A Recovery That We Can Trust? Deducing and Testing the Restrictions of the Recovery Theorem

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Abstract

How reliable is the recovery theorem of Ross (2015)? We explore this question in the context of options on the 30-year Treasury bond futures, allowing us to deduce restrictions that link the risk-neutral and physical distributions. The backbone of these restrictions is that the martingale component of the stochastic discount factor is unity. Our approach and empirical results provide an agnostic view of the claims of the recovery theorem in the long-term bond futures market. Moreover, our theoretical formulation and implementation reveal the presence of a martingale component that exhibits substantial dispersion and is positively correlated with the transitory component.

KEY WORDS: Recovery theorem, state prices, martingale component, transitory component
1. Introduction

Is it possible to recover both a stochastic discount factor (SDF) and the physical probability distribution from option prices? The underlier for the option could be an equity index, an individual stock, or the futures of a 30-year Treasury bond, and the answer from Ross (2015, Theorem 1, pages 622 and 646) is that it is feasible, provided (i) there is a single-state variable that is driven by a finite-state, irreducible Markov chain, and (ii) the pricing kernel satisfies the transition independence property. His solution approach and formalization, based on the Perron-Frobenius theorem, appears to have inspired an across-the-board intellectual conversation.

How reliable and useful is the recovery theorem of Ross (2015) in applied work? This paper builds on Borovicka, Hansen, and Scheinkman (2015), and provides a framework to assess the reliability of the Ross recovery theorem using data on futures of the 30-year Treasury bond and its options. The motivation is that finance theory has derived much of its analytical power and appeal from a few simplifying assumptions (e.g., the Black-Scholes formula), and the defense of a theory eventually resides in its empirical validity.

Our rationale for featuring the long-term bond market and their futures contract, denoted by $f_t$, stems from the following observations. First, under certain conditions, one can express the risk-neutral density, denoted by $q[z_{t+1}]$, with $z_{t+1} \equiv \log \left( \frac{f_{t+1}}{f_t} \right)$, as (see Bakshi, Kapadia, and Madan (2003, equations (11), (12), and (42))):

$$q[z_{t+1}] = \frac{\mathbb{E}(\xi_{t+1}|z_{t+1}) p[z_{t+1}]}{\int_{-\infty}^{\infty} \mathbb{E}(\xi_{t+1}|z_{t+1}) p[z_{t+1}] d z_{t+1}},$$

where $p[z_{t+1}]$ is the physical density of $z_{t+1}$. Moreover, $\xi_{t+1}$ is the unnormalized change of measure SDF and $\mathbb{E}(\xi_{t+1}|z_{t+1})$ is the expectation of $\xi_{t+1}$ conditional on $z_{t+1}$. When we have claims written on the futures of the long-term bond for which we have data, all that is required is $\mathbb{E}(\xi_{t+1}|z_{t+1})$.

Second, we exploit a result in Borovicka, Hansen, and Scheinkman (2015), showing that what is recovered by Ross (2015) via the Perron-Frobenius theorem is the transitory component of the SDF, and this
treatment yields a martingale component that is identically unity. In the context of the long-term bond market, one can show that the SDF can be identified as

$$E(ξ_{t+1} | z_{t+1}) = e^{z_{t+1}} / R_{t+1,rf},$$

for risk-free return $R_{t+1,rf}$, and

$$\int_{-\infty}^{+\infty} E(ξ_{t+1} | z_{t+1}) \ p[z_{t+1}] \ dz_{t+1} = 1 / R_{t+1,rf}.$$ 

Accordingly, the risk-neutral density is

$$q[z_{t+1}] = e^{-z_{t+1}} \ p[z_{t+1}].$$

(2)

The novelty of the link in equation (2) is that it furnishes a set of restrictions between the moment generating function of the risk-neutral distribution and return quantities under the physical probability measure.

We attempt to reconcile the enthusiasm for the recovery theorem versus the potential skepticism from three perspectives. First, we consider a convex optimization problem that minimizes the expectation of a convex function of the martingale component subject to three constraints: the martingale condition is satisfied, the martingale component is nonnegative, and the SDF correctly prices the riskfree bond, the long-term bond, and a finite number of other risky claims in the bond market.

Our approach and empirical implementation indicate that the martingale component is notably volatile. When the test assets include the returns of the riskfree bond, the long-term bond, and a collection of out-of-the-money puts and calls on the 30-year Treasury bond futures, the optimal solution for the martingale component yields a minimum dispersion of around 48% (annualized). Moreover, the martingale component is positively correlated with the transitory component. These solution properties go against the treatment of Ross (2015), and are robust under a bootstrap procedure and to alternative choices for the convex function.

Next, we develop tests related to the adequacy of return and volatility forecasts. Third, we consider the merits of the theory, relying on the generalized method of moments estimation. Both sets of results undermine the implications of the recovery theorem. These findings are of interest, as the long-term bond (futures) is a uniquely important and appropriate object to empirically assess the recovery theorem.

Related literature. There is a collection of theoretical and empirical papers that directly address various aspects of the recovery theorem, including Audrino, Huitema, and Ludwig (2015), Carr and Yu
(2012), Dubynskiy and Goldstein (2013), Liu (2015), Martin and Ross (2013), Qin and Linetsky (2015), Schneider and Trojani (2015), Tran and Xia (2015), and Walden (2014). Our work can be distinguished from the above papers in key ways. First, our paper is devoted to developing testable restrictions and conducting hypothesis testing in the context of the long-term bond market. Second, departing from all of these papers, we present a method to quantitatively characterize the nature of the martingale component.

2. The theoretical framework and testable implications

We center our attention on developing the testable implications of the recovery theorem in a stochastic environment that is somewhat more general than a finite-state Markov chain.

The theoretical framework synthesizes three elements and is intended to reconcile parts of the analysis in Ross (2015) and Borovicka, Hansen, and Scheinkman (2015). Our analysis serves as the basis for the empirical investigation and for setting up a convex problem to extract the martingale component from the market data.

First, Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) show that an SDF, denoted by $m_{t+1}$, can be uniquely decomposed as:

$$m_{t+1} = m_{t+1}^P m_{t+1}^T,$$  \hspace{1cm} (3)

where $m_{t+1}^P$ ($m_{t+1}^T$) is the martingale (transitory) component of $m_{t+1}$. The martingale component satisfies $E_t(m_{t+1}^P) = 1$, and $E_t(.)$ indicates expectation under the physical probability measure. We recognize that $m_{t+1} = M_{t+1}/M_t$, where the random variable $M_{t+1}$ represents the pricing kernel at date $t + 1$. The gross return of a riskfree bond is denoted by $R_{t+1, rf} = \frac{1}{E_t(m_{t+1})}$.

Consider a Markovian environment driven by a vector of variables $z$. The Perron-Frobenius theorem,\footnote{The joint laws of the risk-neutral and physical density have been studied in the context of the equity market under assumptions about the SDFs prior to Ross (2015). For a partial list, see Aït-Sahalia and Lo (2000), Jackwerth (2000), Rosenberg and Engle (2002), Bakshi, Kapadia, and Madan (2003), Chabi-Yo, Garcia, and Renault (2008), Bollerslev and Todorov (2011), Kozhan, Neuberger, and Schneider (2013), and Christoffersen, Jacobs, and Heston (2013), and Chaudhuri and Schroder (2015).}
that is, the eigenfunction problem of Hansen and Scheinkman (2009, Proposition 2), is applied to solve:

\[ \phi[z_t] \text{ and } \rho \text{ are solutions to } E_t \left( m_{t+1} e^{\rho \phi[z_{t+1}]} \right) - 1 = 0. \] (4)

\( \phi[z_t] \) represents the unique eigenfunction, and \( e^\rho \) is the eigenvalue (\( \rho \) is typically negative). With \( \phi[z_t] \) and \( \rho \) determined, the transitory component is \( m^T_{t+1} = e^\rho \phi[z_{t+1}] \) and \( m^P_{t+1} = m_{t+1}/m^T_{t+1} \), as also noted in Borovicka, Hansen, and Scheinkman (2015, equations (6), (9), or (17)).

The nominal discount bond price at date \( t \), denoted by \( V_t[1_{t+k}] \), represents a claim to $1 at date \( t+k \):

\[ V_t[1_{t+k}] = E_t \left( \frac{M_{t+k}}{M_t} \times 1 \right). \] (5)

Accordingly, a \( k \)-period discount bond bought at time \( t \) at price \( V_t[1_{t+k}] \), and sold at time \( t+1 \) at price \( V_{t+1}[1_{t+k}] \), has gross return \( R_{t+1,k} \), given by:

\[ R_{t+1,k} \equiv \frac{V_{t+1}[1_{t+k}]}{V_t[1_{t+k}]} = \frac{E_{t+1} \left( \frac{M_{t+k}}{M_{t+1}} \times 1 \right)}{E_t \left( \frac{M_{t+k}}{M_t} \times 1 \right)}. \] (6)

The return of a long-term bond corresponds to a large \( k \): \( \lim_{k \to \infty} R_{t+1,k} \equiv R_{t+1,\infty} \).

Importantly, the arguments developed in Alvarez and Jermann (2005) reveal that the transitory component of \( m_{t+1} \) can be identified under certain conditions as:

\[ m^T_{t+1} = \frac{1}{R_{t+1,\infty}}, \text{ where } R_{t+1,\infty} \text{ is the gross return of the discount bond with infinite maturity.} \] (7)

Second, Ross (2015) uses the Perron-Frobenius theorem to show that one can identify the transition probabilities from market data in conjunction with option prices. In this regard, Borovicka, Hansen, and Scheinkman (2015, e.g., Section 1.2 and Section 3) show that, for the recovery theorem to hold, the mar-
tingale component of the SDF must be equal to unity (see also Qin and Linetsky (2015, page 4)):

\[ m_{t+1} = 1, \quad \text{for all } t. \quad (8) \]

Specifically, given the assumptions in Ross (2015), one obtains

\[ m_{t+1} = e^{\rho \phi[z]} = m_{t+1}^T, \]

which also satisfies the problem in equation (4). The essential point is that the working of the Perron-Frobenius theorem appears to pin down one particular risk-neutral pricing density, and the analytical tractability of the recovery theorem can be traced to the degenerate case of \( m_{t+1}^P = 1 \).

We ask: What does \( m_{t+1}^P = 1 \) imply for the structure of risk-neutral distributions? When \( m_{t+1}^P = 1 \), are the features of the recovered physical return distribution consistent with those observed in the market?

Third, we recognize that the bond with infinite maturity is not traded, and we surrogate \( R_{t+1,\infty} \) with the returns of the futures on the 30-year Treasury bond using spot-futures arbitrage. Importantly, we develop testable implications, in terms of the relevant observable quantities, and we rely, in particular, on both the futures on the 30-year Treasury bond, and options on the 30-year Treasury bond futures.

To impart empirical content to the derived pricing equations, recall that \( f_t \) represents the time-\( t \) price of a one-period futures contract on \( V_{t+1}[1_{t+k}] \), where \( k \) can be large. Then, from Cox, Ingersoll, and Ross (1981, equation (46)) and for a marked-to-market futures contract,

\[ f_t = \mathbb{E}^Q(V_{t+1}[1_{t+k}]), \quad \text{where } \mathbb{E}^Q(\cdot) \text{ indicates expectation under the risk-neutral measure,} \]

\[ = R_{t+1,rf} V_t[1_{t+k}], \quad \text{for a large } k. \quad (9) \]

Thus, for a marked-to-market futures contract, we obtain the following relationship:

\[ \frac{f_{t+1}}{f_t} = \frac{1}{R_{t+1,rf}} \lim_{k \to \infty} \frac{V_{t+1}[1_{t+k}]}{V_t[1_{t+k}]} = \frac{R_{t+1,\infty}}{R_{t+1,rf}} \quad (11) \]
With the relation in equation (11) and \( m_{t+1}^P = 1 \), the SDF can be expressed as:

\[
m_{t+1} = m_{t+1}^P m_{t+1}^T, \tag{12}
\]

\[
m_{t+1}^T = \frac{1}{R_{t+1,\infty}}, \quad \text{(from equations (7) and (8))} \tag{13}
\]

\[
= \frac{1}{R_{t+1,rf}} \frac{f_t}{f_{t+1}}, \quad \text{(from equation (11))} \tag{14}
\]

\[
= \frac{1}{R_{t+1,rf}} e^{-z_{t+1}}, \quad \text{where defining } z_{t+1} \equiv \log \left( \frac{f_{t+1}}{f_t} \right) \in (-\infty, +\infty). \tag{15}
\]

Observe that the resulting \( m_{t+1} \) is free of any parameterization, and we further note that

\[
\mathbb{E}_t(m_{t+1}) = \frac{1}{R_{t+1,rf}}, \quad \text{which implies that } \mathbb{E}_t(e^{-z_{t+1}}) = 1 \text{ (from equation (14))}. \tag{16}
\]

With this said, consider claims written on the futures of the long-term discount bond, and recall that \( p[z_{t+1}] \) is the physical density of \( z_{t+1} \equiv \log \left( \frac{f_{t+1}}{f_t} \right) \). The associated risk-neutral pricing density is:

\[
q[z_{t+1}] = \frac{e^{-z_{t+1}} p[z_{t+1}]}{\int_{-\infty}^{+\infty} e^{-z_{t+1}} p[z_{t+1}] dz_{t+1}} = e^{-z_{t+1}} p[z_{t+1}], \quad \text{since } \mathbb{E}_t(e^{-z_{t+1}}) = 1, \tag{17}
\]

which can be interpreted as the Esscher transform of the physical density \( p[z_{t+1}] \) (e.g., Gerber and Shiu (1994, equation (2.5))). In the setting of Ross (2015), the risk-neutral density can be seen as exponentially tilted physical density, but the form of tilting is \( e^{-nz_{t+1}} \) with \( n = 1 \). Equation (17) holds in a general economic environment, including when \( z_{t+1} \) follows a finite-state Markov chain.

The form of \( q[z_{t+1}] \) presented in (17) corresponds to the risk-neutral density in Ross (2015, equations (6) and (25)). Guided by this implication, we consider a class of restrictions for an arbitrary parameter \( n \):

\[
\mathbb{E}_t^Q(e^{(n+1)z_{t+1}}) = \int_{-\infty}^{+\infty} e^{(n+1)z_{t+1}} q[z_{t+1}] dz_{t+1}, \tag{18}
\]

\[
= \int_{-\infty}^{+\infty} e^{(n+1)z_{t+1}} e^{-z_{t+1}} p[z_{t+1}] dz_{t+1}, \tag{19}
\]

\[
= \mathbb{E}_t(e^{nz_{t+1}}). \tag{20}
\]
The takeaway is that the Ross recovery theorem implies a restriction on the moment generating function of the physical and the risk-neutral distributions of $z_{t+1}$, provided $\left| E^Q_t \left( e^{(n+1)z_{t+1}} \right) \right| < +\infty$ for suitable choices of $n$. The restriction stems from the underlying theory that the recovered SDF coincides with the transitory component (Ross (2015, equations (6) and (25)) or Tran and Xia (2015, equation (2))). That is, in the context of the long-term bond market, the SDF is the inverse of the gross return of the long-term bond (e.g., Carr and Yu (2012, equation (30)) and Martin and Ross (2013, Result 5)).

The restriction (20) is testable, provided one can compute the risk-neutralized quantity $E^Q_t \left( e^{(n+1)z_{t+1}} \right)$. First, note from equation (20) that $E^Q_t \left( e^{(n+1)z_{t+1}} \right)_{n=0} = 1$. Next, the function $e^{(n+1)z_{t+1}} = \frac{1}{f_{t+1}} f^{n+1}_{t+1}$ is twice continuously differentiable in $f_{t+1}$ for $n \geq 1$. Thus, one can synthesize $E^Q_t \left( e^{(n+1)z_{t+1}} \right)$ from out-of-the-money option prices on the Treasury bond futures as follows (e.g., adapting the relations in Carr and Madan (2001) or Bakshi, Kapadia, and Madan (2003, equation (2)) to stochastic interest rates):

$$E^Q_t \left( e^{(n+1)z_{t+1}} \right) = 1 + \frac{n(n+1)R_{t+1,tf}}{f_t^2} \left( \int_{K>f_t} \left( \frac{K}{f_t} \right)^{n-1} C_t[K] dK + \int_{K<f_t} \left( \frac{K}{f_t} \right)^{n-1} P_t[K] dK \right), \quad (21)$$

where $C_t[K]$ ($P_t[K]$) is the price of the one-period European call (put) option on the futures of a 30-year Treasury bond with strike price $K$. The testable restriction when $m^P_{t+1} = 1$, is that, for $n \geq 1$,

$$E_t \left( \frac{e^{n z_{t+1}}}{x_t^{n+1}} - 1 \right) = 0, \quad \text{where} \quad x_t^{[n+1]} = E^Q_t \left( e^{(n+1)z_{t+1}} \right). \quad (22)$$

The restrictions embedded in (22) are the focus of the goodness-of-fit empirical tests in Sections 3.3 and 3.4.

The market for claims on the long-term bond provides a useful laboratory for testing the recovery theorem. First, the SDF is a specific function of the futures return $z_{t+1}$ when $m^P_{t+1} = 1$. Second, the option payoff is itself contingent on the uncertainty about $z_{t+1}$. Thus, the resulting economic environment is driven by a single-state variable $z_{t+1}$.

Closing, the pricing restrictions of the type presented in equation (22) can be reconciled within the finite-state Markov chain setting of Ross (2015). To elaborate on this feature of our analysis, we set $n = 1$,
and obtain $E_t(e^{zt_{n+1}})|_{n=1} = E_t(\frac{f_{t+1}}{f_t}) = \frac{1}{R_{t+1,rf}} E_t(R_{t+1,\infty}) = E_t^Q(\{\frac{f_{t+1}}{f_t}\}^2)$. We can further deduce that

$$\frac{1}{R_{t+1,rf}} E_t(R_{t+1,\infty}) = E_t^Q \left( \left\{ \frac{f_{t+1}}{f_t} \right\}^2 \right) = \frac{1}{R_{t+1,rf}} E_t^Q \left( R_{t+1,\infty}^2 \right) = \frac{1}{R_{t+1,rf}} \left( \text{Var}_t^Q(R_{t+1,\infty}) + R_{t+1,rf}^2 \right).$$

(23)

Canceling $R_{t+1,rf}$ and rearranging, the restriction on the conditional mean of the long-term bond return is:

$$E_t(R_{t+1,\infty}) - R_{t+1,rf} = \frac{1}{R_{t+1,rf}} \text{Var}_t^Q(R_{t+1,\infty}).$$

(24)

Thus, the result for $n = 1$ in equation (25) matches the corresponding one in Martin and Ross (2013, Result 7) and an equivalence can also be established for $n > 1$. The advantage of our theoretical formulations is that they map to traded quantities and, thus, conform to data realities connected with using risk-neutralized quantities to infer physical measure quantities.

The linchpin of the Ross recovery theorem is $m_{t+1}^P = 1$, and our work is aimed at empirically understanding the nuances underlying $m_{t+1}^P = 1$.

3. Empirical results and interpretation

The characterization in equation (22) shows that one could evaluate whether certain moments of the recovered physical distribution are aligned with the historical record. In the discussion that follows, we describe the data, motivate a convex optimization problem to study the absence of the martingale component, and then consider empirical exercises that investigate the reliability of the recovery theorem.

3.1. Data on the 30-year Treasury bond futures and options

Our empirical investigation features a testing framework that exploits data of options written on the futures of the 30-year Treasury bond. We focus on the 30-year Treasury bond futures, as they manifest
contingent claims with a long tenor. As an aside, there are no exchange traded options on Treasury bonds.

The master file from the Chicago Mercantile Exchange has daily data of options on the 30-year Treasury bond futures, and includes options across all expiration cycles. This data includes (i) the strike price, (ii) the remaining maturity, (iii) the option price, (iv) the identifier for a call or put option, and (v) the futures price. The data is available from October 1982 to December 2013, with 1,092,134 daily option records.

We apply the following steps to process the daily master file. First, we retain out-of-the-money options with the nearest maturity. We define out-of-the-money calls as having moneyness \( \log(f_t/K) < 0 \) and out-of-money puts as having moneyness \( \log(K/f_t) < 0 \). As highlighted in the context of equation (21), the construction of \( E_1^{(n)} \left( e^{(n+1)z_{n+1}} \right) \), for \( n \geq 1 \), requires a snapshot of out-of-the-money option prices with fixed maturity. Next, we omit the data prior to January 1985 to maintain a total of at least eight out-of-the-money options every month, which can enable a more accurate valuation of the payoff curvature via options.

Finally, we build a set of option prices and five time series of option returns, all at the end of the month:

- **Nearest maturity options at the monthly frequency**: These options usually expire on the last Friday, at least two business days from the last business day of the next month. The options so constructed have an average maturity of 27 days. The number of out-of-the-money calls (puts) vary from four (four) to 42 (49), with a total of 9,209 option observations.

- **Returns of a straddle**: The gross return of a straddle on the 30-year Treasury bond futures at the end of each month is constructed as:

\[
R_{t+1, \text{straddle}} = \frac{\max(f_{t+1} - K, 0) + \max(K - f_{t+1}, 0)}{C[K] + P[K]},
\]

where \( C[K] \) and \( P[K] \) are, respectively, the prices of calls and puts, with moneyness closest to zero.

- **Returns of a 3% and 1% out-of-the-money put**: At the end of each month, we search for a put option that is closest to 3% and 1% out-of-the-money, respectively. For example, the gross return of a 3% out-of-the-money put is constructed as:

\[
R_{t+1, 3\% \, \text{otm \ put}} = \frac{\max(K - f_t, 0)}{P[K]}, \quad \text{where } K \approx e^{-0.03 \times f_t}.
\]

- **Returns of a 1% and 3% out-of-the-money call**: At the end of each month, we search for a call option that is closest to 1% and 3% out-of-the-money, respectively. The gross return of a 1% out-of-the-money call is constructed as:

\[
R_{t+1, 1\% \, \text{otm \ call}} = \frac{\max(f_{t+1} - K, 0)}{C[K]}, \quad \text{where } K \text{ solves } f_t/K \approx e^{-0.01}.
\]
The returns of the out-of-the-money puts and calls indexed by strikes and straddle returns are employed as test assets in our procedure to gauge variations in the martingale component of the SDF, which is a constant under the treatment of Ross (2015), as established in Borovicka, Hansen, and Scheinkman (2015).

3.2. The extracted martingale components do not favor the $m_{t+1}^P = 1$ treatment of Ross (2015)

Integral to the recovery theorem of Ross (2015) is the notion of $m_{t+1}^P = 1$. But, how pronounced is the martingale component $m_{t+1}^P$? Even when the time-invariance of the martingale component can be refuted, what can be said about the covariation between the martingale component and the transitory component? In this subsection, our interest lies in isolating the martingale component in the long-term bond market, and this exercise could shed light on the reliability of the recovery theorem.

3.2.1. The framework for the convex optimization problem

In what follows, we write $\mathbb{E}(\cdot)$ to express unconditional expectation. Define

$$
R_{t+1} \equiv (R_{t+1,\infty}, R_{t+1,j})' \quad \text{and} \quad Z_{t+1} \equiv \frac{R_{t+1} - R_{t+1,rf}I}{R_{t+1,\infty}},
$$

(26)

where $I$ is a conformable vector of ones. Moreover, $R_{t+1,j}$, for $j = 1, \ldots, J$, is a $J \times 1$ vector of gross returns of risky assets that excludes the long-term bond return, and $Z_{t+1}$ is a vector of excess returns (over the riskfree return) divided by the gross return of the long-term bond. We assume $\mathbb{E}(|Z|^2) < +\infty$.

Consider the set $\mathcal{M} \equiv \{m_{t+1}^P \geq 0 : \mathbb{E}(m_{t+1}^P R_{t+1,\infty}) = 1 \text{ and } \mathbb{E}(m_{t+1}^P R_{t+1}) = 1\}$. To address the dispersion of the martingale component, we pose the following convex optimization problem:

$$
\inf_{m^P \in \mathcal{M}} \mathbb{E}(\psi[m^P]), \quad \text{for a convex function } \psi[m^P] \text{ satisfying } \mathbb{E}(\psi[m^P]) < +\infty,
$$

(27)

subject to

$$
\mathbb{E}(m^P Z) = 0, \mathbb{E}(m^P) = 1, \text{ and } m^P \geq 0.
$$

(28)
The equality constraints $\mathbb{E}(m^P Z) = 0$ are a statement about the absence of arbitrage and reflect correct pricing, while the equality constraint $\mathbb{E}(m^P) = 1$ is the martingale condition. Additionally, the condition $m^P \geq 0$ is aimed at enforcing the nonnegativity of the martingale component.

We repose the constraints on correct pricing differently from Borovicka, Hansen, and Scheinkman (2015, equation (29)), as it simplifies analytics, and because the inclusion of $R_{t+1,\infty}$ as the first element of $R_{t+1}$ automatically ensures the correct pricing of the riskfree bond. To see this point, $\mathbb{E}(m^P_{t+1}) = 1$ implies $R_{t+1,\infty}/R_{t+1,\infty} = 1$, by virtue of $\mathbb{E}(m^P_{t+1}) = 1$, and in light of $m_{t+1} = m_{t+1}^P m_{t+1}^T = m_{t+1}^P / R_{t+1,\infty}$.

Recognize that the minimization in equation (27) is over a possibly infinite-dimensional space, but solving the dual enables tractability. Let $\lambda$ be the $(J+1) \times 1$ vector of Lagrange multipliers associated with $\mathbb{E}(m^P Z) = 0$ and $\nu$ be the Lagrange multiplier associated with $\mathbb{E}(m^P) = 1$. We explore solutions with two different convex functions to establish robustness (the proofs are in the appendix).

**Case 1** Consider $\psi[m^P] = \frac{1}{2} (m^P)^2$. Since the expectation of $m^P_{t+1}$ is unity, minimizing $\mathbb{E}((m^P)^2)$ is equivalent to minimizing the variance, $\mathbb{E}((m^P - 1)^2)$. The optimal solution can be characterized as

$$m_{t+1}^P = \max \left( \nu^* + \lambda^* Z_{t+1}, 0 \right),$$  \hspace{1cm} (29)

where $(\lambda^*, \nu^*)$ solves

$$\inf_{(\lambda, \nu)} -\nu + \frac{1}{2} \mathbb{E} \left( [\nu + Z^T \lambda] \times 1_{\{\nu + Z^T \lambda \geq 0\}} \right)^2. \hspace{1cm} (30)$$

*In equation (30), $1_{\{a \geq 0\}}$ is an indicator function for the event $\{a \geq 0\}$. ♦*

**Case 2** Consider $\psi[m^P] = m^P \times \log(m^P)$ for $m^P > 0$. The optimal solution can be characterized as

$$m_{t+1}^P = \exp \left( -1 + \nu^* + \lambda^* Z_{t+1} \right),$$ \hspace{1cm} (31)

where $(\lambda^*, \nu^*)$ solves

$$\inf_{(\lambda, \nu)} -\nu + \mathbb{E} \left( \exp \left( -1 + \nu + Z^T \lambda \right) \right). \hspace{1cm} (32)$$
The exponential form of the solution ensures $m^P > 0$. Since $E(\psi[m^P]) - \psi[E(m^P)] = E(\frac{1}{2}(m^P - 1)^2 - \frac{1}{6}(m^P - 1)^3 + \frac{1}{12}(m^P - 1)^4 + O((m^P - 1)^5))$, the objective weights higher-order moments of $m^P$.

3.2.2. The $m^P_{t+1}$ component is not time-invariant and is positively correlated with $m^P_t$

Germane to implementing the solution, either via equation (29) or (31), is the question of which data is suitable to use from the long-term bond market. We opt in favor of parsimony and take $R_{t+1,j}$ to be a four-dimensional vector that contains the returns of 1% and 3% out-of-the-money puts and calls. Moreover, including a collection of option returns offers the possibility to span and mimic the state-space of futures returns each month. With the understanding that $R_{t+1,\infty} = R_{t+1,rf}\frac{f_{t+1}}{f_t}$, we consider the following $Z$:

$$Z_{t+1} = \frac{1}{R_{t+1,rf}\frac{f_{t+1}}{f_t}}\begin{pmatrix}
R_{t+1,rf}\frac{f_{t+1}}{f_t} - R_{t+1,rf} \\
R_{t+1,3\% \text{ otm put}} - R_{t+1,rf} \\
R_{t+1,1\% \text{ otm put}} - R_{t+1,rf} \\
R_{t+1,1\% \text{ otm call}} - R_{t+1,rf} \\
R_{t+1,3\% \text{ otm call}} - R_{t+1,rf}
\end{pmatrix}$$  \hspace{1cm} (33)

We draw on extant approaches and numerically solve the sample analog to equations (30) and (32) by searching over ($\lambda, \nu$), analogous to, for example, Hansen, Heaton, and Luttmer (1995, Sections 2 and 4) and Gospodinov, Kan, and Robotti (2015, Section 4). Panels A and B of Table 1 report the properties of the extracted $m^P_t$ series, according to equations (29) and (31), respectively.

In both cases, we consider a block bootstrap procedure to generate $Z$, and report the 5th, 25th, 50th, 75th, and 95th percentile values of the $m^P_t$ distribution, across the 25,000 bootstrap trials.

The reported results are informative from a number of perspectives. For one, the entries for the solution in Panels A and B are in agreement for a given $Z$, indicating that our choice of the convex functions does not appear to materially affect the properties of the extracted martingale component.
Our results further suggest that there is little justification for the time-invariance of the martingale component. When $\psi[m^P] = \frac{1}{2}(m^P)^2$, the optimal solution generates an annualized minimum $\sqrt{\text{Variance}}$ of 48.4%, with a 50th percentile bootstrap value of 67.5%. When $\psi[m^P] = m^P \log(m^P)$, the $\sqrt{\text{Variance}}$ of the extracted $m^P$ remains in the ballpark of 48%, with a 50th percentile bootstrap value of 68.9%.

Equally crucial, we establish that the martingale component is correlated with the transitory component. The correlation, denoted by $\rho_{PT}$, is 0.41, whereas the 50th percentile bootstrap value is 0.29, offering differentiation from the studies of Alvarez and Jermann (2005) and Bakshi and Chabi-Yo (2012). The hypothesis that $\rho_{PT} = 0$ is rejected in favor of $\rho_{PT} > 0$, with the highest one-sided bootstrap $p$-value of 0.069. Both pieces of evidence, namely, the pronounced volatility of $m^P_t$ and its positive correlation with the transitory component, are not supportive of the $m^P_{t+1} = 1$ treatment of Ross (2015).

The solution shares other attributes that are consistent with economic intuition. For example, inheriting the features of the $Z$ distribution, the $m^P_{t+1}$ distribution is positively skewed and fat-tailed, more so when the convex function to be minimized is $m^P \log(m^P)$. In other words, when the SDF correctly prices the riskfree bond, the long-term bond, and the 1% and 3% out-of-the-money puts and calls, the resulting distribution of the martingale component is volatile, right-skewed, and fat-tailed. It is further shown in Table Appendix-I that a similar conclusion emerges when $R_{t+1,j}$ incorporates either (i) the gross return of a straddle, or (ii) the gross return of a 3% out-of-the-money put, indicating that incorporating additional claims on the downside and the upside tends to increase the volatility of the extracted martingale components.

In summary, our results provide new insights that the extracted martingale component is positively correlated with the transitory component and displays substantial amount of volatility, features that are not present in the $m^P_{t+1} = 1$ framework of Martin and Ross (2013).

What are the consequences of our finding that $m^P_{t+1}$ and $m^T_{t+1}$ are correlated? It turns out that recovery is possible in the long-term bond market under the assumption that the martingale component is independent of the transitory component. To elaborate on this important assertion, we note that $E_t^Q(e^{(n+1)\zeta_{t+1}}) = R_{t+1,n} E_t(m^P_{t+1}m^T_{t+1}e^{(n+1)\zeta_{t+1}}) = E_t(e^{n\zeta_{t+1}})$, where the last equality follows from $m^T_{t+1} = \frac{1}{R_{t+1,n}} e^{-\zeta_{t+1}}$, $E_t(m^P_{t+1}) =$
1, and from the independence of \( m_{t+1}^P \) and \( m_{t+1}^T \). More specifically, the pricing restrictions in equation (20) still hold for each \( n \), when the martingale component is stochastic and independent of \( m_{t+1}^T \).

### 3.3. Adequacy of return and volatility forecasts

Is the treatment \( m_{t+1}^P = 1 \) of Ross (2015) innocuous, when recovering the return quantities under the physical probability measure from option prices?

To develop testable implications of the relation 
\[
\mathbb{E}_t (e^{n z_{t+1}}) = \mathbb{E}_t^Q (e^{(n+1) z_{t+1}})
\]

in equation (22), we define
\[
\begin{align*}
   y_{t+1}^{[n]} & \equiv e^{n z_{t+1}} \\
   \varepsilon_{t+1}^{[n]} & \equiv y_{t+1}^{[n]} - \mathbb{E}_t (y_{t+1}^{[n]}). 
\end{align*}
\]

Then, we can transform the theoretical restriction in (22) into an empirical restriction as
\[
\begin{align*}
   y_{t+1}^{[n]} = \alpha^{[n]} + \beta^{[n]} x_{t}^{[n+1]} + \varepsilon_{t+1}^{[n]}, \quad \text{for } n \geq 1.
\end{align*}
\]

Equation (35) exposes an implication of the recovery theorem in that \( x_{t}^{[n+1]} \), as inferred from all out-of-the-money option prices at the end of month \( t \), helps to forecast \( y_{t+1}^{[n]} \), resembling an approach pursued in different contexts by others, and is in the flavor of Fama (1984, Section 2.1). The two-sided \( p \)-values for the OLS coefficients are constructed based on the Newey and West (1987) standard errors, with lag length chosen automatically according to Newey and West (1994). We also consider the two-sided \( p \)-values based on the Hodrick (1992) 1B covariance estimator under the null of no forecasting ability.

The null hypothesis for a fixed \( n \) in the OLS regression is

\[
\begin{align*}
   \alpha^{[n]} = 0 \quad \text{and} \quad \beta^{[n]} = 1, \quad \text{for } n \geq 1.
\end{align*}
\]

One can construct the time series of \( x_{t}^{[n+1]} = \mathbb{E}_t^Q (e^{(n+1) z_{t+1}}) \) using option data at the end of month \( t \) for \( n \geq 1 \), so these restrictions can be tested using data on the returns of the 30-year Treasury bond futures and
option prices on the 30-year Treasury bond futures.

How much empirical support is there for the link between the observed return distributions and that implied by the recovery theorem (i.e., as postulated by the relation in equation (35))? A related question is which recovered physical return moment should one match?

To address these questions, we focus on \( n = 1, 2 \). For example, \( n = 1 \) addresses the recovery of the mean futures (gross) return, as outlined in equation (25), whereas \( n = 2 \) implies the recovery of \( \mathbb{E}_t(\{ \frac{f_{t+1}}{f_t} \}^2) \) from \( \mathbb{E}_Q^Q(\{ \frac{f_{t+1}}{f_t} \}^3) \). We can, thus, extract the recovery theorem implied variance of the futures return as

\[
\text{Var}_t \left( \frac{f_{t+1}}{f_t} - 1 \right) = \mathbb{E}_t \left( \left\{ \frac{f_{t+1}}{f_t} \right\}^2 \right) - \left\{ \mathbb{E}_t \left( \frac{f_{t+1}}{f_t} \right) \right\}^2,
\]

(37)

\[
= x_t^{[3]} - \{x_t^{[2]}\}^2.
\]

(38)

All these links stem specifically when \( m_t^P = 1 \), which yields the form of the risk-neutral return density in equation (17). Following Andersen, Bollerslev, Diebold, and Labys (2003), we estimate \( \text{Var}_t \left( \frac{f_{t+1}}{f_t} - 1 \right) \) as the subsequent sum of squared demeaned daily returns, with the number of days in the sum matched to the remaining days to expiration of the options at the end of month \( t \).

In a nutshell, we employ option prices to compute \( \mathbb{E}_t(\{ \frac{f_{t+1}}{f_t} \}^n) \), as per the recovery theorem, and this recovered quantity could be compared to the counterparts in the futures market.

Table 2 (Panels A and B) presents the point estimates of \( \alpha \) and \( \beta \) when \( x_t^{[2]} \) and \( x_t^{[3]} \) are constructed using nearest maturity options. The takeaway is that the restrictions imposed by the recovery theorem are not supported in the data. In other words, when \( m_{t+1}^P = 1 \), the physical return moments determined from the Arrow-Debreu state prices do not line up with the actual counterparts.

In Panel A of Table 2, the point estimates of \( \beta \) is 7.631 and the \( \hat{\beta} \) estimate is statistically significant. The estimate of \( \alpha \) is \(-6.635\), and the null hypothesis that \( \alpha = 0 \) is rejected. The correlation between \( y_{t+1}^{[1]} \) and \( x_t^{[2]} \) is 0.21. Panel B of Table 2 assesses the forecast of variance using options data, and we obtain \( \alpha \) and \( \beta \) estimates of 0.001 and 0.604, respectively. Both \( \alpha \) and \( \beta \) estimates are statistically significant. The
correlation between the realized variance and the options-inferred \( x_t^3 - \{ x_t^2 \}^2 \) is 0.57. The January 1990 to December 2013 subsample (288 observations) yield similar inferences about \( \alpha \) and \( \beta \). Given that the big picture remains the same, these results are not reported.

Affirming our findings from a different angle, the Wald test statistics, which are \( \chi^2 \)-distributed with 2 degrees of freedom, reject the null hypothesis that \( \alpha = 0 \) and \( \beta = 1 \). The \( p \)-values for the Wald statistics, shown in parentheses, are not higher than 0.03.

While the adjusted \( R^2 \) for the mean return is 4.4\% and the adjusted \( R^2 \) rises to 32.1\% for return variance, our emphasis is on the restrictions \( \alpha = 0 \) and \( \beta = 1 \). Still, how does one benchmark the adjusted \( R^2 \)? As an illustration, we consider option implied variance as an ad-hoc forecasting variable for subsequent return variance. The option implied variance, computed using an at-the-money option on the 30-year Treasury bond futures and the model of Black (1976), achieves an adjusted \( R^2 \) of 44\% and a slope coefficient of 0.93 (H[\( p \)] of 0.00). Such a comparison provides a finding that option implied variance encodes information that helps to forecast return variance with comparable explanatory power.

To further examine the reliability of the recovery theorem, we compute the deviations:

\[
\begin{align*}
\varepsilon_{t \rightarrow t+1}^{\text{mean}} & \equiv \mathcal{X}_{t+1}^{[1]} - \mathcal{X}_{t}^{[2]}, \\
\varepsilon_{t \rightarrow t+1}^{\text{volatility}} & \equiv \left( \sqrt{\text{Var}_{t} \left( f_{t+1} \right) - 1} - \sqrt{\mathcal{X}_{t}^{[3]} - \{ \mathcal{X}_{t}^{[2]} \}^2} \right) / \sqrt{\text{Var}_{t} \left( f_{t+1} \right) - 1},
\end{align*}
\]

which reflects the difference between the realized value and the theoretical counterparts recovered from option prices. We tabulate the distribution of the deviations (the mean, the standard deviation, and some percentiles) below:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std.</th>
<th>5th</th>
<th>25th</th>
<th>50th</th>
<th>75th</th>
<th>95th</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon_{t \rightarrow t+1}^{\text{mean}} )</td>
<td>0.0013</td>
<td>0.0303</td>
<td>-0.0473</td>
<td>-0.0166</td>
<td>0.0027</td>
<td>0.0185</td>
<td>0.0471</td>
</tr>
<tr>
<td>( \varepsilon_{t \rightarrow t+1}^{\text{volatility}} )</td>
<td>0.1750</td>
<td>0.2453</td>
<td>-0.2938</td>
<td>0.0513</td>
<td>0.2262</td>
<td>0.3388</td>
<td>0.4938</td>
</tr>
</tbody>
</table>

While the average \( \varepsilon_{t \rightarrow t+1}^{\text{mean}} \) is small, the (5th) 95th percentile value is not small on a monthly basis. Moreover,
the average $e_{t \rightarrow t+1}^{\text{volatility}}$ is 17.5%, where the scaling by the realized return volatility imparts a percentage deviation interpretation to $e_{t \rightarrow t+1}^{\text{volatility}}$.

Along another yardstick, the magnitudes of $e_{t \rightarrow t+1}^{\text{mean}}$ are large relative to the monthly mean return of 0.0022 (annualized 2.64%). Our analysis also suggests that the less than adequate performance of the recovery theorem is not confined to one direction, with both underprediction and overprediction.

Are deviations from the recovery theorem related to the number of available option strikes? Probing this possibility, we examine the absolute deviations binned across the quintiles of the number of strikes:

<table>
<thead>
<tr>
<th>Number of puts and calls</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>e_{t \rightarrow t+1}^{\text{mean}}</td>
<td>$</td>
<td>0.0179</td>
<td>0.0241</td>
<td>0.0264</td>
</tr>
<tr>
<td>$</td>
<td>e_{t \rightarrow t+1}^{\text{volatility}}</td>
<td>$</td>
<td>0.2982</td>
<td>0.2800</td>
<td>0.2893</td>
</tr>
</tbody>
</table>

As seen, the absolute deviations are not declining across the quintiles of option strikes. Specifically, when we have an average of 50 calls and puts to compute a forecast of mean return, the absolute deviation is larger compared to when there are an average of 11 calls and puts.

The main takeaway is that our results present a doubtful picture about the reliability and the practical usefulness of the recovery theorem in the context of the long-term bond market. It appears that the $n_{t+1}^P = 1$ treatment is not innocuous and can potentially misalign the mapping between the physical density and the risk-neutral density (as posited in equation (17)).

### 3.4. Consistency between unconditional moments implied by the recovery theorem and data

The purpose of this subsection is to apply the generalized method of moments (GMM) estimation of Hansen (1982) to examine whether the physical return moments implied by the recovery theorem are compatible with their actual counterparts. Our goal is to provide another perspective on a theory that recovers the physical distribution from the risk-neutral distribution in conjunction with the market determined option prices.
Consider the disturbance terms in light of an encompassing specification of equation (22):

\[ u_{t+1}^{[1]} \equiv \eta_1 e^{(1-\delta_1)z_{t+1}} x_t^{[2]} - 1 \quad \text{and} \quad u_{t+1}^{[2]} \equiv \eta_2 e^{(2-\delta_2)z_{t+1}} x_t^{[3]} - 1, \]  \hspace{1cm} (41)

for some parameters \((\eta_1, \delta_1)\) and \((\eta_2, \delta_2)\). Reposing the restrictions of subsection 3.3 and using unconditional expectations, we consider the following implications for a set of instruments \(I_t\):

\[ \mathbb{E} \left( u_{t+1}^{[1]} \otimes I_t \right) = 0 \quad \text{and} \quad \mathbb{E} \left( u_{t+1}^{[2]} \otimes I_t \right) = 0. \]  \hspace{1cm} (42)

The following parametric restrictions hold when the recovery theorem correctly inverts the physical density from the knowledge of the risk-neutral density:

\[ \eta_j = 1.0, \quad \text{and} \quad \delta_j = 0.0, \quad \text{for} \quad j = 1, 2. \]  \hspace{1cm} (43)

In the GMM estimations, the choice of instruments is often critical, and we take \(I_t\) to include a constant and the first and second lags of the futures return. With this choice, there are three moment conditions and two parameters to be estimated.

The minimized value of the GMM criterion multiplied by \(T\) (the number of time-series observations), denoted by \(J_T\), is \(\chi^2\)-distributed under the null of correctly recovering both the SDF and the physical probabilities, with degrees of freedom \((df)\) equal to the number of orthogonality conditions minus the number of estimated parameters.

Table 3 reports the GMM estimation results. Let us start with the GMM estimation associated with the mean of the physical distribution, which corresponds to \(x_t^{[2]}\), that is, when we span and value the payoff \(e^{2z_{t+1}} = (f_{t+1}/f_t)^2\). We emphasize that \(\delta_1\) is a free parameter with a hypothesized value of zero.

Contradicting the above implication, the point estimate of \(\delta_1\) is 0.993. Additionally, the point estimate departs many standard errors away from zero with a \(p\)-value of 0.00, and the hypothesis that \(\delta_1\) is zero is
rejected. At the same time, the point estimate of \( \eta_1 \) is close to one and is statistically significant.

The results with \( x_i^{[3]} \) provide additional confirmatory evidence that the recovery theorem is unable to capture the behavior of the second moment of futures return. The reported estimate of \( \delta_2 \) is 1.981 and is statistically distinct from zero. Both estimations indicate a rejection of an implication of the recovery theorem.

The lowest p-value for the J-statistic is 0.97. The inability to reject the extended model may not be surprising, as the recovery theorem imposes a value of \( \delta_j = 0 \), whereas we have kept \( \delta_j \) to be a freely determined parameter when minimizing the GMM criterion function. Specifically, the Ross recovery theorem assigns a value of \( n = 1 \) to the SDF specification \( \frac{1}{R_{t+1,n}} e^{-n\zeta_{t+1}} \) (see equation (15)). In contrast, our estimates suggest that the coefficient \( n \) on \( e^{-n\zeta_{t+1}} \) is a magnified version, assigning a weight that tends to exaggerate the tails of the SDF.

The overall interpretation is that the treatment in Ross (2015) can compromise the structure of risk and pricing in the long-term bond market, whereby the state prices are not aligned with \( \mathbb{E}(\{f_{t+1}/f_t\}^n) \) in a manner that is dictated by the recovery theorem. Moreover, the message from the GMM estimations agrees with the findings from the preceding subsections, calling into question the empirical viability of the recovery theorem.

4. Concluding remarks

The recovery theorem of Ross (2015) provides an elegant method to extract the physical probabilities and the SDF simultaneously from option prices, and Ross’s work has ignited a stream of research and controversy. In particular, the theoretical study of Borovicka, Hansen, and Scheinkman (2015) shows that the notion of a constant martingale component is indispensable to the recovery theorem of Ross (2015).

The concept of a constant martingale component seems abstract and confounding to many trying to understand what all of this means for risk-neutral and physical return distributions. Motivated by this...
ambivalence, this paper formalized the theoretical restrictions between the risk-neutral and the physical distributions when the SDF is driven entirely by the transitory (the non-martingale) component. Featuring claims in the long-term bond market, we expand on this idea to develop the empirical implications of the recovery theorem.

To reconcile and interpret our empirical findings, we solve a convex minimization problem to extract the martingale component from the data. Our investigation shows that the extracted martingale component exhibits considerable dispersion and tail asymmetries. The additional crucial finding is that the extracted martingale component is positively correlated with the transitory component.

Exploring the reliability of the recovery theorem, our empirical approaches and results provide an agnostic view of the practical usefulness of the recovery theorem in the long-term bond market. In one specific exercise, we study the accuracy of return and volatility forecasts when reality generates the empirical data points. Still, the question on how to satisfactorily recover the physical distributions from option prices (if feasible) remains unresolved, and this is where more theoretical and empirical work should be done. Whether the glass is half full or half empty is left to the judgment of the reader.
References


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Appendix: Proof of the solutions in Case 1 and Case 2 of Section 3.2

Proof of Case 1. To streamline equation presentation in what follows next, we define

\[ w_t \equiv m_t^p \quad \text{for all} \quad t, \quad \text{and} \quad \psi[w_t] \equiv \frac{1}{2} w_t^2, \quad (A1) \]

and write \( \mathbb{E}(\cdot) \) to indicate unconditional expectation. We assume \( \mathbb{E}[w^2] < +\infty \).

Let \( \lambda \in \mathbb{R}^{J+1} \) be a \( J \)-dimensional vector of Lagrange multipliers for the equality constraints \( \mathbb{E}(wZ) = 0 \) and \( \nu \in \mathbb{R} \) be the Lagrange multiplier for the equality constraint \( \mathbb{E}(w) = 1 \). The associated Lagrangian is

\[ \mathcal{L}[w, \lambda, \nu] = \mathbb{E}(\psi[w]) + \lambda^T(0 - \mathbb{E}(wZ)) + \nu(1 - \mathbb{E}(w)). \quad (A2) \]

The Lagrange dual problem is

\[ \sup_{(\lambda, \nu) \in \mathbb{R}^{J+2}} \left( \inf_w \mathcal{L}[w, \lambda, \nu] \right). \quad (A3) \]

If we denote \( w^* \) as the optimal solution,

\[ \inf_w \mathcal{L}[w, \lambda, \nu] = - \left( -\mathbb{E}(\psi[w^*]) + \lambda^T \mathbb{E}(w^*Z) + \nu \mathbb{E}(w^*) \right) + \lambda^T 0 + \nu. \quad (A4) \]

Henceforth, observe that

\[ -\mathbb{E}(\psi[w^*]) + \lambda^T \mathbb{E}(w^*Z) + \nu \mathbb{E}(w^*) = \mathbb{E}(\underbrace{(\lambda^T Z_t + \nu)w_t^*}_{\psi^*[\lambda^T Z_t + \nu]} - \psi[w_t^*]). \quad (A5) \]

The Fenchel conjugate, denoted by \( \psi^*[h] \), of a function \( \psi[\omega] \), is defined as (Borwein and Zhu (2005, Section 4.1.1 and also Theorem 4.4.3))

\[ \psi^*[h] \equiv \sup_{\omega \in [0, \infty] \cap \text{domain } \psi} h \omega - \psi[\omega]. \quad (A6) \]
Accordingly, we can write equation (A4) as

\[
\inf_w L[w, \lambda, v] = \lambda^0 + v - \mathbb{E}(\psi^*[Z'\lambda + v]).
\]  

(A7)

The dual problem is

\[
\sup_{(\lambda, v)} \theta^'\lambda + v - \mathbb{E}(\psi^*[Z'\lambda + v]).
\]  

(A8)

The task is to derive the form of \(\psi^*[h]\), while noting that \(\psi[\omega] = \frac{1}{2} \omega^2\) for Case 1.

To find the value of \(\omega\) that maximizes (A6), we equate to zero its derivative with respect to \(\omega\):

\[
\frac{d(h \omega - \psi[\omega])}{d\omega} = h - \omega = 0.
\]  

(A9)

If \(h \geq 0\), \(\omega = h\) is the unique solution. Hence, \(\omega = h \times 1_{\{h \geq 0\}}\),

(A10)

where \(1_{\{h \geq 0\}}\) is an indicator function for the event \(\{h \geq 0\}\).

The asserted solution holds since domain \((h \omega - \psi[\omega]) = [0, \infty)\). Otherwise, if \(h \leq 0\), then \(h - \omega \leq 0\). We recognize that since \(h \omega - \psi[\omega]\) is a decreasing function of \(\omega\), the sup is obtained for \(\omega = 0\).

Replacing \(\omega\) in the original equation (A6), we obtain the conjugate \(\psi^*[h]\) as

\[
\psi^*[h] = \frac{1}{2} (h \times 1_{\{h \geq 0\}})^2.
\]  

(A11)

We can now obtain \(w^*\) from the conjugate evaluated at the optimal \((\lambda^*, v^*)\), that is,

\[
w^* = \frac{d\psi^*[h]}{dh}\bigg|_{h = v^* + \lambda^*Z}
\]  

(A12)

\[
= (v^* + \lambda^*Z) \times 1_{\{v^* + \lambda^*Z \geq 0\}} = \max(v^* + \lambda^*Z, 0).
\]  

(A13)
Completing the description, the vector of Lagrange multipliers \((\lambda, \nu)\) is a solution to
\[
\sup_{(\lambda, \nu)} \nu - \mathbb{E}(\psi^*[Z'\lambda + \nu]). \tag{A14}
\]

Equivalently, we have
\[
\sup_{(\lambda, \nu)} \nu - \frac{1}{2} \mathbb{E}(\mathbb{E}((Z'\lambda + \nu) \times \mathbb{1}_{\{Z\lambda + \nu \geq 0\}})^2). \tag{A15}
\]

The characterizations in equations (A13) and (A15) constitute the solution to the martingale component of the SDF subject to the nonnegativity constraint. We have proved the expressions in Case 1.

**Proof of Case 2.** The steps of the proof are similar to how we did it before, except for the form of the conjugate function. The considered convex function is now
\[
\psi[w] = w \log(w). \quad \text{Assume } \mathbb{E}[w \log(w)] < +\infty. \tag{A16}
\]

The problem is \(\inf_w \mathbb{E}(\psi[w])\) subject to the constraints in (28). Let \(\lambda \in \mathbb{R}^{J+1}\) be a \(J\)-dimensional vector of Lagrange multipliers for the equality constraints \(\mathbb{E}(wZ) = 0\) and \(\nu \in \mathbb{R}\) be the Lagrange multiplier for the equality constraint \(\mathbb{E}(w) = 1\). The Fenchel conjugate, denoted by \(\psi^*[h]\), of \(\psi[\omega]\), is
\[
\psi^*[h] \equiv \sup_{\omega \in (0, \infty) \cap \text{domain } \psi} h \omega - \psi[\omega]. \tag{A17}
\]

Equating to zero the derivative of the preceding expression with respect to \(\omega\), we have
\[
\frac{d\psi^*[h]}{d\omega} = h - (1 + \log(\omega)) = 0. \tag{A18}
\]

The solution to equation (A18) is \(\omega = \exp(h - 1)\). \tag{A19}
Substituting \( \omega \) back into equation (A17), we obtain the form of the conjugate as

\[
\psi^* [h] = h \exp(h - 1) - \exp(h - 1) \log(\exp(h - 1)), \quad (A20)
\]

\[
= \exp(h - 1). \quad (A21)
\]

The martingale component that minimizes the objective function can be obtained from the conjugate as follows:

\[
w^* = \frac{d\psi^*[h]}{dh} \bigg|_{h = \nu^* + \lambda' Z} = \exp\left(-1 + \nu^* + \lambda' Z\right). \quad (A22)
\]

The Lagrange multipliers, therefore, are a solution to

\[
\sup_{(\lambda, \nu)} \nu - E(\psi^*[Z' \lambda + \nu]). \quad (A23)
\]

Equivalently,

\[
\sup_{(\lambda, \nu)} \nu - E\left(\exp\left(-1 + \nu + \lambda' Z\right)\right). \quad (A24)
\]

The martingale condition is satisfied, since the derivative of equation (A24) respect to \( \nu \) is

\[
1 - E\left(\exp\left(-1 + \nu + \lambda' Z\right)\right) = 0. \quad (A25)
\]

We have verified that the solution forming the system in equations (31) and (32) holds.
Table 1
Properties of the martingale component of the stochastic discount factor

We numerically solve (using proc fminsearch in Matlab) for the Lagrange multipliers \((\lambda^*, \nu^*)\) in equations (30) and (32). Then we extract the martingale component according to
\[
m_{t+1}^* = \max \left( \nu^* + \lambda^* Z_{t+1}^{\prime}, 0 \right) \quad \text{or} \quad m_{t+1}^* = \exp \left( -1 + \nu^* + \lambda^* Z_{t+1}^{\prime} \right).
\]
Reported are the annualized standard deviation, monthly skewness, and monthly kurtosis, of the extracted martingale components. \(\rho_{PT}\) is the correlation between the martingale component and the transitory component (i.e., \(1/R_{t+1,\infty}\)). We use the following \(Z_{t+1}\) in our calculations:
\[
Z_{t+1} = \frac{1}{R_{t+1,\infty}} \left( \begin{array}{c}
R_{t+1,\text{rf}} - R_{t+1,\text{rf}} \\
R_{t+1,1\% \text{ otm put}} - R_{t+1,\text{rf}} \\
R_{t+1,1\% \text{ otm call}} - R_{t+1,\text{rf}} \\
R_{t+1,3\% \text{ otm put}} - R_{t+1,\text{rf}} \\
R_{t+1,3\% \text{ otm call}} - R_{t+1,\text{rf}}
\end{array} \right)
\]

We adopt a block bootstrap procedure (block size of 20) to generate 25,000 bootstrap samples and report the mean, standard deviation, and percentiles of the respective \(m_{t+1}^*\) statistics. Reported also are the \(p\)-values, in curly brackets, for the hypothesis \(\rho_{PT} = 0\), which represents the proportion of replications for which the correlation \(\rho_{PT} < 0\). The sample period is January 1985 to December 2013, for a total of 348 observations.

<table>
<thead>
<tr>
<th>Panel A: The convex function (\psi[m^P]) is (\frac{1}{2} (m^P)^2)</th>
<th>Panel B: The convex function (\psi[m^P]) is (m^P \times \log(m^P))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Martingale component</td>
<td>Martingale component</td>
</tr>
<tr>
<td>(\sqrt{\text{Variance}})</td>
<td>(\sqrt{\text{Variance}})</td>
</tr>
<tr>
<td>Skewness</td>
<td>Skewness</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>Kurtosis</td>
</tr>
<tr>
<td>(\rho_{PT})</td>
<td>(\rho_{PT})</td>
</tr>
<tr>
<td>Solution</td>
<td>0.484</td>
</tr>
<tr>
<td></td>
<td>1.88</td>
</tr>
<tr>
<td></td>
<td>11.47</td>
</tr>
<tr>
<td></td>
<td>0.41</td>
</tr>
<tr>
<td>Block bootstrap</td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.744</td>
</tr>
<tr>
<td>Std.</td>
<td>0.335</td>
</tr>
<tr>
<td>5th</td>
<td>0.351</td>
</tr>
<tr>
<td>25th</td>
<td>0.521</td>
</tr>
<tr>
<td>50th</td>
<td>0.675</td>
</tr>
<tr>
<td>75th</td>
<td>0.884</td>
</tr>
<tr>
<td>95th</td>
<td>1.377</td>
</tr>
<tr>
<td>(p)-val., (\rho_{PT} = 0)</td>
<td>{0.064}</td>
</tr>
</tbody>
</table>
Table 2
Adequacy of return and volatility forecasts relying on the recovery theorem and the options on the 30-year Treasury bond futures

Reported are the results from the OLS regressions:

\[ y_{t+1}^{[n]} = \alpha^{[n]} + \beta^{[n]} x_t^{[n+1]} + \varepsilon_{t+1}^{[n]}, \quad \text{for } n \geq 1, \]

where \( x_t^{[n+1]} \equiv E_t^{\mathbb{Q}} \left( e^{(n+1)z_{t+1}} \right) \) and is synthesized using option prices, as described in equation (21), at the end of month \( t \). The variance of the futures return consistent with the recovery theorem is:

\[ \text{Var} \left( \frac{f_{t+1} - f_t}{f_t} - 1 \right) = x_t^{[3]} - \left( x_t^{[2]} \right)^2. \]

The variance (the dependent variable in Panel B) is calculated as the sum of squared demeaned daily futures returns, and the number of days in the sum match the remaining days to expiration of the options contract. We report the coefficient estimates, as well as the two-sided \( p \)-values (in square brackets, denoted by NW\([p]\)) based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994). Reported also are the two-sided \( p \)-values (denoted by H\([p]\)) based on the Hodrick (1992) 1B covariance estimator under the null of no forecasting ability. The adjusted \( R^2 \) (in \%) is denoted by \( \overline{R^2} \), and DW is the Durbin-Watson statistic. We perform the Wald test for the hypothesis \( \alpha = 0 \) and \( \beta = 1 \) and report the \( \chi^2(2) \) statistics, with \( p \)-values in parenthesis. The sample period is January 1985 to December 2013. With nearest-maturity options, there are 348 monthly observations.

<table>
<thead>
<tr>
<th>Dependent variable</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \overline{R^2} )</th>
<th>DW</th>
<th>Wald test</th>
<th>CORR(( y_{t+1}^{[n]}, x_t^{[n+1]} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross futures return:</td>
<td>-6.635</td>
<td>7.631</td>
<td>4.4</td>
<td>1.47</td>
<td>7.21</td>
<td>0.21</td>
</tr>
<tr>
<td>NW([p]]</td>
<td>[0.010]</td>
<td>[0.000]</td>
<td>[0.03]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H([p]]</td>
<td>[0.067]</td>
<td>[0.035]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Panel B: Using options data to recover the physical return variance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Variance of futures return: | 0.001 | 0.604 | 32.9 | 1.28 | 115.63 | 0.57 |
| NW\([p]\] | [0.000] | [0.000] | [0.00] | | | |
| H\([p]\] | [0.000] | [0.000] | | | | |
Table 3
Testing the link between the return quantities implied by the recovery theorem and the data: GMM estimation results

The estimation is for a single equation using Hansen’s (1982) GMM:

\[ E \left( u_{t+1}^{[1]} \otimes I_t \right) = 0 \quad \text{and} \quad E \left( u_{t+1}^{[2]} \otimes I_t \right) = 0, \]

where the disturbance terms are defined as

\[ u_{t+1}^{[1]} \equiv \eta_1 e^{(1-\delta_1)z_{t+1}} x_t^{[2]} - 1 \quad \text{and} \quad u_{t+1}^{[2]} \equiv \eta_2 e^{(2-\delta_2)z_{t+1}} x_t^{[3]} - 1, \]

for some parameters \((\eta_1, \delta_1)\) and \((\eta_2, \delta_2)\). Our transformation of equation (22) is aimed at testing whether the data supports \(\delta_1 = 0\) and \(\delta_2 = 0\). We perform the GMM estimation and report the two-sided \(p\)-values for the coefficients in square brackets. Reported also is the \(J_T\) statistic of Hansen (1982), with the \(p\)-value in parenthesis. The instruments are a constant and one and two lags of the futures return. There are 348 monthly observations, and the sample period is January 1985 to December 2013.

<table>
<thead>
<tr>
<th></th>
<th>(\eta_1)</th>
<th>(\delta_1)</th>
<th>(J_T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMM estimates</td>
<td>(E \left( u_{t+1}^{[1]} \otimes I_t \right) = 0)</td>
<td>1.001</td>
<td>0.993</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[0.00]</td>
<td>[0.00]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(\eta_2)</th>
<th>(\delta_2)</th>
<th>(J_T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMM estimates</td>
<td>(E \left( u_{t+1}^{[2]} \otimes I_t \right) = 0)</td>
<td>1.003</td>
<td>1.981</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[0.00]</td>
<td>[0.00]</td>
</tr>
</tbody>
</table>
**Table Appendix-I**

**Properties of the martingale component of the SDF using alternative sets of \(Z_{t+1}\)**

We numerically solve for the Lagrange multipliers \( (\lambda^*, \nu^*)\) in equations (30) and (32). Then we extract \( m^p_{t+1} = \max \left( \nu^* + \lambda^* Z_{t+1}, 0 \right) \) or \( m^p_{t+1} = \exp \left( -1 + \nu^* + \lambda^* Z_{t+1} \right) \). Reported are the *annualized* standard deviation, monthly skewness, and monthly kurtosis, of the extracted martingale components. \( \rho_{P,T} \) is the correlation between the martingale component and the transitory component (i.e., \( 1/R_{t+1,\infty} \)). We use two different \( Z_{t+1} \) in our calculations:

\[
Z_{t+1} = \frac{1}{R_{t+1,\text{str}} \frac{R_{t+1}}{R_{t+1,\text{str}}}} \left( R_{t+1,\text{str}} - R_{t+1,\text{str}} \right) \quad \text{or} \quad Z_{t+1} = \frac{1}{R_{t+1,\text{str}} \frac{R_{t+1}}{R_{t+1,\text{str}}}} \left( R_{t+1,\text{str}} - R_{t+1,\text{str}} \right).
\]

We adopt a block bootstrap procedure (block size of 20) to generate 25,000 bootstrap samples and report the mean, standard deviation, and percentiles of the respective \( m^p_{t+1} \) statistics. Reported also are the \( p \)-values, in curly brackets, for the hypothesis \( \rho_{P,T} = 0 \), which represents the proportion of replications for which the correlation \( \rho_{P,T} < 0 \). The sample period is January 1985 to December 2013, for a total of 348 observations.

<table>
<thead>
<tr>
<th>( Z ) contains the long-term bond and the straddle</th>
<th>( Z ) contains the long-term bond and the 3% out-of-the-money put option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Martingale component</td>
<td>Martingale component</td>
</tr>
<tr>
<td>( \sqrt{\text{Variance}} )</td>
<td>( \sqrt{\text{Variance}} )</td>
</tr>
<tr>
<td>( \text{Skewness} )</td>
<td>( \text{Skewness} )</td>
</tr>
<tr>
<td>( \text{Kurtosis} )</td>
<td>( \text{Kurtosis} )</td>
</tr>
<tr>
<td>( \rho_{P,T} )</td>
<td>( \rho_{P,T} )</td>
</tr>
</tbody>
</table>

### Panel A: The convex function \( \psi[m^p] \) is \( \frac{1}{2}(m^p)^2 \)

<table>
<thead>
<tr>
<th>Solution</th>
<th>Block bootstrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Std.</td>
</tr>
<tr>
<td>0.266</td>
<td>0.153</td>
</tr>
<tr>
<td>0.344</td>
<td>0.113</td>
</tr>
<tr>
<td>0.345</td>
<td>0.105</td>
</tr>
<tr>
<td>0.235</td>
<td>0.63</td>
</tr>
<tr>
<td>0.331</td>
<td>1.59</td>
</tr>
<tr>
<td>0.438</td>
<td>1.85</td>
</tr>
<tr>
<td>0.618</td>
<td>2.13</td>
</tr>
</tbody>
</table>

\( p \)-val., \( \rho_{P,T} = 0 \)

\{0.073\} \quad \{0.070\}

### Panel B: The convex function \( \psi[m^p] \) is \( m^p \times \log(m^p) \)

<table>
<thead>
<tr>
<th>Solution</th>
<th>Block bootstrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Std.</td>
</tr>
<tr>
<td>0.267</td>
<td>0.157</td>
</tr>
<tr>
<td>0.346</td>
<td>0.113</td>
</tr>
<tr>
<td>0.234</td>
<td>0.91</td>
</tr>
<tr>
<td>0.331</td>
<td>1.96</td>
</tr>
<tr>
<td>0.441</td>
<td>2.35</td>
</tr>
<tr>
<td>0.628</td>
<td>2.90</td>
</tr>
</tbody>
</table>

\( p \)-val., \( \rho_{P,T} = 0 \)

\{0.074\} \quad \{0.073\}