.3. Adequacy of return and variance forecasts

Is the treatment $m^P_{t+1} = 1$ of Ross (2015) innocuous, when recovering return quantities under the physical probability measure from option prices? To develop testable implications of the relation $\mathbb{E}_t\left(e^{n\, z_{t+1,\infty}}\right) = \underbrace{\mathbb{E}^{\mathbb{Q}}_t(e^{(n+1)\, z_{t+1,\infty}})}_{x_t^{[n+1]}}$ in equation (11), we define

$$y^{[n]}_{t+1} \ \equiv \ e^{n\, z_{t+1,\infty}} \qquad \text{and} \qquad \varepsilon^{[n]}_{t+1} \ \equiv \ y^{[n]}_{t+1} \ - \ \mathbb{E}_t(y^{[n]}_{t+1}). \tag{20}$$

Then, we can transform the theoretical restriction in equation (11) into an empirical restriction as

$$y^{[n]}_{t+1} \ = \ \alpha^{[n]} \ + \ \beta^{[n]}\, x^{[n+1]}_t \ + \ \varepsilon^{[n]}_{t+1}, \quad \text{for } n \geq 1. \tag{21}$$

Equation (21) exposes an implication of the recovery theorem in that $x^{[n+1]}_t$, as inferred from option prices at the end of month t, helps to forecast $y^{[n]}_{t+1}$, resembling an approach pursued in different contexts by others, and is in the flavor of Fama (1984, Section 2.1).

The null hypothesis for a fixed n in the OLS regression is

$$\alpha^{[n]} = 0 \qquad \text{and} \qquad \beta^{[n]} = 1, \qquad \text{for } n \geq 1. \tag{22}$$

These restrictions can be tested using data on the returns of the 30-year Treasury bond futures and option prices on the 30-year Treasury bond futures. The two-sided p-values for the OLS coefficients $\alpha^{[n]}$ and $\beta^{[n]}$ are constructed based on the Newey and West (1987) standard errors, with lag length chosen automatically according to Newey and West (1994). We also consider the two-sided p-values based on the Hodrick (1992) 1B covariance estimator under the null of no forecasting ability.

How much empirical support is there for the link between the observed return distributions and that implied by the recovery theorem, i.e., as postulated by the relation in equation (21)? When $n = 1$, it addresses the recovery of the mean futures (gross) return, whereas $n = 2$ addresses the recovery of $\mathbb{E}_t(e^{2z_{t+1,\infty}})$ from $\mathbb{E}_t^{\mathbb{Q}}(e^{3z_{t+1,\infty}})$. Equivalently, we can express the return variance that is implied by the recovery theorem as

$$\begin{aligned}
\mathrm{Var}_t\left(e^{z_{t+1,\infty}} - 1\right) &= \mathbb{E}_t\left(e^{2z_{t+1,\infty}}\right) - \left\{\mathbb{E}_t\left(e^{z_{t+1,\infty}}\right)\right\}^2, & (23)\\
&= x_t^{[3]} - \{x_t^{[2]}\}^2. & (24)
\end{aligned}$$

Following Andersen, Bollerslev, Diebold, and Labys (2003), we compute the realized variance as the subsequent sum of squared demeaned daily futures returns, with the number of days in the sum matched to the remaining days to expiration of the options at the end of month t.

Panels A and B of Table 2 present the estimates of α and β, when $x_t^{[2]}$ and $x_t^{[3]}$ are constructed using the nearest maturity options. The takeaway is that the restrictions imposed by the recovery theorem are not supported in the data. In other words, when $m^P_{t+1} = 1$, the physical return moments determined from the Arrow-Debreu state prices do not line up with the actual counterparts.

The forecast of futures return with $x_t^{[2]}$ is considered in Panel A of Table 2. The point estimate of β is 7.340 and the β estimate is statistically significant. Moreover, the estimate of α is -6.343, and the hypothesis that $\alpha = 0$ is rejected. The correlation between $y_{t+1}^{[1]}$ and $x_t^{[2]}$ is 0.20.

Panel B of Table 2 assesses the forecast of return variance using $x_t^{[3]} - \{x_t^{[2]}\}^2$, and we obtain α and β estimates of 0.001 and 0.644, respectively. Both α and β estimates are statistically different from zero. The correlation between the realized variance and $x_t^{[3]} - \{x_t^{[2]}\}^2$ is 0.58.

Affirming our findings from a different angle, the Wald test statistics, which are χ^2-distributed with 2 degrees of freedom, reject the null hypothesis that $\alpha = 0$ and $\beta = 1$. The p-values for the Wald statistics, shown in parentheses, are not higher than 0.04. While the adjusted R^2 for the mean return is 4.0%, and the adjusted R^2 rises to 33.4% for return variance, our emphasis is on the restrictions $\alpha = 0$ and $\beta = 1$. Still, how does one benchmark the adjusted R^2?

First, the level, slope, and curvature variables (the first, second, and third principal components extracted from available Treasury yields (e.g., Cochrane (2015))) produce an adjusted R^2 of 4.4% when forecasting the futures return. Going further, Internet Appendix (Section II and Table Internet Appendix-III) explores three bivariate predictive regressions of $x_t^{[2]}$ with level, slope, or curvature as forecasting variables. While our results imply that $x_t^{[2]}$ remains a statistically significant predictor, the coefficient on the slope variable is also statistically significant.

Next, we consider option-implied variance as an ad hoc (univariate) forecasting variable for return variance. The option-implied variance, computed using an at-the-money option on the 30-year Treasury bond futures and the model of Black (1976), achieves an adjusted R^2 of 43.5%.

Thus, a readily constructed variable encodes information that helps to forecast return variance with comparable explanatory power. Additionally, our bivariate predictive regression results (see Internet Appendix (Section II) and Table Internet Appendix-III) show that while $x_t^{[3]} - \{x_t^{[2]}\}^2$ is a statistically significant predictor, the component of option-implied variance that is orthogonal to $x_t^{[3]} - \{x_t^{[2]}\}^2$ is also a statistically significant predictor. We consider this bivariate predictive regression, as option-implied variance has a correlation of 0.73 with $x_t^{[3]} - \{x_t^{[2]}\}^2$.

To further examine the reliability of the recovery theorem, we compute the following deviations:

$$e^{\text{mean}}_{t\to t+1} \equiv y^{[1]}_{t+1} - x^{[2]}_t \text{ and} \tag{25}$$

$$e^{\text{volatility}}_{t\to t+1} \equiv \left(\sqrt{\text{Var}_t\left(e^{z_{t+1,\infty}}-1\right)} - \sqrt{x^{[3]}_t - \{x^{[2]}_t\}^2}\right) \Big/ \sqrt{\text{Var}_t\left(e^{z_{t+1,\infty}}-1\right)}, \tag{26}$$

which reflect the difference between the realized value and the theoretical counterparts implied by the recovery theorem in combination with option prices. We tabulate the distribution of the deviations (the mean, the standard deviation, and some percentiles) below:

	Mean	Std.	5th	25th	50th	75th	95th
$e^{\text{mean}}_{t\to t+1}$	0.0014	0.0299	-0.0485	-0.0163	0.0029	0.0185	0.0465
$e^{\text{volatility}}_{t\to t+1}$	0.1788	0.2415	-0.2908	0.0538	0.2287	0.3388	0.4938

While the average $e^{\text{mean}}_{t\to t+1}$ is small, the (5th) 95th percentile value is not small on a monthly basis. Moreover, the average $e^{\text{volatility}}_{t\to t+1}$ is 17.88%, in which the scaling by the realized return volatility imparts a percentage deviation interpretation to $e^{\text{volatility}}_{t\to t+1}$.

Exploring a bit further, we also consider the expression for the Sharpe ratio, computed as $(\mathbb{E}_t(R_{t+1,\infty}) - R_{t+1,\text{rf}})/\sqrt{\text{Var}_t(R_{t+1,\infty})} = (\mathbb{E}^{\mathbb{Q}}_t(e^{2z_{t+1,\infty}}) - 1)/\sqrt{\mathbb{E}^{\mathbb{Q}}_t(e^{3z_{t+1,\infty}}) - \{\mathbb{E}^{\mathbb{Q}}_t(e^{2z_{t+1,\infty}})\}^2}$. Under the recovery theorem, our implementations obtain an average Sharpe ratio of 0.104, which deviates from the unconditional value of 0.254 computed from the data.

The main takeaway is that our results present a doubtful picture about the reliability and the practical usefulness of the recovery theorem in the context of the long-term bond market. It appears that the $m_{t+1}^P = 1$ treatment is not innocuous and can potentially misalign the mapping between the physical return density and the risk-neutral return density (as in equation (6)).

3.4. Consistency between unconditional return moments implied by the recovery theorem and data

We apply the GMM estimation of Hansen (1982) to examine whether the (unconditional) physical return moments implied by the recovery theorem are compatible with their actual counterparts. Our goal is to provide another perspective on a theory that postulates an exact relation between the physical return distribution and the risk-neutral return distribution in the long-term bond market.

Consider the disturbance terms in light of an encompassing specification of equation (11):

$$u_{t+1}^{[1]} \equiv \eta_1 \frac{e^{(1-\delta_1)\, z_{t+1,\infty}}}{x_t^{[2]}} - 1 \qquad \text{and} \qquad u_{t+1}^{[2]} \equiv \eta_2 \frac{e^{(2-\delta_2)\, z_{t+1,\infty}}}{x_t^{[3]}} - 1, \tag{27}$$

for some parameters (η_1, δ_1) and (η_2, δ_2). Reposing the restrictions of Subsection 3.3 and using unconditional expectations, we consider the following implications for a set of instruments $\mathbb{I}_t$:

$$\mathbb{E}\left(u_{t+1}^{[1]} \otimes \mathbb{I}_t\right) = 0 \qquad \text{and} \qquad \mathbb{E}\left(u_{t+1}^{[2]} \otimes \mathbb{I}_t\right) = 0. \tag{28}$$

The following parametric restrictions hold when the recovery theorem correctly inverts the physical density from the knowledge of the risk-neutral density:

$$\eta_j = 1, \qquad \text{and} \qquad \delta_j = 0, \qquad \text{for } j = 1,2. \tag{29}$$

In the GMM estimation results reported in Table 3, we explore several sets of instruments. (i) **Set A**: a constant and the first lag of level, slope, and curvature variables, (ii) **Set B**: a constant and the first and second lags of the level variable, (iii) **Set C**: a constant and the first and second lags of the slope variable, (iv) **Set D**: a constant and the first and second lags of the curvature variable, and (v) **Set E**: a constant and the first and second lags of the option-implied variance. The minimized value of the GMM criterion multiplied by $\mathcal{T}$ (the number of time-series observations), denoted by $J_{\mathcal{T}}$, is χ^2-distributed under the null of correctly recovering both the SDF and the physical probabilities, with degrees of freedom (df) equal to the number of orthogonality conditions minus the number of estimated parameters.

Consider the GMM estimations associated with the disturbance $u_{t+1}^{[1]} = \eta_1 \frac{e^{(1-\delta_1) z_{t+1,\infty}}}{x_t^{[2]}} - 1$, where δ_1 is a free parameter with a hypothesized value of zero. Contradicting this implication, the δ_1 estimates are between 0.851 and 0.992. The p-values are consistently 0.00 and the hypothesis of $\delta_1 = 0$ is rejected. Our results associated with the disturbance $u_{t+1}^{[2]} = \eta_2 \frac{e^{(2-\delta_2) z_{t+1,\infty}}}{x_t^{[3]}} - 1$ also indicate a rejection of an implication of the recovery theorem. The δ_2 estimates are between 1.553 and 1.976 and are statistically distinct from zero. At the same time, the estimates of η_1 and η_2 are close to one and statistically different from zero.

While our focus is on testing the restrictions $\delta_j = 0$, the p-values for the J-statistic are higher than 0.05 (0.10) in nine (six) out of 10 estimations. The inability to reject the extended model with some sets of instruments arises as the recovery theorem imposes a value of $\delta_j = 0$, whereas we have kept δ_j to be a freely determined parameter when minimizing the GMM criterion function.

The overall interpretation is that the treatment in Ross (2015) can compromise the structure of risk and pricing in the long-term bond market, whereby $\mathbb{E}^{\mathbb{Q}}(e^{(n+1)\,z_{t+1,\infty}})$ is not aligned with $\mathbb{E}(e^{n\,z_{t+1,\infty}})$, for $n = 1,2$, in a manner that is dictated by the recovery theorem.[9] Our message from the GMM estimations agrees with the findings from the preceding subsections, calling into question the empirical viability of the recovery theorem.

4. Concluding remarks

The recovery theorem of Ross (2015) provides an elegant method to extract the physical probabilities and the SDF simultaneously from option prices, and the work of Ross has ignited a stream of research and controversy. In particular, the theoretical study of Borovička, Hansen, and Scheinkman (2016) shows that the notion of martingale component of SDFs identical to unity is indispensable to the recovery theorem of Ross (2015).

The concept of martingale component of SDFs identical to unity seems abstract and confounding to many trying to understand what all of this means for extracted physical probabilities. Motivated by this ambivalence, this paper formalizes the theoretical restrictions between the physical and risk-neutral return distributions when the SDF is driven by the transitory (the non-martingale) component. Featuring claims in the long-term bond market, we expand on this idea to develop the empirical implications of the recovery theorem.

The environment of long-term bond futures (and the options on futures) is uniquely important to assess the recovery theorem, for two reasons. First, it is theoretically tractable to isolate the form of the SDF in the long-term bond market when the martingale component is degenerate. Second, the futures prices are determined in an arbitrage-free manner consistent with the observed

[9] We also consider surrogating $z_{t+1,\infty}$ with the returns of 10-year Treasury bond futures, and construct the corresponding options data and returns of out-of-the-money puts and calls. Our conclusions remain robust and are akin to those based on Tables 1 through 3.

yield curve and the (perceived) evolution of the interest rates.

To reconcile and interpret our empirical findings, we solve two convex minimization problems to extract the martingale component of SDFs from the returns data. Our key result is that the assumption of a degenerate (identical to unity) martingale component is violated in the long-term bond market. The extracted martingale component exhibits considerable dispersion, and this extracted component is positively correlated with the transitory component. Both sets of results reject an implicit assumption underlying the recovery theorem.

The recovery theorem distills the idea that the first (second) physical return moment can be recovered from the second (third) risk-neutral return moment. We conduct predictive regressions of this restriction and reject the null hypothesis that the intercept is zero and the slope coefficient is one. Complementing the predictive regression exercises, we additionally reject the hypothesis that unconditional physical return moments implied by the recovery theorem and option prices are compatible with the data in a GMM setting. Thus, our empirical approaches and results provide an agnostic view of the practical usefulness of the recovery theorem in the long-term bond market.

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Table 1
Properties of the extracted martingale component of SDFs

We numerically solve for the Lagrange multipliers $(\boldsymbol{\lambda}^*, \nu^*)$ in equations (16) and (18). Then we extract the martingale component according to $m_{t+1}^{P*} = \max(\nu^* + \boldsymbol{\lambda}^{*\prime}\mathbf{Z}_{t+1}, 0)$ or $m_{t+1}^{P*} = \exp(-1 + \nu^* + \boldsymbol{\lambda}^{*\prime}\mathbf{Z}_{t+1})$. Reported are the *annualized* standard deviation, monthly skewness, and monthly kurtosis, of the extracted martingale component. $\rho_{P,T}$ is the correlation between m_{t+1}^{P*} and m_{t+1}^{T} (i.e., $1/R_{t+1,\infty}$). We use the following $\mathbf{Z}_{t+1}$ in our calculations:

$$\mathbf{Z}_{t+1} = \frac{1}{R_{t+1,\mathrm{rf}}\, e^{z_{t+1,\infty}}} \begin{pmatrix} R_{t+1,\mathrm{rf}}\, e^{z_{t+1,\infty}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,3\%\ \mathrm{otm\ put}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,1\%\ \mathrm{otm\ put}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,1\%\ \mathrm{otm\ call}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,3\%\ \mathrm{otm\ call}} - R_{t+1,\mathrm{rf}} \end{pmatrix}.$$

We adopt a block bootstrap procedure (block size of 20) to generate 25,000 bootstrap samples and report the mean, standard deviation, and percentiles of the respective m_{t+1}^{P} statistics. Reported also are the p-values, in curly brackets, for the null hypothesis $\rho_{P,T} \leq 0$, which represents the proportion of replications for which the estimates of correlation $\rho_{P,T} \leq 0$. The sample period is January 1985 to December 2015, for a total of 372 observations.

	Panel A: The convex function $\psi[m^P]$ is $\frac{1}{2}(m^P)^2$				*Panel B:* The convex function $\psi[m^P]$ is $m^P \log(m^P)$			
	Martingale component				Martingale component			
	$\sqrt{\text{Variance}}$	Skewness	Kurtosis	$\rho_{P,T}$	$\sqrt{\text{Variance}}$	Skewness	Kurtosis	$\rho_{P,T}$
Solution	0.487	1.55	9.08	0.53	0.490	2.22	12.32	0.53
Block bootstrap								
Mean	0.726	2.19	18.57	0.41	0.745	3.69	36.34	0.40
Std.	0.296	1.77	18.22	0.23	0.325	2.82	42.46	0.23
5th	0.377	-0.08	4.37	0.05	0.375	0.50	5.16	0.04
25th	0.526	0.86	6.34	0.23	0.530	1.60	8.89	0.23
50th	0.664	1.89	12.55	0.39	0.673	2.93	19.37	0.38
75th	0.852	3.27	23.87	0.57	0.871	5.18	46.32	0.56
95th	1.295	5.38	53.28	0.81	1.383	9.36	129.22	0.81
Bootstrap p-val. H_0: $\rho_{P,T} \leq 0$				{0.026}				{0.027}

Table 2

Adequacy of return and volatility forecasts relying on the recovery theorem and the options on the 30-year Treasury bond futures

Reported are the results from the OLS regressions:

$$y_{t+1}^{[n]} = \alpha^{[n]} + \beta^{[n]} x_t^{[n+1]} + \varepsilon_{t+1}^{[n]}, \quad \text{for } n \geq 1,$$

where $x_t^{[n+1]} \equiv \mathbb{E}_t^{\mathbb{Q}}(e^{(n+1)\, z_{t+1,\infty}})$ and is synthesized using option prices, as described in equation (10), at the end of month t. The variance of the futures return consistent with the recovery theorem is

$$\text{Var}_t(e^{z_{t+1,\infty}} - 1) = x_t^{[3]} - \{x_t^{[2]}\}^2.$$

The realized variance (the dependent variable in Panel B) is calculated as the sum of squared demeaned daily futures returns, and the number of days in the sum match the remaining days to expiration of the options contract. We report the coefficient estimates, as well as the two-sided p-values (in square brackets, denoted by NW[p]) based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994). Reported also are the two-sided p-values (denoted by H[p]) based on the Hodrick (1992) 1B covariance estimator under the null of no forecasting ability. The adjusted R^2 (in %) is denoted by $\overline{R}^2$, and DW is the Durbin-Watson statistic. We perform the Wald test for the hypothesis $\alpha = 0$ and $\beta = 1$ and report the $\chi^2(2)$ statistics, with p-values in parentheses. The sample period is January 1985 to December 2015. With nearest-maturity options, there are 372 monthly observations.

Dependent variable		α	β	$\overline{R}^2$ (%)	DW	Wald test $\alpha = 0$, $\beta = 1$	CORR($y_{t+1}^{[n]}, x_t^{[n+1]}$)
Panel A: Using options data to recover the first physical return moment							
Gross futures return:		-6.343	7.340	4.0	1.51	6.44	0.20
	NW[p]	[0.01]	[0.00]			(0.04)	
	H[p]	$\lfloor 0.08 \rfloor$	$\lfloor 0.04 \rfloor$				
Panel B: Using options data to recover the physical return variance							
Variance of futures return:		0.001	0.644	33.4	1.25	110.87	0.58
	NW[p]	[0.00]	[0.00]			(0.00)	
	H[p]	$\lfloor 0.00 \rfloor$	$\lfloor 0.00 \rfloor$				

Table 3

Testing the link between the return quantities implied by the recovery theorem and the data: GMM estimation results

The estimation is for a single equation using Hansen's (1982) GMM: $\mathbb{E}\left(u_{t+1}^{[1]} \otimes \mathbb{I}_t\right) = 0$ and $\mathbb{E}\left(u_{t+1}^{[2]} \otimes \mathbb{I}_t\right) = 0$, where the disturbance terms are defined as $u_{t+1}^{[1]} \equiv \eta_1 \frac{e^{(1-\delta_1) z_{t+1,\infty}}}{x_t^{[2]}} - 1$, and $u_{t+1}^{[2]} \equiv \eta_2 \frac{e^{(2-\delta_2) z_{t+1,\infty}}}{x_t^{[3]}} - 1$, for some parameters (η_1, δ_1) and (η_2, δ_2). We test whether the data supports $\delta_1 = 0$ and $\delta_2 = 0$, and report the two-sided p-values in square brackets. Reported also is the $J_{\mathcal{T}}$ statistic of Hansen (1982), with the p-value in parentheses. We consider several sets of instruments. Set A: a constant and the first lag of level, slope, and curvature variables; Set B: a constant and the first and second lags of the level variable; Set C: a constant and the first and second lags of the slope variable; Set D: a constant and the first and second lags of the curvature variable, and Set E: a constant and the first and second lags of the option-implied variance. The level, slope, and curvature variables are the first three principal components extracted from available Treasury yields. The sample period is January 1985 to December 2015, for a total of 372 monthly observations.

			η_1 [p-val.]	δ_1 [p-val.]	$J_{\mathcal{T}}$ (p-val.)	df
GMM estimates	$\mathbb{E}\left(u_{t+1}^{[1]} \otimes \mathbb{I}_t\right) = 0$	SET A	1.001	0.965	0.51	2
			[0.00]	[0.00]	(0.777)	
	$\mathbb{E}\left(u_{t+1}^{[1]} \otimes \mathbb{I}_t\right) = 0$	SET B	1.001	0.966	2.83	1
			[0.00]	[0.00]	(0.093)	
	$\mathbb{E}\left(u_{t+1}^{[1]} \otimes \mathbb{I}_t\right) = 0$	SET C	1.001	0.973	1.02	1
			[0.00]	[0.00]	(0.313)	
	$\mathbb{E}\left(u_{t+1}^{[1]} \otimes \mathbb{I}_t\right) = 0$	SET D	1.001	0.992	2.14	1
			[0.00]	[0.00]	(0.143)	
	$\mathbb{E}\left(u_{t+1}^{[1]} \otimes \mathbb{I}_t\right) = 0$	SET E	1.006	0.851	3.85	1
			[0.00]	[0.00]	(0.050)	

			η_2 [p-val.]	δ_2 [p-val.]	$J_{\mathcal{T}}$ (p-val.)	
GMM estimates	$\mathbb{E}\left(u_{t+1}^{[2]} \otimes \mathbb{I}_t\right) = 0$	SET A	1.002	1.895	0.46	2
			[0.00]	[0.00]	(0.795)	
	$\mathbb{E}\left(u_{t+1}^{[2]} \otimes \mathbb{I}_t\right) = 0$	SET B	1.002	1.901	2.72	1
			[0.00]	[0.00]	(0.099)	
	$\mathbb{E}\left(u_{t+1}^{[2]} \otimes \mathbb{I}_t\right) = 0$	SET C	1.002	1.920	0.97	1
			[0.00]	[0.00]	(0.326)	
	$\mathbb{E}\left(u_{t+1}^{[2]} \otimes \mathbb{I}_t\right) = 0$	SET D	1.003	1.976	2.17	1
			[0.00]	[0.00]	(0.141)	
	$\mathbb{E}\left(u_{t+1}^{[2]} \otimes \mathbb{I}_t\right) = 0$	SET E	1.002	1.553	3.72	1
			[0.00]	[0.00]	(0.054)	

Appendix: Proof of the solutions in Case 1 and Case 2

Proof of Case 1. To streamline equation presentation in what follows next, we define

$$w_t \equiv m_t^P \text{ for all } t, \qquad \text{and} \qquad \psi[w_t] \equiv \frac{1}{2}w_t^2, \tag{A1}$$

and write $\mathbb{E}(.)$ to indicate unconditional expectation. We assume $\mathbb{E}(w^2) < +\infty$.

Let $\boldsymbol{\lambda} \in \mathbb{R}^{J+1}$ be a J-dimensional vector of Lagrange multipliers for the equality constraints $\mathbb{E}(w\mathbf{Z}) = \mathbf{0}$ and $\nu \in \mathbb{R}$ be the Lagrange multiplier for the equality constraint $\mathbb{E}(w) = 1$. The associated Lagrangian is

$$\mathcal{L}[w, \boldsymbol{\lambda}, \nu] = \mathbb{E}(\psi[w]) + \boldsymbol{\lambda}^{'}(\mathbf{0} - \mathbb{E}(w\mathbf{Z})) + \nu(1 - \mathbb{E}(w)). \tag{A2}$$

The Lagrange dual problem is

$$\sup_{(\boldsymbol{\lambda},\nu)\in\mathbb{R}^{J+2}} \left(\inf_w \ \mathcal{L}[w, \boldsymbol{\lambda}, \nu] \right). \tag{A3}$$

If we denote w^* as the optimal solution,

$$\inf_w \ \mathcal{L}[w, \boldsymbol{\lambda}, \nu] = -\left(-\mathbb{E}(\psi[w^*]) + \boldsymbol{\lambda}^{'}\mathbb{E}(w^*\mathbf{Z}) + \nu\mathbb{E}(w^*) \right) + \boldsymbol{\lambda}'\mathbf{0} + \nu. \tag{A4}$$

Henceforth, observe that

$$-\mathbb{E}(\psi[w^*]) + \boldsymbol{\lambda}^{'}\mathbb{E}(w^*\mathbf{Z}) + \nu\mathbb{E}(w^*) = \mathbb{E}(\underbrace{(\lambda^{'}\mathbf{Z} + \nu)w^* - \psi[w^*]}_{\equiv\psi^\bullet[\boldsymbol{\lambda}^{'}\mathbf{Z}+\nu]}). \tag{A5}$$

The Fenchel conjugate, denoted by $\psi^\bullet[h]$, of a function $\psi[\omega]$, is defined as (Borwein and Zhu

(2005, Section 4.1.1 and also Theorem 4.4.3))

$$\psi^{\bullet}[h] \;\equiv \sup_{\omega \in [0,\infty] \,\cap\, \text{domain}\, \psi} h\omega \;-\; \psi[\omega]. \tag{A6}$$

Accordingly, we can write equation (A4) as

$$\inf_{w} \; \mathcal{L}[w, \boldsymbol{\lambda}, \nu] \;=\; \boldsymbol{\lambda}'\mathbf{0} \;+\; \nu \;-\; \mathbb{E}(\psi^{\bullet}[\mathbf{Z}'\boldsymbol{\lambda} + \nu]). \tag{A7}$$

The dual problem is

$$\sup_{(\boldsymbol{\lambda},\nu)} \quad \mathbf{0}'\boldsymbol{\lambda} \;+\; \nu \;-\; \mathbb{E}(\psi^{\bullet}[\mathbf{Z}'\boldsymbol{\lambda} + \nu]). \tag{A8}$$

The task is to derive the form of $\psi^{\bullet}[h]$, while noting that $\psi[\omega] = \frac{1}{2}\omega^2$ for Case 1.

To find the value of ω that maximizes (A6), we equate to zero its derivative with respect to ω,

$$\frac{d\,(h\,\omega - \psi\,[\omega])}{d\omega} \;=\; h \;-\; \omega \;=\; 0. \tag{A9}$$

$$\text{If } h \geq 0,\;\; \omega = h \text{ is the unique solution. Hence, } \; \omega = h \times 1_{\{h \geq 0\}}, \tag{A10}$$

where $1_{\{h \geq 0\}}$ is an indicator function for the event $\{h \geq 0\}$.

The asserted solution holds since *domain* $(h\omega - \psi[\omega])$=$[0,\infty)$. Otherwise, if $h \leq 0$, then $h - \omega \leq 0$. We recognize that since $h\,\omega - \psi[\omega]$ is a decreasing function of ω, the sup is obtained for $\omega = 0$.

Replacing ω in the original equation (A6), we obtain the conjugate $\psi^{\bullet}\,[h]$ as

$$\psi^{\bullet}[h] \;=\; \frac{1}{2}\left(h \times 1_{\{h \geq 0\}}\right)^2. \tag{A11}$$

We can now obtain w^* from the conjugate evaluated at the optimal $(\boldsymbol{\lambda}^*, \nu^*)$, that is,

$$w^* = \left.\frac{d\psi^\bullet[h]}{dh}\right|_{h=\nu^*+\boldsymbol{\lambda}^{*\prime}\mathbf{Z}} \tag{A12}$$
$$= \left(\nu^* + \boldsymbol{\lambda}^{*\prime}\mathbf{Z}\right) \times 1_{\{\nu^*+\boldsymbol{\lambda}^{*\prime}\mathbf{Z}\geq 0\}} = \max(\nu^* + \boldsymbol{\lambda}^{*\prime}\mathbf{Z}, 0). \tag{A13}$$

Completing the description, the vector of Lagrange multipliers $(\boldsymbol{\lambda}, \nu)$ is a solution to

$$\sup_{(\boldsymbol{\lambda},\nu)} \quad \nu - \mathbb{E}(\psi^\bullet[\mathbf{Z}'\boldsymbol{\lambda} + \nu]). \tag{A14}$$

Equivalently, we have

$$\sup_{(\boldsymbol{\lambda},\nu)} \quad \nu - \frac{1}{2}\mathbb{E}\left([(\mathbf{Z}'\boldsymbol{\lambda} + \nu) \times 1_{\{\mathbf{Z}'\boldsymbol{\lambda}+\nu\geq 0\}}]^2\right). \tag{A15}$$

The characterizations in equations (A13) and (A15) constitute the solution to the martingale component of the SDF subject to the nonnegativity constraint. We have proved the expressions in Case 1. ■

Proof of Case 2. The steps of the proof are similar to how we did it previously, except for the form of the conjugate function. The considered convex function is now

$$\psi[w] = w\log(w). \quad \text{Assume } \mathbb{E}(w\log(w)) < +\infty. \tag{A16}$$

The problem is $\inf_w \ \mathbb{E}(\psi[w])$ subject to the constraints in (13). Let $\boldsymbol{\lambda} \in \mathbb{R}^{J+1}$ be a J-dimensional vector of Lagrange multipliers for the equality constraints $\mathbb{E}(w\mathbf{Z}) = \mathbf{0}$ and $\nu \in \mathbb{R}$ be the Lagrange multiplier for the equality constraint $\mathbb{E}(w) = 1$. The Fenchel conjugate, denoted by $\psi^\bullet[h]$, of

$\psi[\omega]$, is

$$\psi^{\bullet}[h] \;\equiv\; \sup_{\omega\in(0,\infty]\,\cap\,\text{domain}\,\psi} \; h\omega \;-\; \psi[\omega]. \tag{A17}$$

Equating to zero the derivative of the preceding expression with respect to ω, we have

$$\frac{d\psi^{\bullet}[h]}{d\omega} \;=\; h \;-\; (1+\log(\omega)) \;=\; 0. \tag{A18}$$

$$\text{The solution to equation (A18) is } \;\omega = \exp(h\;-1). \tag{A19}$$

Substituting ω back into equation (A17), we obtain the form of the conjugate as

$$\psi^{\bullet}\,[h] \;\;=\;\; h\exp(h\;-\;1)\;-\;\exp(h\;-\;1)\log(\exp(h\;-\;1)) \tag{A20}$$

$$=\;\; \exp(h\;-\;1). \tag{A21}$$

The martingale component that minimizes the objective function can be obtained from the conjugate as follows:

$$w^{*} = \left.\frac{d\psi^{\bullet}[h]}{dh}\right|_{h=\nu^{*}+\boldsymbol{\lambda}^{*\prime}\mathbf{Z}} \;=\; \exp\left(-1+\nu^{*}+\boldsymbol{\lambda}^{*\prime}\mathbf{Z}\right). \tag{A22}$$

The Lagrange multipliers, therefore, are a solution to

$$\sup_{(\boldsymbol{\lambda},\nu)} \;\; \nu-\mathbb{E}(\psi^{\bullet}[\mathbf{Z}'\boldsymbol{\lambda}+\nu]), \;\text{ or, equivalently, }\; \sup_{(\boldsymbol{\lambda},\nu)} \;\; \nu-\mathbb{E}\left(\exp\left(-1+\nu+\boldsymbol{\lambda}'\mathbf{Z}\right)\right). \tag{A23}$$

The martingale condition is satisfied, since the derivative of equation (A23) respect to ν is

$$1 \;-\; \mathbb{E}\left(\exp\left(-1+\nu+\boldsymbol{\lambda}'\mathbf{Z}\right)\right) = 0. \tag{A24}$$

We have verified that the solution forming the system in equations (17) and (18) holds. ■

A Recovery That We Can Trust? Deducing and Testing the Restrictions of the Recovery Theorem

Internet Appendix: Not for Publication

Abstract

This Internet Appendix presents additional empirical results. Section I analyzes the impact of altering the sets of test assets on (i) the variance of the extracted m^P_{t+1}, and (ii) the correlation between the extracted m^P_{t+1} and the transitory component m^T_{t+1}. Section II considers return and variance forecasts in the presence of additional predictors. Section III shows that deviations from the recovery theorem are not declining across the quintiles of option strikes.

I. Sets of test assets and the properties of the extracted m^P_{t+1}

Our Table 1 is based on $\mathbf{Z}_{t+1}$ (as in equation (19)) that has a dimension of five. For the discussions here, we denote this extracted martingale component (obtained by solving the minimum discrepancy problems in Case 1 and Case 2) by $m^P_{t+1}|_{\dim(5)}$.

We consider the consequences of changing the dimension of $\mathbf{Z}_{t+1}$ from three to five or from five to six, and evaluate its impact on the variance of the extracted m^P_{t+1}.

The additional impetus is to study how the correlation, $\rho_{P,T}$, between the extracted m^P_{t+1} and m^T_{t+1} changes with the dimension of $\mathbf{Z}_{t+1}$. Our proposed exercises recognize the possibility that the variance of the extracted m^P_{t+1} could constitute a small part of the variance of the true m^P_{t+1}. In this case, the $\rho_{P,T}$ may be a poor estimate of the true correlation.

Panel A of Table Internet Appendix-I presents the results when the set of test assets in equation (19) is augmented to include straddle returns:

$$\mathbf{Z}_{t+1} = \frac{1}{R_{t+1,\mathrm{rf}}\, e^{z_{t+1,\infty}}} \begin{pmatrix} R_{t+1,\mathrm{rf}}\, e^{z_{t+1,\infty}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,3\%\ \mathrm{otm\ put}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,1\%\ \mathrm{otm\ put}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,1\%\ \mathrm{otm\ call}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,3\%\ \mathrm{otm\ call}} - R_{t+1,\mathrm{rf}} \\ R_{t+1,\mathrm{straddle}} - R_{t+1,\mathrm{rf}} \end{pmatrix}. \tag{IA-1}$$

The gross return of a straddle on the 30-year Treasury bond futures is constructed as $R_{t+1,\mathrm{straddle}} \equiv \frac{\max(F^{[360m]}_{t+1}-K,0)+\max(K-F^{[360m]}_{t+1},0)}{C_t[K]+P_t[K]}$, where $C_t[K]$ and $P_t[K]$ are the prices of calls and puts with moneyness closest to zero, respectively. We denote this extracted martingale component by $m^P_{t+1}|_{\dim(6)}$.

Next, Panel B of Table Internet Appendix-I presents the results when the set of test assets in equation (19) excludes the returns of 3% out-of-the-money puts and calls:

$$\mathbf{Z}_{t+1} = \frac{1}{R_{t+1,\text{rf}}\, e^{z_{t+1,\infty}}} \begin{pmatrix} R_{t+1,\text{rf}}\, e^{z_{t+1,\infty}} \;-\; R_{t+1,\text{rf}} \\ R_{t+1,1\%\text{ otm put}} \;-\; R_{t+1,\text{rf}} \\ R_{t+1,1\%\text{ otm call}} \;-\; R_{t+1,\text{rf}} \end{pmatrix}. \tag{IA-2}$$

We denote this extracted martingale component by $m^P_{t+1}\big|_{\dim(3)}$.[1]

To gauge the differences in the extracted martingale components obtained using different sets of test assets, we construct the time-series of squared demeaned martingale components:

$$v_{t+1}\big|_{\dim(j)} \equiv (m^P_{t+1}\big|_{\dim(j)} - 1)^2, \quad \text{for } j = 3, 5, 6. \tag{IA-3}$$

Additionally, we construct the time-series

$$\Delta v^a_{t+1} \equiv v_{t+1}\big|_{\dim(5)} - v_{t+1}\big|_{\dim(3)} \qquad \text{and} \qquad \Delta v^b_{t+1} \equiv v_{t+1}\big|_{\dim(6)} - v_{t+1}\big|_{\dim(5)}. \tag{IA-4}$$

Consider the regressions of Δv^a_{t+1} and Δv^b_{t+1} on a constant, that is, $\Delta v^a_{t+1} = \mho^a + \varepsilon_{t+1}$ and $\Delta v^b_{t+1} = \mho^b + \varepsilon_{t+1}$. The hypotheses of $\mho^a = 0$ and $\mho^b = 0$ amount to testing whether the variances of the extracted martingale components are equal. We report the results in Table Internet Appendix-II.

Our results indicate that increasing the dimensionality of $\mathbf{Z}_{t+1}$ from three to five increases the variance of the extracted m^P_{t+1} and results in an estimate of $\mho^a$ that is positive and statistically significant (the p-values are 0.009 and 0.022).[2] In contrast, increasing the dimensionality of $\mathbf{Z}_{t+1}$ from five to six does not generate a statistically significant increase in the variance of the extracted

[1] When $\mathbf{Z}_{t+1}$ only contains the return of the long-term bond, the volatility of the extracted martingale component is 26%, and this extracted martingale component has a correlation of 0.995 with the transitory component.

[2] The two-sided p-values on the estimates of $\mho^a$ and $\mho^b$ are based on Newey and West (1987), with automatically selected lags.

m^P_{t+1} (the p-values are 0.096 and 0.234).

Our exercises also allow us to assess the impact of changing the sets of test assets on the estimates of $\rho_{P,T}$. The estimate of $\rho_{P,T}$ changes from 0.64 to 0.53 and then to 0.49 (with $\psi[m^P] = \frac{1}{2}(m^P)^2$) when the dimensionality of $\mathbf{Z}_{t+1}$ changes from three to five and then from five to six. Moreover, the null hypothesis $\rho_{P,T} \leq 0$ is rejected in each of the three cases.

Additionally, we consider the null hypothesis that $\rho_{P,T}$ estimates are increasing with the number of test assets. To do so, we resort to a block bootstrap (size 20) procedure and randomly select, with replacement, raw returns and recompute $\mathbf{Z}_{t+1}$ across the three, five, and six test assets. We then solve the minimum discrepancy problems and compute $m^P_{t+1}\big|_{\text{dim}(3)}$, $m^P_{t+1}\big|_{\text{dim}(5)}$, and $m^P_{t+1}\big|_{\text{dim}(6)}$. The respective correlations are denoted by $\rho_{P,T}\big|_{\text{dim}(3)}$, $\rho_{P,T}\big|_{\text{dim}(5)}$, and $\rho_{P,T}\big|_{\text{dim}(6)}$. We further compute (in each bootstrap draw)

$$\Delta\rho^a_{P,T} \equiv \rho_{P,T}\big|_{\text{dim}(5)} - \rho_{P,T}\big|_{\text{dim}(3)} \qquad \text{and} \qquad \Delta\rho^b_{P,T} \equiv \rho_{P,T}\big|_{\text{dim}(6)} - \rho_{P,T}\big|_{\text{dim}(5)}. \tag{IA-5}$$

Table Internet Appendix-II illustrates that one can reject the null hypothesis that $\Delta\rho^a_{P,T} \geq 0$ but is unable to reject $\Delta\rho^b_{P,T} \geq 0$. The bootstrap p-value (in curly brackets) is the proportion (across the 25,000 bootstraps) in which $\Delta\rho^a_{P,T} \geq 0$ (or $\Delta\rho^b_{P,T} \geq 0$). ■

II. Forecasting return and variance with additional predictors

The additional variables used in our bivariate forecasting exercises are constructed as follows:

Level$_t$: The first principal component extracted from available Treasury yields (maturities of 1 month, 3 month, 1, 2, 5, 7, 10, 20, and 30 years. Source: the CRSP Risk Free Rates File and the CRSP Fixed Term Indices Files);

Slope$_t$: The second principal component extracted from available Treasury yields;

Curvature$_t$: The third principal component extracted from available Treasury yields; and

Option-Implied Variance$_t$: The option-implied variance is computed from an at-the-money option on the 30-year Treasury bond futures and the model of Black (1976).

We examine whether $x_t^{[2]}$ is a statistically significant predictor of futures return in the bivariate predictive regressions involving Level$_t$, Slope$_t$, or Curvature$_t$:

$$e^{z_{t+1,\infty}} = \alpha + \beta x_t^{[2]} + \theta_1 \text{Level}_t + \varepsilon_{t+1}, \qquad \text{(IA-6)}$$

$$e^{z_{t+1,\infty}} = \alpha + \beta x_t^{[2]} + \theta_2 \text{Slope}_t + \varepsilon_{t+1}, \qquad \text{(IA-7)}$$

$$e^{z_{t+1,\infty}} = \alpha + \beta x_t^{[2]} + \theta_3 \text{Curvature}_t + \varepsilon_{t+1}. \qquad \text{(IA-8)}$$

The correlation between $x_t^{[2]}$ and Level$_t$, Slope$_t$, and Curvature$_t$ is 0.17, 0.24, and -0.14, respectively.

Table Internet Appendix-III shows that $x_t^{[2]}$ is a statistically significant predictor in the presence of Level$_t$ or Curvature$_t$. Additionally, both θ_1 and θ_3 are individually insignificant (the NW[p] is 0.35 and 0.41, respectively). Moreover, our results imply that $x_t^{[2]}$ and Slope$_t$ are both statistically significant predictors, with NW[p] (H[p]) of 0.01 (0.10) and 0.00 (0.00), respectively. In the bivariate regression with $x_t^{[2]}$ and Slope$_t$, the adjusted R^2 is 6.2%, as opposed to 4.0% in the univariate regression with $x_t^{[2]}$.

Next, we consider the predictive content of Option-Implied Variance$_t$ for the realized variance of futures return. In this regard, we note that Option-Implied Variance$_t$ and $x_t^{[3]} - \{x_t^{[2]}\}^2$ have a correlation of 0.73. Hence, we consider the following bivariate predictive regression:

$$\text{RV}_{t\to t+1} = \alpha + \beta\,(x_t^{[3]} - \{x_t^{[2]}\}^2) + \theta_4\,\text{Option-Implied Variance}_t^{\perp} + \varepsilon_{t+1}, \qquad \text{(IA-9)}$$

where Option-Implied Variance$_t^\perp$ is the component of Option-Implied Variance$_t$ that is orthogonal to $x_t^{[3]} - \{x_t^{[2]}\}^2$. The realized variance, here denoted by $\text{RV}_{t\to t+1}$, is calculated as the sum of the squared demeaned daily futures returns (the number of days in the sum match the remaining days to expiration of the options contract).

When forecasting $\text{RV}_{t\to t+1}$ in equation (IA-9), the coefficient on Option-Implied Variance$_t^\perp$ is positive with a NW[p] (H[p]) of 0.00 (0.00). The adjusted R^2 is 47.6%, which contrasts 33.4% obtained with $x_t^{[3]} - \{x_t^{[2]}\}^2$ alone. ■

III. Number of option strikes and deviations from the recovery theorem

The concern that we address is whether deviations from the recovery theorem are sensitive to the number of available option strikes. Probing this possibility, we examine the absolute deviations binned across the *quintiles* of the number of strikes:

	Q1	Q2	Q3	Q4	Q5
Number of puts and calls	12	18	24	33	53
$\left\|e_{t\to t+1}^{\text{mean}}\right\|$	0.0186	0.0274	0.0228	0.0167	0.0245
$\left\|e_{t\to t+1}^{\text{volatility}}\right\|$	0.2900	0.3072	0.2651	0.2209	0.2180

The absolute deviations are not declining across the quintiles of option strikes. For example, when we have an average of 53 calls and puts to compute a forecast of mean return, the absolute deviation is larger compared to when there is an average of 12 calls and puts. ■

Table Internet Appendix-I

Properties of the extracted martingale component of SDFs using alternative sets of $\mathbf{Z}_{t+1}$

We numerically solve for the Lagrange multipliers $(\boldsymbol{\lambda}^*, \nu^*)$ in equations (16) and (18). Then we extract $m^{P*}_{t+1} = \max(\nu^* + \boldsymbol{\lambda}^{*\prime}\mathbf{Z}_{t+1}, 0)$ or $m^{P*}_{t+1} = \exp(-1 + \nu^* + \boldsymbol{\lambda}^{*\prime}\mathbf{Z}_{t+1})$. Reported are the *annualized* standard deviation, monthly skewness, and monthly kurtosis, of the extracted martingale component. $\rho_{P,T}$ is the correlation between m^{P*}_{t+1} and m^{T}_{t+1} (i.e., $1/R_{t+1,\infty}$). We use two different $\mathbf{Z}_{t+1}$ in our calculations:

$$\mathbf{Z}_{t+1} = \frac{1}{R_{t+1,\text{rf}}\, e^{z_{t+1,\infty}}} \begin{pmatrix} R_{t+1,\text{rf}}\, e^{z_{t+1,\infty}} - R_{t+1,\text{rf}} \\ R_{t+1,3\%\text{ otm put}} - R_{t+1,\text{rf}} \\ R_{t+1,1\%\text{ otm put}} - R_{t+1,\text{rf}} \\ R_{t+1,1\%\text{ otm call}} - R_{t+1,\text{rf}} \\ R_{t+1,3\%\text{ otm call}} - R_{t+1,\text{rf}} \\ R_{t+1,\text{straddle}} - R_{t+1,\text{rf}} \end{pmatrix} \quad \text{or} \quad \mathbf{Z}_{t+1} = \frac{1}{R_{t+1,\text{rf}}\, e^{z_{t+1,\infty}}} \begin{pmatrix} R_{t+1,\text{rf}}\, e^{z_{t+1,\infty}} - R_{t+1,\text{rf}} \\ R_{t+1,1\%\text{ otm put}} - R_{t+1,\text{rf}} \\ R_{t+1,1\%\text{ otm call}} - R_{t+1,\text{rf}} \end{pmatrix}.$$

The gross return of a straddle on the 30-year Treasury bond futures is $R_{t+1,\text{straddle}} \equiv \frac{\max(F^{[360m]}_{t+1} - K, 0) + \max(K - F^{[360m]}_{t+1}, 0)}{C_t[K] + P_t[K]}$, where $C_t[K]$ and $P_t[K]$ are, respectively, the prices of calls and puts, with moneyness closest to zero. We adopt a block bootstrap procedure (block size of 20) to generate 25,000 bootstrap samples and report the mean, standard deviation, and percentiles of the respective m^{P}_{t+1} statistics. Reported also are the p-values, in curly brackets, for the null hypothesis $\rho_{P,T} \leq 0$, which represents the proportion of replications for which the estimates of correlation $\rho_{P,T} \leq 0$. The sample period is January 1985 to December 2015, for a total of 372 observations.

	Panel A: **Z** contains the long-term bond, 1% and 3% out-of-the-money puts and calls, and straddles				Panel B: **Z** contains the long-term bond, and 1% out-of-the-money put and call			
	Martingale component				Martingale component			
	$\sqrt{\text{Variance}}$	Skewness	Kurtosis	$\rho_{P,T}$	$\sqrt{\text{Variance}}$	Skewness	Kurtosis	$\rho_{P,T}$
The convex function $\psi[m^P]$ is $\frac{1}{2}(m^P)^2$								
Solution	0.527	2.52	14.23	0.49	0.406	1.13	6.10	0.64
Mean	0.810	3.80	36.11	0.36	0.503	1.44	7.73	0.55
Std.	0.329	2.54	37.33	0.21	0.149	0.71	3.16	0.27
5th	0.440	0.59	5.46	0.04	0.282	0.35	4.39	0.08
50th	0.737	3.29	22.68	0.35	0.491	1.40	7.01	0.57
95th	1.429	8.76	113.66	0.74	0.763	2.65	13.45	0.94
Bootstrap p-val. H_0: $\rho_{P,T} \leq 0$				{0.024}				{0.026}
The convex function $\psi[m^P]$ is $m^P \log(m^P)$								
Solution	0.522	1.74	9.76	0.50	0.404	0.55	4.69	0.64
Mean	0.788	2.12	17.42	0.37	0.499	0.81	5.45	0.54
Std.	0.300	1.57	15.20	0.21	0.144	0.67	1.37	0.27
5th	0.434	0.08	4.53	0.05	0.281	-0.20	3.98	0.08
50th	0.727	2.00	12.95	0.36	0.486	0.79	5.10	0.56
95th	1.371	4.84	45.64	0.74	0.748	1.97	8.32	0.95
Bootstrap p-val. H_0: $\rho_{P,T} \leq 0$				{0.027}				{0.027}

Table Internet Appendix-II

Impact of altering the number of test assets in the minimum discrepancy problems

Denoted by $m^P_{t+1}\big|_{\dim(3)}$, $m^P_{t+1}\big|_{\dim(5)}$, and $m^P_{t+1}\big|_{\dim(6)}$ are the martingale components that solve the minimum discrepancy problems (i.e., Case 1 and Case 2), when the number of test assets in $\mathbf{Z}_{t+1}$ is increased from three to five and then from five to six. We further denote the correlation between the extracted martingale component and the transitory component by $\rho_{P,T}\big|_{\dim(3)}$, $\rho_{P,T}\big|_{\dim(5)}$, and $\rho_{P,T}\big|_{\dim(6)}$, respectively. For our purposes, we construct the time-series (multiplied by 12 to annualize)

$$v_{t+1}\big|_{\dim(j)} \equiv (m^P_{t+1}\big|_{\dim(j)} - 1)^2, \quad \text{for } j = 3,5,6.$$

Define

$$\Delta v^a_{t+1} \equiv v_{t+1}\big|_{\dim(5)} - v_{t+1}\big|_{\dim(3)}, \qquad \text{and} \qquad \Delta v^b_{t+1} \equiv v_{t+1}\big|_{\dim(6)} - v_{t+1}\big|_{\dim(5)}.$$

The hypothesis of equal variances of the extracted martingale components is equivalent to testing the statistical significance of the intercepts in the regressions $\Delta v^a_{t+1} = \mho^a + \varepsilon_{t+1}$ and $\Delta v^b_{t+1} = \mho^b + \varepsilon_{t+1}$. The associated two-sided p-values are based on the procedure of Newey and West (1987) with lags automatically selected. Additionally, our aim is to evaluate the null hypothesis that the correlation between the extracted martingale component and the transitory component is increasing with the number of test assets. We consider a block bootstrap procedure (block size is 20) and randomly select, with replacement, raw returns and recompute $\mathbf{Z}_{t+1}$ across the three, five, and six test assets. We then solve the respective minimum discrepancy problems. For each set of the bootstrap draws, we compute

$$\Delta\rho^a_{P,T} \equiv \rho_{P,T}\big|_{\dim(5)} - \rho_{P,T}\big|_{\dim(3)} \qquad \text{and} \qquad \Delta\rho^b_{P,T} \equiv \rho_{P,T}\big|_{\dim(6)} - \rho_{P,T}\big|_{\dim(5)}.$$

Reported are the p-values, in curly brackets, for the null hypothesis $\Delta\rho^a_{P,T} \geq 0$ (or $\Delta\rho^b_{P,T} \geq 0$), which represents the proportion of replications for which $\Delta\rho^a_{P,T} \geq 0$ (or $\Delta\rho^b_{P,T} \geq 0$).

	Three assets versus Five assets		Five assets versus Six assets	
	$\mho^a$		$\mho^b$	
$\psi[m^P]$	$\frac{1}{2}(m^P)^2$	$m^P \log(m^P)$	$\frac{1}{2}(m^P)^2$	$m^P \log(m^P)$
Differences in variances of extracted m^P_{t+1} across the sets of test assets				
Coeff.	0.073	0.075	0.036	0.037
p-value	[0.009]	[0.022]	[0.096]	[0.234]
	$\Delta\rho^a_{P,T}$		$\Delta\rho^b_{P,T}$	
$\psi[m^P]$	$\frac{1}{2}(m^P)^2$	$m^P \log(m^P)$	$\frac{1}{2}(m^P)^2$	$m^P \log(m^P)$
Differences in estimated correlations across the sets of test assets				
Estimate	-0.11	-0.11	-0.03	-0.03
Bootstrap p-val.	{0.028}	{0.026}	{0.050}	{0.080}
H_0: $\Delta\rho_{P,T} \geq 0$				

Table Internet Appendix-III
Return and volatility forecasts using additional predictors

Reported are the results from the following bivariate predictive regressions:

$$\begin{aligned} e^{z_{t+1,\infty}} &= \alpha + \beta x_t^{[2]} + \theta_1 \text{Level}_t + \varepsilon_{t+1}, \\ e^{z_{t+1,\infty}} &= \alpha + \beta x_t^{[2]} + \theta_2 \text{Slope}_t + \varepsilon_{t+1}, \\ e^{z_{t+1,\infty}} &= \alpha + \beta x_t^{[2]} + \theta_3 \text{Curvature}_t + \varepsilon_{t+1}, \text{ and} \\ \text{RV}_{t\to t+1} &= \alpha + \beta (x_t^{[3]} - \{x_t^{[2]}\}^2) + \theta_4 \text{Option-Implied Variance}_t^{\perp} + \varepsilon_{t+1}, \end{aligned}$$

where $\text{RV}_{t\to t+1}$ is realized variance, calculated as the sum of the squared demeaned daily futures returns, and the number of days in the sum match the remaining days to expiration of the options contract. The predictors $x_t^{[n+1]} \equiv \mathbb{E}_t^{\mathbb{Q}}(e^{(n+1)z_{t+1,\infty}})$ are synthesized using option prices (as described in equation (10)) at the end of month t. The additional predictors are constructed as follows:

- Level_t: The first principal component extracted from available Treasury yields;
- Slope_t: The second principal component extracted from available Treasury yields;
- Curvature_t: The third principal component extracted from available Treasury yields; and
- Option-Implied $\text{Variance}_t^{\perp}$: The component of option-implied variance that is orthogonal to $x_t^{[3]} - \{x_t^{[2]}\}^2$.

We report the coefficient estimates, as well as the two-sided p-values (in square brackets, denoted by NW[p]) based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994). Reported also are the two-sided p-values (denoted by H[p]) based on the Hodrick (1992) 1B covariance estimator under the null of no forecasting ability. The adjusted R^2 (in %) is denoted by $\overline{R}^2$, and DW is the Durbin-Watson statistic. The sample period is January 1985 to December 2015 (372 monthly observations).

Dependent variable		α	β	θ_1	θ_2	θ_3	$\overline{R}^2$ (%)	DW
Gross futures return		-5.904	6.900	0.001			4.5	1.51
	NW[p]	[0.02]	[0.01]	[0.35]				
	H[p]	⌊0.10⌋	⌊0.06⌋	⌊0.27⌋				
		-5.001	5.998		0.008		6.2	1.51
	NW[p]	[0.04]	[0.01]		[0.00]			
	H[p]	⌊0.17⌋	⌊0.10⌋		⌊0.00⌋			
		-6.124	7.120			-0.009	4.2	1.51
	NW[p]	[0.02]	[0.00]			[0.41]		
	H[p]	⌊0.09⌋	⌊0.05⌋			⌊0.46⌋		
		α	β	θ_4			$\overline{R}^2$ (%)	DW
Variance of futures return		0.001	0.644	0.806			47.6	1.48
	NW[p]	[0.00]	[0.00]	[0.00]				
	H[p]	⌊0.00⌋	⌊0.00⌋	⌊0.00⌋				