An Inquiry into the Nature and Sources of Variation in the Expected Excess Return of a Long-Term Bond

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Abstract

This paper proposes an approach to study the expected excess return of a long-term bond and focuses on a lower bound. This lower bound is a crucial number, as it represents the minimum expected excess return demanded by investors. The derived bound is model-independent and can be extracted from options on the 30-year Treasury bond futures. Our implementation reveals that the annualized lower bound ranges from 0.22\% to 6.07\%, with an unconditional average of 1.18\%. The ideas and developed results are useful for thinking about cost of debt, allocation between equities and bonds, and measuring investor reaction to monetary policy shocks.

KEY WORDS: Long-term Treasury bond, expected excess return, lower bound, options on bond futures.

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1. Introduction

A fundamental question in financial economics is how to estimate the expected excess return of an asset. The challenging nature of this problem is eloquently stated by Black (1993):

*The key issue in investments is estimating expected return. It is neither explaining return nor, as Fama and French suggest, explaining average return. ... Estimating expected return is hard.*

Merton (1980) more generally recognized early on the importance of estimating expected return.

The focus of this paper is on the expected excess return of the long-term Treasury bond, a topic that sits at the intersection of investments, asset pricing, and corporate finance. The estimate of the expected excess return of the long-term Treasury bond is central to numerous calculations involving net present values, capital budgeting, and investment planning. Shocks to the expected excess return could impact the borrowing cost of the Treasury, which could percolate to the timing and size of Treasury issuances and to the public ownership of Treasury bonds. Additionally, our study is connected to the ongoing discussion over the supply and demand of sovereign long-term safe assets.

Our approach to studying the nature and sources of variation in the expected excess return of a long-term bond is novel, since we focus on the lower bound of the expected excess return. The knowledge of the lower bound is conceptually important, as it quantifies the minimum expected compensation for investing in the long-term bond. Our methodological innovation is to impute the lower bound by relying on prices of call and put options written on the 30-year Treasury bond futures (which are one of the most liquid options markets). An essential feature of the framework is that the lower bound is model-free and circumvents the use of past return realizations to estimate expected excess returns.

In particular, we address two key questions. First, how can one adapt the asset pricing theory to derive a model-free lower bound on the expected excess return of the long-term bond? Second, could one empirically discern the attributes of the variation in the minimum expected compensation demanded by
investors for investing in the long-term bond? For example, is the variation in the lower bound linked to changes in the slope of the Treasury yield curve, or, alternatively, to proxies for marginal utility of wealth, or worldwide demand for hedging assets?

Our core idea is linked to a theory that postulates that the stochastic discount factor can be uniquely decomposed into its transitory and permanent components and that the return of a long-term discount bond is the inverse of the transitory component (Alvarez and Jermann (2005) and Hansen and Scheinkman (2009)). We specifically show that the lower bound on the long-term bond is equal to its discounted return variance under the risk-neutral measure. The intuition is that suppliers of capital dislike shocks that make the risk-neutral return distributions more volatile, thereby raising their minimum expected excess return.

The paper develops the view that the minimum expected excess return can be extracted from a positioning in options on the 30-year Treasury bond futures. Each period, the estimates are forward looking, incorporate the risk and pricing of interest rate movements, and respond to time-varying market conditions (e.g., monetary and fiscal policy). Moreover, the offered theory is cast in nominal terms, which renders it useful in economic applications. Our characterizations do not rely on the functional form of the stochastic discount factor or the modeling of economic fundamentals.

Many economists have proposed conceptually innovative mechanisms to generate the expected excess return of the long-term bond (see the survey papers by Piazzesi (2010) and Gurkaynak and Wright (2011)). In contrast, we offer the distinction of exploiting the information content of traded option prices for the expected excess return of the bond. Implementation shows that the lower bound has an unconditional average of 1.18% over the sample period of 1982:10 to 2013:12. The minimum required compensation is importantly never negative, ranging between 0.22% to 6.07%.

What are the channels that underlie the variations in the minimum expected excess return of a long-term bond? Our approach is to link the patterns of expected excess returns to three economically motivated variables. First, we show that a one-standard-deviation increase in the slope of the Treasury yield curve over the previous month increases the lower bound by an average of 0.07% (all units are annualized), and
the effect is statistically significant. Next, we construct the return of a security with the payoff equal to the stochastic discount factor (SDF). This channel operates through the effect that the return of the SDF security is high during bad times, and we find that a one-standard-deviation movement in this return raises the lower bound by 0.18%. We also explore an asset that is often considered to be a hedge against economic tail risks. Specifically, we take the returns of a fully collateralized futures position in gold, and find that it exerts a positive and statistically significant effect on the expected excess returns of a bond over certain periods. A one-standard-deviation increase in the return of gold lifts the lower bound by 0.10%. Moreover, the effects remain robust when we control for a set of variables that are thought to impact the expected excess return of a bond – specifically, aggregate output growth, inflation, and the supply of Treasury bonds.

The line of inquiry pursued here traverses other strands of research. First, our theoretical approach to the determination of expected excess return is related to the status of Treasury bonds as a global safe asset (e.g., Gorton and Ordonez (2013) and Gourinchas and Jeanne (2013)), so the time-varying nature of the lower bound could help to benchmark the expected return of other assets belonging to this class, and risky assets in general. Second, the focus on option-inferred expected excess returns departs from studies that exploit a parametric model to understand model-implied expected excess returns of a bond (e.g., Campbell, Sunderam, and Viceira (2013, Table 2 and Figure 10)). Despite the fact that much has been written on the Treasury bond market, we are far from finding a consensus on what could be point estimates of the conditional and unconditional expected excess return of the long-term bond.

2. Expected excess return of a long-term discount bond

The key new theoretical result is the derivation of the lower bound on the expected excess return of the long-term discount bond. We additionally show how to operationalize the lower bound in a model-free manner, using options and futures data. Our study has implications for the modeling of both the term structure of interest rates and the stochastic discount factor, and, particularly, the transitory component of the stochastic discount factor, which determines risk and pricing in the long-term bond market.
2.1. **Return of the long-term bond and the transitory component of the stochastic discount factor**

Consider an economy outlined in Hansen and Scheinkman (2009), in which the absence of arbitrage guarantees the existence of a pricing kernel at time $t$, denoted $M_t$. Let $z_t$ represent the set of state variables.

The work of Hansen and Scheinkman (2009, page 179) shows that $M_t$ can be expressed as:

$$M_t = \exp(\rho t) \frac{\phi[z_0]}{\phi[z_t]}.$$  

Hence, over two dates,

$$\frac{M_{t+1}}{M_t} = \exp(\rho) \frac{\phi[z_t]}{\phi[z_{t+1}]},$$

(1)

where $\rho$ is the deterministic growth rate component (the eigenvalue), $\hat{M}_t$ is a martingale, and $\log(\hat{M}_t)$ has stationary increments. Moreover, $\phi[z_t]$ is a positive function and $\frac{\phi[z_t]}{\phi[z_{t+1}]}$ represents the transitory contribution to the stochastic discount factor, that is, $m_{t+1} \equiv \frac{M_{t+1}}{M_t} > 0$. The analysis maintains that $m_{t+1}$ is stationary.

Hansen and Scheinkman (2009, Proposition 2, page 201) formalize how an appropriately solved eigenfunction problem could ensure the uniqueness of the decomposition in equation (1), specifically,

$$\phi[z_t] \text{ and } \rho \text{ are solutions to } E_t(M_{t+1} \phi[z_{t+1}]) = e^\rho M_t \phi[z_t],$$

(2)

where $E_t(.)$ indicates conditional expectation under the physical probability measure. The eigenfunction problem may yield distinct eigenvalues $\rho$ leading to different martingales (different permanent components). In addition, positive martingales with unit expectations can be used to induce alternative change of measures. However, Hansen (2012, Sections 6.2–6.3), and Hansen (2013, Sections 2.2-2.3) show that there is at most one such decomposition for which the martingale induces stochastically stable dynamics. In such case, the permanent component of the SDF is $\frac{M_{t+1}^P}{M_t^P} = \frac{\hat{M}_{t+1}}{\hat{M}_t}$, together with:

$$\frac{M_{t+1}^P}{M_t^P} = \frac{M_{t+1}}{M_t} / \frac{M_{t+1}^T}{M_t^T} > 0, \quad \text{where the transitory component is } \frac{M_{t+1}^T}{M_t^T} = e^\rho \frac{\phi[z_t]}{\phi[z_{t+1}]} > 0.$$  

(3)

$\phi[z_t]$ represents the unique eigenfunction that guarantees stable dynamics. The permanent and the transitory components of the SDF are correlated.
The discount bond price \( B_{t,k} \) is a claim to $1 at date \( t+k \). The gross return of the bond is \( R_{t+1,k} = B_{t+1,k}/B_{t,k} \). The absence of arbitrage implies the pricing relations (e.g., Cochrane (2005, Chapter 1)):

\[
E_t(m_{t+1}) = \frac{1}{R_{t+1,f}} \quad \text{and} \quad E_t(m_{t+1}B_{t+1,\infty}) = B_{t,\infty},
\]

(4)

where \( B_{t,\infty} \) is the price of an infinite-maturity discount bond. The gross return of a risk-free bond is denoted by \( R_{t+1,f} \) (which is known at date \( t \)), and the gross return of an infinite-maturity discount bond by \( R_{t+1,\infty} \equiv \lim_{k \to \infty} B_{t+1,k}/B_{t,k} \). The stationarity of \( m_{t+1} \) implies that \( R_{t+1,\infty} \) is stationary.

To link the return of the infinite-maturity discount bond and the transitory component of the stochastic discount factor, we invoke a result due to Alvarez and Jermann (2005, proof of Proposition 2, page 2007):

\[
\left( \frac{MT_{t+1}}{M^T_t} \right)^{-1} = R_{t+1,\infty}.
\]

(5)

When equation (5) is satisfied, the decomposition in equation (1) is unique (e.g., Alvarez and Jermann (2005, Proposition 1)).

2.2. The lower bound on the expected excess return of the long-term bond

Using the relation \( E_t(uv) - \text{Cov}_t(u,v) = E_t(u)E_t(v) \), we obtain \( \frac{1}{R_{t+1,f}} - \text{Cov}_t \left( \frac{M_{t+1}^p}{M^T_{t+1}}, \frac{M_{t+1}^T}{M^T_t} \right) = E_t \left( \frac{1}{R_{t+1,\infty}} \right) \).

Then by the convexity of \( 1/R_{t+1,\infty} \) and Jensen’s inequality, it holds that \( E_t \left( 1/R_{t+1,\infty} \right) \leq 1/E_t \left( R_{t+1,\infty} \right) \).

Writing \( m_{t+1}^{p} = \frac{M_{t+1}^p}{M^T_{t+1}} \) and \( m_{t+1}^{T} = \frac{M_{t+1}^T}{M^T_t} \) for brevity, we deduce that

\[
\frac{1}{R_{t+1,f}} - \text{Cov}_t \left( m_{t+1}^{p}, m_{t+1}^{T} \right) \geq \frac{1}{E_t \left( R_{t+1,\infty} \right)}. \]

(6)

Equation (6) allows us to establish that \( \frac{E_t(R_{t+1,\infty}) - R_{t+1,f}}{E_t(R_{t+1,\infty})} \geq \text{Cov}_t \left( m_{t+1}^{p}, m_{t+1}^{T} \right) \). Hence,

\[
E_t \left( R_{t+1,\infty} \right) - R_{t+1,f} \geq 0, \quad \text{provided that} \quad \text{Cov}_t \left( m_{t+1}^{p}, m_{t+1}^{T} \right) \geq 0. \]

(7)
The sufficient condition for the expected excess return of the long-term bond to be positive is that the conditional covariance between the permanent and transitory components of $m_{t+1}$ be positive.

The economic question of interest is how to characterize the behavior of the expected excess return of the long-term bond in a model-free manner, in particular, the lower bound.

To proceed with our developments, let $E_t^*(H[B_{t+1\infty}])$ indicate the expectation of a generic payoff $H[B_{t+1\infty}]$ under the risk-neutral (pricing) measure, defined as (e.g., Singleton (2006, pages 202-203)):

$$E_t^*(H[B_{t+1\infty}]) = E_t \left( \frac{m_{t+1}}{E_t(m_{t+1})} H[B_{t+1\infty}] \right),$$

where the normalization $E_t(m_{t+1})$ on the right-hand side of equation (8) ensures that the risk-neutral density integrates to unity.

We suppose that the transitory component of the stochastic discount factor satisfies the eigenfunction problem outlined in equation (2), and $E_t^*((R_{t+1\infty} - R_{t+1,f})^2) < \infty$. We now show the following result on the expected excess return of a long-term discount bond.

**Proposition 1** The expected excess return of the long-term discount bond satisfies the lower bound:

$$E_t(R_{t+1\infty}) - R_{t+1,f} \geq \frac{1}{R_{t+1,f}} E_t^*\left( (R_{t+1\infty} - R_{t+1,f})^2 \right),$$

where $E_t^*\left( (R_{t+1\infty} - R_{t+1,f})^2 \right)$ is the risk-neutral return variance of the long-term discount bond.

**Proof.** See Appendix A. □

A lower bound on the expected excess return is an important number, as it conveys the minimum expected compensation demanded by investors for committing their capital to the long-term bond at date $t$, as opposed to the upper bound, which conveys the maximum compensation. Accordingly, the lower bound in equation (9) must be distinguished from the unconditional upper bound in Cochrane (2005, equation 1.17, page 17) and must not be confused with the restriction $|E(R_{t+1\infty}) - R_{t+1,f}| \leq \frac{1}{E(m_{t+1})} \sqrt{\text{Var}(m_{t+1}) \text{Var}(R_{t+1\infty})}$. 
which holds when $m_{t+1}$ prices the long-term bond. Such an upper bound requires knowledge of the unconditional variance of $m_{t+1}$ (or its Hansen and Jagannathan (1991, equation (12)) lower bound).

Proposition 1 and Appendix A develop the view that the expected excess return is bounded by the discounted conditional return variance of the long-term bond under the risk-neutral measure. The rationale follows from the relation $E_t (R_t R_{t+1}; ¥) = E_t (m_{t+1} R_{t}^2; ¥) - \text{Cov}_t (m_{t+1}, m_{t+1})$, which guides the intuition that return movements in the tails exert a positive influence on $E_t (R_t R_{t+1}; ¥)$. Ceteris paribus, the more negative is the variance risk premium implicit in $E_t (m_{t+1} R_{t}^2; ¥)$, the higher the minimum required return. Moreover, the derived lower bound in equation (9) relies on a property of $m_t$ that $\text{Cov}_t (m_{t+1}, m_{t+1}) \geq 0$.

The result in Proposition 1 is general and does not hinge on assumptions about the dynamics of the permanent and transitory components of $m_{t+1}$, when $m_{t+1}$ can be decomposed into its permanent and transitory components. The result also does not depend on the shape and parametric form of $m_t$. Hence, the derived bound is free of the $m_t$ specification, and it is also consistent with a positive expected excess return of the long-term bond.

The lower bound is meant to be a data-inferred quantity, so the question is how to determine the discounted risk-neutral return variance $\frac{1}{R_{t+1,f}} E_t^s ((R_{t+1,\infty} - R_{t+1,f})^2)$? The answer lies in recognizing that options on the futures of a long-term Treasury bond are traded. Our approach is twofold. First, we map the quantity $\frac{1}{R_{t+1,f}} E_t^s ((R_{t+1,\infty} - R_{t+1,f})^2)$ to the expectation of the squared futures return under the risk-neutral measure. Second, we extract the arbitrage-free value of the squared futures return using the methods of Bakshi and Madan (2000, Appendix A.3) and Carr and Madan (2001, equation (1)).

To do so, we first exploit the internal link between the prices of futures and the long-term bond. Let $f_t$ represent the time-$t$ price of the one-period futures contract on $B_{t+1,\infty}$ and, hence, at time $t + 1$, $f_{t+1} = B_{t+1,\infty}$. Then $f_t = E_t^s (B_{t+1,\infty}) = R_{t+1,f} B_{t,\infty}$, akin to a cost-of-carry relationship (e.g., Cox, Ingersoll, and Ross (1981, equation (46))). Therefore, we may equivalently write $R_{t+1,\infty} - R_{t+1,f}$ as:

$$R_{t+1,\infty} - R_{t+1,f} = \frac{B_{t+1,\infty}}{B_{t,\infty}} - R_{t+1,f} = R_{t+1,f} \left( \frac{f_{t+1}}{f_t} - 1 \right). \quad (10)$$
We can now formalize the lower bound, which is at the center of the empirical investigation.

**Proposition 2** The lower bound on the expected excess return of the long-term discount bond can be expressed in terms of the risk-neutral variance of the futures return:

\[
E_t (R_{t+1,\infty}) - R_{t+1,f} \geq R_{t+1,f} E_t^* \left( \frac{f_{t+1}}{f_t} - 1 \right)^2, \tag{11}
\]

The lower bound can be computed in a model-free manner at each date \( t \) as:

\[
R_{t+1,f} E_t^* \left( \frac{f_{t+1}}{f_t} - 1 \right)^2 = 2R_{t+1,f}^2 \left( \int_{\{K>f_t\}} G_t[K] dK + \int_{\{K<f_t\}} P_t[K] dK \right), \tag{12}
\]

where \( G_t[K] (P_t[K]) \) is the time \( t \) price of the out-of-the-money call (put) written on the futures of the long-term bond with the strike price \( K \).

**Proof.** See Appendix B.

Our result in equation (12) reveals that the discounted return variance of the long-term bond under the risk-neutral measure (as in equation (9)) can be statically obtained through a positioning of \( \frac{2R_{t+1,f}^2}{f_t^2} \) in all out-of-the-money calls and puts on the long-term bond futures. Therefore, the lower bound in equation (12) is computable.

Note that the gross return of a fully collateralized long futures position is \( R_{t+1,\text{futures}} \equiv \frac{1}{f_t} (f_{t+1} + (R_{t+1,f} - 1)f_t) \) and, hence, \( R_{t+1,\text{futures}} - R_{t+1,f} = \frac{f_{t+1}}{f_t} - 1 \). Therefore, in economic terms, \( E_t^* \left( \frac{f_{t+1}}{f_t} - 1 \right)^2 \) corresponds to the risk-neutral variance of the futures return.

The options positioning is derived in a general setting of a stochastic discount factor, allows for stochastic interest rates, and does not invoke assumptions about the stochastic process for the futures price (e.g., diffusion uncertainty). Importantly, the changing nature of the expected excess return of the long-term bond is reflected in the changing nature of the risk-neutral variance of the futures return, scaled by \( R_{t+1,f} \).
One could argue that if one follows Black (1993), then the expected excess return of the long-term bond can also be obtained by relying on the Capital Asset Pricing Model, as in \( E_t (R_{t+1,.}) - R_{t+1,f} = \beta (E_t (R_{t+1,M}) - R_{t+1,f}) \). Even when the beta of the long-term bond can be estimated with some level of sophistication from the historical data (e.g., Ross, Westerfield, and Jaffe (2010, Chapter 13)), the difficulty is to infer the conditional expected excess return of the equity market (i.e., \( E_t (R_{t+1,M}) - R_{t+1,f} \)). Merton (1980) argues that estimating expected return can be challenging when it is believed that the expected return is shifting with economic conditions. Our approach provides an estimate of the minimum expected excess return of the long-term bond using an options portfolio (as in equation (12)) and, hence, circumvents estimation steps.

There is one additional way in which our theoretical results in (11)–(12) could prove useful. To elaborate, suppose \( m_{t+1} = 1/R_{t+1,.} \). This candidate for \( m_{t+1} \) satisfies the Euler equation for the long-term bond return; it also postulates that \( m_{t+1} \) is monotonically declining in \( R_{t+1,.} \). Under this parametric assumption, we have the relation \( E_t (R_{t+1,i}) - R_{t+1,f} = -R_{t+1,f} \text{Cov}_t \left( \frac{1}{R_{t+1,.}}, R_{t+1,i} \right) \) for \( i = 1, \ldots, N \). Therefore,

\[
E_t (R_{t+1,i}) - R_{t+1,f} = \xi_i \underbrace{E_t (R_{t+1,.}) - R_{t+1,f}}_{\text{Expected excess return}}
\]

where \( \xi_i \equiv \frac{\text{Cov}_t \left( \frac{1}{R_{t+1,.}}, R_{t+1,i} \right)}{\text{Cov}_t \left( \frac{1}{R_{t+1,.}}, R_{t+1,.} \right)} \). Equation (13) imparts empirical content by using estimated \( \xi_i \) with the lower bound in equation (12), which yields an estimate of the lower bound on the expected excess return for asset \( i \).

Like the conditional expected return of the equity market (e.g., Merton (1980)), the expected return of a long-term bond is an important concept, prompting many authors, including Jarrow (1978), Fama and French (1989), Ilmanen (1995), Campbell and Viceira (2001), and Campbell, Sunderam, and Viceira (2013) to study the problem from theoretical and empirical perspectives. For instance, Campbell, Sunderam, and Viceira (2013, equation (6), Figure 10) estimate a model where the yields are linear-quadratic functions of the state variables, and derive the conditional expected excess return of a discount bond. Proposition 2 offers an alternative in that the lower bound on the expected excess return can be inferred from a portfolio...
of options on the futures of the long-term bond (indexed by strikes and fixing the maturity). Furthermore, the Online Appendix considers a parameterized model and illustrates the empirical tractability of obtaining the lower bound on the expected excess return of the long-term bond from the options data.

Departing from our setting, Kazemi (1992, equation (5)) shows that $E_t (R_{t+1,\infty} - R_{t+1,f}) = \text{Var}_t (R_{t+1,\infty})$ (see Proposition 5, Kazemi (1992, equation (15))). This result is driven by the assumption that the state variables have a long-run stationary joint distribution. In contrast, our lower bound in Proposition 2 does not require distributional assumptions.

3. Description of the options data on the 30-year Treasury bond futures

There are two sources of variation in the expected excess returns of a long-term bond, according to the derived equation (23): $E_t (R_{t+1,\infty} - R_{t+1,f}) = \frac{1}{R_{t+1,f}} E_t^* \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right) - \text{Cov}_t \left( m_{t+1}^P, \frac{1}{m_{t+1}^P} \right)$. First, shocks to expected excess returns can be traced to shocks to $\frac{1}{R_{t+1,f}} E_t^* \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right)$. The generality of our analysis comes from two features: (i) we do not specify the form of the stochastic discount factor, and (ii) $E_t^* \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right)$ can be directly recovered from options written on the long-term bond futures. Second, shocks to the expected excess returns can be traced to shocks to $\text{Cov}_t \left( m_{t+1}^P, \frac{1}{m_{t+1}^P} \right)$. However, the conditional covariance term cannot be determined without specifying and estimating a model.

Our approach motivates a lower bound on $E_t (R_{t+1,\infty} - R_{t+1,f})$. Three inputs are required to compute the lower bound in equation (12) of Proposition 2. Specifically, the implementation requires the futures price, the prices of out-of-the-money calls and puts on the bond futures, and the risk-free return.

The 30-year Treasury bond is used to proxy for the long-term bond in the empirical analysis (as also analyzed by Alvarez and Jermann (2005, Table 1)), as it is the longest maturity available, and the corresponding futures and options on the futures are actively traded in the market.

We compute the lower bound at the end of each month over the sample period of 1982:10 to 2013:12 (375 observations), which coincides with the introduction of options on the futures of the 30-year Treas-
sury bond. This 31-year sample period encompasses many of the important events and developments in monetary and fiscal policy, and includes four recessions.

The construction of the options positioning underlying the square of the futures excess return, specifically, $E_t^*(\frac{f_{t+1}}{f_t} - 1)^2$, requires a careful handling of the options data on the bond futures, which are reported by the DataMine End-of-Day from the Chicago Mercantile Exchange. The options data is daily, and the contract months are the first three consecutive (two serial expirations and one quarterly expiration) months, plus the next two months in the March, June, September, and December quarterly cycle.

The futures data on the 30-year Treasury bond is also obtained from the Chicago Mercantile Exchange (CME). The data is daily and the futures contract months are the first three consecutive contracts in the March, June, September, and December quarterly cycle.\(^1\) Online Appendix describes how we convert the CME quotes on futures and options to the dollar prices.

Several criterion were applied to the daily options data to construct the end-of-month observations on calls and puts. First, we sample options at the end of each month and focus on options with maturity closest to 30 days. Let the corresponding closing price of the futures contract at the end-of-the-month be $f_t$. Second, we keep out-of-the-money calls, i.e., those with $f_t/K < 1$, and out-of-the-money puts, i.e., those with $K/f_t < 1$, where $K$ is the strike price. Using out-of-the-money options also mitigates the impact of the early exercise feature of American options (as shown by Mueller, Vedolin, and Yen (2013, page 15)). Third, we omit 31 option quotes prior to 1993:07, at which point the strike price is more than $4,000 away from the adjacent strike price. Moreover, we omit 175 option quotes after 1993:07, when the strike price is more than $2,000 away from the adjacent strike price. Each of the omitted options is extremely deep out-of-the-money, has a settlement price of $15.625 (the minimum tick size), and has zero trading volume.

The final options sample contains 9,471 observations with 4,331 out-of-the-money calls, 5,140 out-of-the-money puts, and matched prices of the underlying bond futures. On average, our sample contains

\(^1\)At the end of each month, we use the settlement price of the nearest maturity contract to compute the futures return, while taking into account the first notice day. For example, the March 2010 futures has a first notice day on February 26, 2010 (Friday). Accordingly, we use the June 2010 futures to compute the return at the end of March 2010.
11.55 calls and 13.71 puts in each month. The average maturity of options is 29.16 days. Table 1 provides a snapshot of the monthly average of the number of strikes and option open interest from 1982 to 2013.

Complementing our empirical analysis, we also construct the daily and monthly returns data on one-month Treasury bills and 30-year Treasury bonds. The source of the one-month returns is the data library of Ken French, whereas the 30-year Treasury bond data is taken from CRSP Fixed-Term Indexes.

4. Understanding the expected excess return of the long-term bond

The key step is to compute the lower bound in equation (12) of Proposition 2, hereby denoted as:

\[ \text{LB}_{t \rightarrow t+1} \equiv \frac{2 R_{t+1}^2}{f_{t+1}^2} \left( \int_{\{K > f_t\}} C_t[K] dK + \int_{\{K < f_t\}} P_t[K] dK \right). \] (14)

The subscript notation \( \{t \rightarrow t+1\} \) is meant to emphasize that the options positioning is computed at the end of month \( t \) and embodies expectation about excess returns over the subsequent month \( t+1 \). Hence, \( \text{LB}_{t \rightarrow t+1} \) is the forward-looking component of expected excess returns between the end of month \( t \) and the end of month \( t+1 \), which we extract from options on the 30-year Treasury bond futures.

The integral representation of the lower bound in equation (14) is tractable and can be reliably approximated using the trapezoidal rule (e.g., Bakshi, Kapadia, and Madan (2003), Jiang and Tian (2005), and Carr and Wu (2009)). We calculate the integral of the call as (e.g., Lindfield and Penny (1995, page 125)):

\[ \int_{\{K > f_t\}} C[K] dK \approx (C[K_{\min}] + 2 C[K_{\min} + \Delta K] + 2 C[K_{\min} + 2 \Delta K] + \ldots + C[K_{\max}]) \frac{\Delta K}{2}, \] (15)

where \( K_{\min} > f_t \). The analogous calculation for the integral of the put is:

\[ \int_{\{K < f_t\}} P[K] dK \approx (P[K_{\max}] + 2 P[K_{\max} - \Delta K] + 2 P[K_{\max} - 2 \Delta K] + \ldots + P[K_{\min}]) \frac{\Delta K}{2}, \] (16)

where \( K_{\max} < f_t \). The options positioning that statically spans the square of the excess futures return differs
from the one in Mueller, Vedolin, and Yen (2013, Proposition 2) and Mele and Obayashi (2013, equation (15)). This distinction emerges because these authors focus on the square of the log futures return.

4.1. The estimated lower bound on the expected excess return is 1.18% on average

Table 2 reports the average $\text{LB}_{(t \rightarrow t+1)}$ over the full 31-year sample and also across several subsamples. Our investment horizon for the expected excess return is monthly (with an average of 29.16 days), but the reported numbers are expressed in annualized percentage units.

The mean lower bound over the entire sample is 1.18% (118 basis points), and the Politis, White, and Patton (2009) 95% bootstrap confidence intervals are between 1.00% and 1.38%. Importantly, the lower bound reflects the investors’ minimum expected excess return when they invest in the long-term bond.

Moreover, the mean lower bound ranges between 1.09% and 1.20% across the four subsamples. In particular, we obtain a mean lower bound of 1.09% prior to the financial crisis (i.e., 1982:10 to 2007:12).\footnote{We are aware of a potential concern that there are fewer option strikes in the first few years of trading. If we compile our results starting in January 1985, the mean lower bound is 1.17%.}

Whereas the average $\text{LB}_{(t \rightarrow t+1)}$ appears to be stable across the considered subsamples, there is substantial time variation in $\text{LB}_{(t \rightarrow t+1)}$, as depicted in Figure 1.\footnote{Our confidence in the estimated lower bound is reinforced by the fact that the mean lower bound is 1.12%, using the second-nearest maturity options (with an average maturity of 84.31 days). The Online Appendix provides additional discussion.} The salient aspect of the lower bound appears to be its countercyclic nature. Notably, the lower bound spikes following events of concern in the financial markets, reflecting an increase in the minimum expected excess return demanded by investors. Prominent among these events are the stock market crash in October 1987, the financial crisis, and the Federal Reserve lowering the discount rate from 1.25% to 1% (June 25, 2003). Our method is flexible and it allows us to recover the minimum conditional expected excess return from options prices at each point in time.

The period of Federal Reserve policy of continued quantitative easing, namely, 2012:01 to 2013:12, is associated with a downward trending pattern of $\text{LB}_{(t \rightarrow t+1)}$, with an average of 0.89%. This observation can

[Fig. 1 about here.]
be understood from two angles. First, there was a decline in the number of sovereigns whose debt can be regarded as safe during this period (e.g., SEC (2014, page 3)). Second, the actions of the Federal Reserve lowered market uncertainty. Overall, LB_{t-r+1} lies between a low of 0.22% in 1991:09 (after a series of cuts in the federal funds rate) to a high of 6.07% in 2008:11 (in the aftermath of the Lehman collapse).

One question is: how can we judge the reasonableness of the lower bound on the expected excess return of the long-term bond? We benchmark our estimates in three ways.

<table>
<thead>
<tr>
<th></th>
<th>Nobs.</th>
<th>Mean</th>
<th>Std.</th>
<th>Min.</th>
<th>Max.</th>
<th>ACF1</th>
<th>TB30</th>
<th>TB20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess return, 30-year bond futures</td>
<td>375</td>
<td>3.02</td>
<td>10.58</td>
<td>-10.17</td>
<td>13.39</td>
<td>0.02</td>
<td>0.94</td>
<td>0.97</td>
</tr>
<tr>
<td>Excess return, 30-year bond (TB30)</td>
<td>375</td>
<td>4.95</td>
<td>12.53</td>
<td>-14.75</td>
<td>17.41</td>
<td>0.07</td>
<td></td>
<td>0.98</td>
</tr>
<tr>
<td>Excess return, 20-year bond (TB20)</td>
<td>375</td>
<td>5.48</td>
<td>10.63</td>
<td>-10.60</td>
<td>14.45</td>
<td>0.03</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

First, a fully collateralized long position in the 30-year Treasury bond futures delivers an average annual return of 3.02%. Second, the coupon-inclusive average returns of the 30-year and the 20-year Treasury bonds are 4.95% and 5.48%, respectively, with a correlation of 0.98. In sum, the average LB_{t-r+1} appears aligned with long-run average returns in the bond market while accounting for the effect of coupons. Finally, the full sample average LB_{t-r+1} of 1.18% corresponds to an average annualized risk-neutral futures return volatility of 10.27% (i.e., $\sqrt{E_t^2((\frac{f_{t+1}}{f_t}-1)^2)}$).

Two additional clarifications are in order. First, our approach moves away from using realized returns to capture the expected excess return. This is conceptually crucial since Elton (1999) argues that the past realized returns do not necessarily equate well with expected returns (see also Black (1993)). Second, our bound corresponds to a monthly holding period, and it is not to be compared with the promised return of a coupon-paying Treasury bond, which captures the entry yield and tends to vary over time with the prevailing interest rate environment.

---

4CME lowered the notional coupon rate of the Treasury bond underlying the futures contract from 8% to 6% in 1999. This change sharply reduced the futures prices of contracts maturing in and after March 2000, in comparison with those maturing in and before December 1999. We avoid the mechanical price drop by switching to the second-nearest maturity futures contract when computing the return in the month of December 1999. The option calculations were unaffected.

What are the ways in which the option-inferred estimates of LB_{t→t+1} could prove useful in empirical work? For instance, one could test whether expected excess returns implied by a term structure model are consistent with our lower bound. In particular, the lower bound restriction on the expected excess return could complement the modeling approaches featured in Gurkaynak and Wright (2011).

Finally, Table 2 shows that the lower bound exhibits a first-order autocorrelation (reported in the column ACF) of 0.74. We also examine the persistence properties of LB_{t→t+1} and fit all ARMA(p,q) models with p ≤ 3 and q ≤ 3. Our findings from model selection indicate that ARMA(1,1) is the best model for LB_{t→t+1}, according to the Bayesian information criterion. Relevant to our regression analysis that follows, this evidence shows that the time series of the lower bound LB_{t→t+1} is stationary.

4.2. Sources of the variation in the lower bound

Asset pricing theories hypothesize that investors demand higher expected return on risky assets in states perceived to be bad, specifically those associated with high marginal utility of wealth (e.g., Merton (1973)). Such a hypothesis has generally proven difficult to implement in an empirical setting given that the conditional expected return is unobservable.

Our approach offers the advantage that the conditional expected excess return can be decomposed into two nonnegative parts, which helps to isolate possible drivers of the expected excess returns. We can do so by assessing which macroeconomic determinants could be underlying the time variation in LB_{t→t+1}. Specifically, we employ a linear specification of the type:

\[
LB_{t→t+1} = \alpha + X'_{t→t+1} \beta + e_{t→t+1}, \quad \text{with,}
\]

\[
X_{t→t+1} = [\Delta \text{Spread}_{t→t+1} r_{t→t}^{\text{SDF}} er_{t→t}^{\text{gold}}]' ,
\]

(17)

where \( \beta \) is the vector of sensitivity coefficients and \( \alpha \) is the intercept. Contained within \( X_{t→t+1} \) are variables of potentially different economic nature, which could reveal the possibly distinct determinants of
expected excess returns. We motivate the choice of our explanatory variables $X_{t-1:t}$ as follows:

**Changes in yield spread** ($\Delta \text{Spread}_{t-1:t}$): Fama (1990, page 1091) and Fama and French (1989, pages 26 and 27) have argued that the yield spread tracks expected returns. The yield spread is the differential between the 30-year Treasury bond yield and the one-month Treasury bill counterpart. We employ the first difference of the yield spread in our regression equation (17).

**Return of a security with SDF payoff** ($r^{\text{SDF}}_{t-1:t}$): This variable captures the return of a security that pays the stochastic discount factor (SDF). We adopt a specification where $m_{t-1:t} = 1/R^M_{t-1:t}$ and, hence, $m_{t-1:t}$ is the inverse of the gross return of the equity market (e.g., Cochrane (2005, equation 9.12, page 160)). The price of a security that pays the SDF is $E_{t-1} \left( m_{t-1:t} \times m_{t-1:t} \right)$. Therefore, the return of a security with the SDF payoff is (see also Cochrane (2005, page 18)):

$$r^{\text{SDF}}_{t-1:t} \equiv \frac{m_{t-1:t}}{E_{t-1} \left( m^2_{t-1:t} \right)} - 1 = \frac{1}{E_{t-1} \left( (R^M_{t-1:t})^{-2} \right)} - 1. \quad (18)$$

To estimate $E_{t-1} ((R^M_{t-1:t})^{-2})$ in each month, we use daily returns and the moment analog $\sum_{i=1}^{22} (R^M_{t-i-1:t+i})^{-2}$. The $r^{\text{SDF}}_{t-1:t}$ series so constructed has the feature that returns are positive during bad times, implying a positive relation between $r^{\text{SDF}}_{t-1:t}$ and $\text{LB}_{t-1:t+1}$. We proxy the market by the value-weighted equity index.

**Return of gold** ($e^{\text{gold}}_{t-1:t}$): Following Yellen (2012) and Barro and Misra (2013), we next explore the returns of an asset that investors rotate into when they are fearful about impending financial market catastrophe or economic tail risks. We consider in particular the monthly excess returns of a fully collateralized long futures position in gold (data source: CME):

$$e^{\text{gold}}_{t-1:t} \equiv \frac{1}{f_{t-1}} \left( f^g_{t} - f^g_{t-1} \right). \quad (19)$$

The hypothesis to consider is whether a rise in gold returns is associated with a rise in expected excess return of the long-term bond.

Tables 3 and 4 present results from univariate and multivariate regressions, respectively, while Table 5...
presents results from multivariate regressions with controls. We focus on three controls: (i) industrial production growth, (ii) inflation, and (iii) log changes in the supply of long-term bonds. Reported are the estimates of individual slope coefficients along with the two-sided \( p \)-values \( NW[p] \), based on the heteroskedasticity and autocorrelation consistent covariance estimator from Newey and West (1987). Our procedure relies on the Bartlett kernel and no prewhitening, with lag length selected automatically, according to Newey and West (1994).

Our results impart several insights about variations in the expected excess returns of the long-term bond.

First, Table 3 shows that each of the three featured variables is positively correlated with \( LB_{(t-\tau+1)} \) over various sample periods. The pairwise correlation ranges between 0.07 to 0.15 for \( \Delta \text{Spread}_{(t-1-\tau)} \); between 0.18 and 0.31 for \( r_{SDF}^{(t-1-\tau)} \); and between 0.04 and 0.16 for \( \varepsilon_{\text{gold}}^{(t-1-\tau)} \). At the same time, the explanatory variables do not share a large common covariation, as the maximum contemporaneous cross-correlation is 0.08 between \( \Delta \text{Spread}_{(t-1-\tau)} \) and \( r_{SDF}^{(t-1-\tau)} \). Figure 2 depicts the time-variation in these variables.

Second, our results indicate that the variables are significantly associated with \( LB_{(t-\tau+1)} \) over different samples. Consider the univariate regression results in Table 3, where (i) the \( NW[p] \) value is below 0.1 in four out of five subsamples for \( \Delta \text{Spread}_{(t-1-\tau)} \), (ii) the \( NW[p] \) value is below 0.1 in three out of five subsamples for \( r_{SDF}^{(t-1-\tau)} \), and (iii) \( \varepsilon_{\text{gold}}^{(t-1-\tau)} \) retains individual parameter significance in two out of five subsamples. The adjusted \( R^2 \)’s of the univariate regressions are the highest with \( r_{SDF}^{(t-1-\tau)} \) and range from 2.8% to 8.8%.

The impact of the featured variables appears economically nontrivial. In particular, a one-standard-deviation increase in \( r_{SDF}^{(t-1-\tau)} \) is seen associated with an annualized increase of 0.18% (18 basis points) in \( LB_{(t-\tau+1)} \). An analogous calculation reveals that a one-standard-deviation increase in \( \Delta \text{Spread}_{(t-1-\tau)} \) \( (\varepsilon_{\text{gold}}^{(t-1-\tau)}) \) is associated with an increase of 0.07% (0.10%) in \( LB_{(t-\tau+1)} \). Compared with the average \( LB_{(t-\tau+1)} \) of 1.18%, these impacts are economically meaningful. The interpretation is that a deterioration in the investor opportunity set is linked to an increase in the minimum expected excess returns, and this
effect is the strongest in response to equity market declines.

Table 4 presents the multivariate regression results combining two or all three economic variables. The coefficient estimate for $\Delta\text{Spread}_{t-1 \rightarrow t}$ is uniformly positive and is statistically significant in 12 out of 15 regressions (i.e., $NW[p] \leq 0.1$). The impact of $r_{t \rightarrow t-1}^{\text{SDF}}$ remains positive in the presence of other variables, and the coefficient estimate is statistically significant in 11 out of 15 regressions. In contrast, the coefficient on gold$_{t-1 \rightarrow t}$, while always positive, is statistically significant in five of 15 regressions, and is generally driven out by $\Delta\text{Spread}_{t-1 \rightarrow t}$ and $r_{t \rightarrow t-1}^{\text{SDF}}$. For instance, over the entire sample, we obtain slope coefficient estimates of 0.09, 0.34%, and 0.16% in the multivariate regression with $X_{t \rightarrow t-1} = [\Delta\text{Spread}_{t-1 \rightarrow t} \ r_{t \rightarrow t-1}^{\text{SDF}} \ \text{er}^{\text{gold}}_{t-1 \rightarrow t}]^\prime$, with $p$-values of 0.05, 0.02, and 0.11, respectively. The three variables together track variations in expected excess returns with an adjusted $R^2$ of 7.8%.

What is the impact of including plausible controls? There are two points worth garnering from Table 5, which presents our results over several samples. First, each of the controls, namely, industrial production growth, inflation, and the log change in bond supply, are insignificant, with $NW[p]$ values higher than 0.05. Second, the statistical significance of the yield spread and the return of the SDF security does not diminish in the presence of controls. Thus, our evidence remains consistent with the view that a rise in the slope of the yield curve and the return of the SDF security tend to increase the expected excess return of the long-term bond. The focus on expected excess returns distinguishes our study from works that consider the possible role of economic variables for understanding the contemporaneous, or next-month, realized excess bond returns (e.g., Fama and French (1989), Ilmanen (1995), and Ludvigson and Ng (2009)).

To probe the importance of $r_{t \rightarrow t-1}^{\text{SDF}}$ in explaining expected excess returns, we also conduct an exercise in the fashion of Lakonishok, Shleifer, and Vishny (1994). Specifically, we classify economic bad states by dividing the realizations of $r_{t \rightarrow t-1}^{\text{SDF}}$ into four groups: (i) the 25 months with the highest positive returns (6.7% of the sample), (ii) the months with positive returns, excluding the 25 highest months (29.6%), (iii) the months with negative returns, excluding the 25 most negative months (57.1%), and (iv) the 25 months with the most negative returns (6.7%). We compute the average lower bound over the subsequent
month within each of the four groups. The takeaway from Table 6 is that the lower bound averages 2.14% during the highest positive 25 months, whereas it averages 1.29% during the most negative 25 months. Importantly, the hypothesis that the average LB\(_{t\rightarrow t+1}\) during the highest positive 25 months equals the average during the lowest negative 25 months is rejected, with a \(p\)-value of 0.02. Overall, our evidence suggests that investors tend to reshape their expectation of excess returns of a long-term bond following negative shocks to their marginal utility of wealth.

5. Concluding remarks

Characterizing the expected excess return of a long-term bond is at the core of finance and economics. Variations in the expected excess return are thought to influence the functioning of financial markets and the valuation of risky cash flows. The expected excess return of a long-term bond can alter the strategic asset allocation decision between equities and long-term bonds. Moreover, it can also impact the corporate sector by changing the cost of corporate debt, incentives for corporate investment and the holding of cash. Recognizing the possible pernicious effects of high bond risk premiums on the macroeconomy, the Federal Reserve has often sought to reduce the risk premium on the long-term bond through policy actions.

Our approach in this paper is to derive a lower bound on the expected excess return of a long-term Treasury bond. The economic interpretation of the lower bound is that it reflects the minimum expected excess return demanded by investors. We show that options on the 30-year Treasury bond futures – which is one of the most liquid markets – are informative about the expected excess return of the Treasury bond. Specifically, we develop an options positioning that extracts the lower bound on the expected excess return of the long-term bond from traded calls and puts on the 30-year Treasury bond futures.

The options data provide the insight that the annualized lower bound on the expected excess return of the long-term bond ranges between 0.22% and 6.07% with an unconditional average of 1.18%. In addition, our investigation shows that investors respond to increases in the slope of the Treasury yield curve by demanding a higher minimum compensation for holding the long-term bond. The empirical analysis also
uncovers a positive relation between the expected excess return of a long-term bond and the return of a security that pays the stochastic discount factor. This finding conveys the insight that the lower bound increases during bad times when the marginal utility of wealth is perceived to be high. Lastly, investors in the long-term bond market dislike increases in the return of gold futures over certain business conditions and price bonds with higher expected returns.

Finally, our Propositions 1 and 2 could be used to judge the reasonableness of conditional and unconditional estimates of expected excess returns from alternative term structure models of interest rate. Overall, the pattern of options prices on the Treasury bond futures can help in understanding the market reaction to monetary policy, for example, forward guidance in a zero lower bound interest rate policy environment. We leave these extensions to follow-up research.

6Our work also connects to Arrow (1995), who, among others, elaborates on the importance of employing an appropriate discount rate for evaluating the costs and benefits of social investments (see also the panel report of Arrow, Cropper, Gollier, Groom, Heal, Newell, Nordhaus, Pindyck, Pizer, Portney, Sterner, Tol, and Weitzman (2012)). Specifically, our framework quantifies expected returns using market prices of options and could be viewed as an alternative to the Ramsey formulation of discount rates.
References


Appendix A: Proof of Proposition 1

For brevity of presentation, we maintain that $m^p_{t+1} \equiv \frac{M^p_{t+1}}{M^f_t}$ and $m^T_{t+1} \equiv \frac{M^T_{t+1}}{M^f_t}$. Hence, $m_{t+1} = m^p_{t+1} m^T_{t+1}$.

Consider $E_t^* \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right)$, which is the return variance of the long-term bond under the risk-neutral (pricing) measure:

$$ E_t^* \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right) = E_t \left( \frac{m^p_{t+1} R^2_{t+1,\infty}}{E_t \left( m^p_{t+1} \right)} \right) - R^2_{t+1,f}, \quad (20) $$

$$ = R_{t+1,f} \left( \text{Cov}_t \left( m^p_{t+1}, \frac{1}{m^f_{t+1}} \right) + E_t \left( R_{t+1,\infty} \right) \right) - R^2_{t+1,f}, \quad (21) $$

$$ = R_{t+1,f} \text{Cov}_t \left( m^p_{t+1}, \frac{1}{m^f_{t+1}} \right) + R_{t+1,f} \left( E_t \left( R_{t+1,\infty} \right) - R_{t+1,f} \right). \quad (22) $$

Expressions in equation (21) follow, since $E_t \left( m^p_{t+1} R^2_{t+1,\infty} \right) = E_t \left( m^p_{t+1} m^T_{t+1} (m^T_{t+1})^{-2} \right)$, $E_t \left( m^p_{t+1} \right) = 1/R_{t+1,f}$, and $E_t \left( m^T_{t+1} \right) = 1$ by the martingale property of the permanent component. Therefore,

$$ E_t (R_{t+1,\infty}) - R_{t+1,f} = \frac{1}{R_{t+1,f}} E_t^* \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right) - \text{Cov}_t \left( m^p_{t+1}, \frac{1}{m^f_{t+1}} \right), \quad (23) $$

$$ \leq 0 \text{ from equation (32)} $$

$$ \geq \frac{1}{R_{t+1,f}} E_t^* \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right). \quad (24) $$

The negative sign assigned to $\text{Cov}_t \left( m^p_{t+1}, \frac{1}{m^f_{t+1}} \right)$ in equation (23) follows by invoking the result that $\text{Cov}_t \left( m^p_{t+1}, m^T_{t+1} \right) \geq 0$ (as in equation (7)) and equation (32) of Lemma 1.

To complete the above argument, we consider the covariance:

$$ \text{Cov}_t \left( m^p_{t+1}, \frac{1}{m^f_{t+1}} \right) = E_t \left( m^p_{t+1} \frac{1}{m^f_{t+1}} \right) - E_t \left( \frac{1}{m^f_{t+1}} \right) = E_t \left( \frac{m^p_{t+1}}{m^f_{t+1}} \right) - E_t \left( \frac{1}{m^f_{t+1}} \right), \quad (25) $$

$$ = E_t \left( m^p_{t+1} R^2_{t+1,\infty} \right) - E_t \left( R_{t+1,\infty} \right), \quad (26) $$

$$ = \frac{1}{R_{t+1,f}} E_t^* \left( R^2_{t+1,\infty} \right) - E_t \left( R_{t+1,\infty} \right), \quad (27) $$

$$ = \frac{1}{R_{t+1,f}} \left( E_t^* \left( R^2_{t+1,\infty} \right) - R^2_{t+1,f} \right) + R_{t+1,f} - E_t \left( R_{t+1,\infty} \right), \quad (28) $$

$$ = \frac{1}{R_{t+1,f}} E_t^* \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right) - \left( E_t \left( R_{t+1,\infty} \right) - R_{t+1,f} \right). \quad (29) $$
Hence, using $C \Leftrightarrow D$ to denote that $C$ is mathematically equivalent to $D$, we have

$$\text{Cov}_t \left( m_{t+1}^p, \frac{1}{m_{t+1}^r} \right) \leq 0 \Leftrightarrow \frac{1}{R_{f,t+1}} E_t^+ \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right) - (E_t (R_{t+1,\infty}) - R_{t+1,f}) \leq 0. \quad (30)$$

Equation (30) is also a statement that

$$\text{Cov}_t \left( m_{t+1}^p, \frac{1}{m_{t+1}^r} \right) \leq 0 \Leftrightarrow E_t (R_{t+1,\infty}) - R_{t+1,f} \geq \frac{1}{R_{f,t+1}} E_t^+ \left( (R_{t+1,\infty} - R_{t+1,f})^2 \right). \quad (31)$$

With this step, we have a proof of Proposition 1.

Lemma 1 is stated next.

**Lemma 1** Suppose $m_{t+1}^p > 0$ and $m_{t+1}^r > 0$. Then

$$\text{Cov}_t \left( m_{t+1}^p, \frac{1}{m_{t+1}^r} \right) \leq 0, \quad \text{if} \quad \text{Cov}_t (m_{t+1}^p, m_{t+1}^r) \geq 0. \quad (32)$$

**Proof:** Both $m_{t+1}^p > 0$ and $m_{t+1}^r > 0$ are positive random variables, hence, $\frac{1}{m_{t+1}^r}$ exists, and is well-defined. Observe that $\log(x)$ is monotonic in $x$. It follows from Egozcue, Garcia, and Wong (2009, Theorem 2.4) that $\text{Cov}_t (\log(m_{t+1}^p), \log(m_{t+1}^r)) \geq 0$. Moreover, it follows that $\text{Cov}_t (\log(m_{t+1}^p), -\log(m_{t+1}^r)) \leq 0$.

The equivalent statement is that $\text{Cov}_t \left( \log(m_{t+1}^p), \log \left( \frac{1}{m_{t+1}^r} \right) \right) \leq 0$. From the covariance inequality results in Egozcue, Garcia, and Wong (2009, Theorem 2.4), it then holds that $\text{Cov}_t \left( m_{t+1}^p, \frac{1}{m_{t+1}^r} \right) \leq 0$. ■

**Appendix B: Proof of Proposition 2**

Any twice-continuously differentiable payoff function $H[f_{t+1,\infty}]$ can be synthesized as per Bakshi and Madan (2000, Appendix A.3) and Carr and Madan (2001, equation (1)):

$$H[f_{t+1}] = H[f_t] - f_t H[f_t] + f_{t+1} H[f_t]$$

$$+ \int_{f_t < K} H_{ff}[K] (f_{t+1} - K)^+ dK + \int_{K < f_t} H_{ff}[K] (K - f_{t+1})^+ dK,$$  \quad (33)
where \( a^+ \equiv \max(a, 0) \), \( H_f[f_t] \) is the first-order derivative of the payoff \( H[f_{t+1}] \) with respect to \( f_{t+1} \) evaluated at \( f_t \), and \( H_{ff}[K] \) is the second-order derivative with respect to \( f_{t+1} \) evaluated at \( K \).

Setting \( H[f_{t+1}] = (\frac{f_{t+1}}{f_t} - 1)^2 \), we obtain:

\[
H[f_t] = \left( \frac{f_{t+1}}{f_t} - 1 \right)^2 \bigg|_{f_{t+1}=f_t} = 0, \tag{34}
\]

\[
H_f[f_t] = \frac{dH[f_{t+1}]}{df_{t+1}} \bigg|_{f_{t+1}=f_t} = \frac{2}{f_t} \left( \frac{f_{t+1}}{f_t} - 1 \right) \bigg|_{f_{t+1}=f_t} = 0, \tag{35}
\]

\[
H_{ff}[K] = \frac{d^2H[f_{t+1}]}{df_{t+1}^2} \bigg|_{f_{t+1}=K} = \frac{2}{f_t^2}. \tag{36}
\]

Therefore, we can write the arbitrage-free value of \( H[f_{t+1}] \) at date \( t \) as:

\[
E_t(m_{t+1}H[f_{t+1}]) = \int_{\{K>f_t\}} H_{ff}[K] C_t[K] dK + \int_{\{K<f_t\}} H_{ff}[K] P_t[K] dK, \tag{37}
\]

where the date-\( t \) value of the European call option, denoted by \( C_t[K] \), and the European put option, denoted by \( P_t[K] \), with strike price \( K \), satisfies:

\[
C_t[K] = E_t(m_{t+1}(f_{t+1} - K, 0)^+) \quad \text{and} \quad P_t[K] = E_t(m_{t+1}(K - f_{t+1}, 0)^+). \tag{38}
\]

Recalling that our object of interest is the payoff \( H[f_{t+1}] = (\frac{f_{t+1}}{f_t} - 1)^2 \), we can now express:

\[
E_t^*(H[f_{t+1}]) = \frac{1}{m_{t+1}} E_t(m_{t+1}H[f_{t+1}]), \tag{39}
\]

\[
= R_{t+1,f} \left( \int_{\{K>f_t\}} \frac{2}{f_t^2} C_t[K] dK + \int_{\{K<f_t\}} \frac{2}{f_t^2} P_t[K] dK \right). \tag{40}
\]

The right-hand side of equation (40) is computable at date \( t \), given the observed futures price and the prices of out-of-the-money European calls and puts written on the bond futures.

The desired expression for \( R_{t+1,f} E_t^*(\left(\frac{f_{t+1}}{f_t} - 1\right)^2) \) is then as displayed in (12) of Proposition 2. 

\[\blacksquare\]
Table 1
Summary statistics for out-of-the-money call and put options on the 30-year Treasury bond futures

The underlying unit of one options contract is one 30-year U.S. Treasury bond with a face value at maturity of $100,000. CME (2012) provides a more detailed description of the contract specifications for the Treasury bond futures and options. Among all available option strikes and maturities at the end of each month, we first retain options with maturity closest to 30 days. Next, we keep out-of-the-money calls, i.e., \( f_t / K < 1 \), and out-of-the-money puts, i.e., \( K / f_t < 1 \), where \( f_t \) is the futures price and \( K \) is the strike price. Reported is a snapshot of (i) the number of strikes for calls, (ii) the open interest for calls, (iii) the number of strikes for puts, and (iv) the open interest for puts. The average maturity of the out-of-money options is 29.16 days. The sample period is 1982:10 to 2013:12, for a total of 375 monthly observations.

<table>
<thead>
<tr>
<th>Year</th>
<th>Calls on 30-year Treasury bond futures</th>
<th>Puts on 30-year Treasury bond futures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Calls (end of month)</td>
<td>Open Interest (end of month)</td>
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<td>3037</td>
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<tr>
<td>2013</td>
<td>21</td>
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Table 2
Lower bound on the expected excess returns of a long-term bond (annualized, in percentage)

Reported are the lower bounds on the expected excess returns of a long-term bond, computed as:

\[
\text{LB}_{t\rightarrow t+1} = R_{t+1,f} E_t^s \left( \frac{f_{t+1}}{f_t} - 1 \right)^2 = \frac{2R_{t+1,f}^2}{f_t^2} \left( \int_{\{K>f_t\}} C_t[K] dK + \int_{\{K<f_t\}} P_t[K] dK \right),
\]

where \(C_t[K]\) (\(P_t[K]\)) are the prices of the out-of-the-money calls (puts) written on the 30-year Treasury bond futures, reported at the end of month \(t\). The futures price is \(f_t\) and \(R_{t+1,f}\) is the gross return of the one-month Treasury bill that is known at the end of month \(t\). The average maturity of the options in our sample is 29.16 days, and the sample period is 1982:10 to 2013:12, with a total of 375 monthly observations. The lower and upper 95% confidence intervals are constructed following the stationary bootstrap procedure of Politis, White, and Patton (2009), with optimal block size, and are based on 10,000 bootstrap draws. All the statistics are annualized and expressed in percentages. For example, over the full sample, the mean lower bound on the expected excess return of the long-term bond is 1.18% per year. ACF\(_1\) is the first-order autocorrelation.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Mean</th>
<th>Std.</th>
<th>Bootstrap 95% CI</th>
<th>Lower</th>
<th>Upper</th>
<th>Min.</th>
<th>Max.</th>
<th>ACF(_1)</th>
<th>Nobs.</th>
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<td></td>
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<tr>
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<td><strong>1.18</strong></td>
<td>0.71</td>
<td>1.00</td>
<td>1.38</td>
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<td>0.74</td>
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<td><strong>Panel B: Subsamples</strong></td>
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<td></td>
<td></td>
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<tr>
<td>1982:10-1997:12</td>
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<td>0.62</td>
<td>0.99</td>
<td>1.45</td>
<td>0.22</td>
<td>4.12</td>
<td>0.66</td>
<td>183</td>
<td></td>
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<td>1998:01-2013:12</td>
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<td>0.79</td>
<td>0.91</td>
<td>1.49</td>
<td>0.33</td>
<td>6.07</td>
<td>0.79</td>
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<tr>
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<td>0.22</td>
<td>6.07</td>
<td>0.74</td>
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<td>1982:10-2007:12</td>
<td><strong>1.09</strong></td>
<td>0.57</td>
<td>0.93</td>
<td>1.29</td>
<td>0.22</td>
<td>4.12</td>
<td>0.68</td>
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Univariate regression analysis of the lower bound on the expected excess return

We perform univariate regressions of the type: \( \text{LB}_{(t-\tau+1)} = \alpha + \beta \times X_{(t-\tau)} + e_{(t-\tau+1)} \), where \( \text{LB}_{(t-\tau+1)} \) is the estimated lower bound on the expected excess return of the long-term bond (computed at the end of month \( t \)) and \( X_{(t-\tau)} \) is an economic variable. Each variable is annualized and in percentage terms. Reported are the coefficient estimates, as well as the two-sided \( p \)-values, denoted by NW\( p \), based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994), reported in the “optimal lag” column. The adjusted \( R^2 \) (in %) is denoted by \( R^2 \), DW is the Durbin-Watson statistic, and CORR is the correlation coefficient between \( \text{LB}_{(t-\tau+1)} \) and \( X_{(t-\tau)} \).

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</thead>
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<tr>
<td>( \Delta \text{Spread}_{(t-\tau)} )</td>
<td>1.17</td>
<td>0.00</td>
<td>0.12</td>
<td>0.02</td>
<td>0.8</td>
<td>0.49</td>
</tr>
<tr>
<td>( r^\text{SDF}_{(t-\tau)} \times 100 )</td>
<td>1.20</td>
<td>0.00</td>
<td>0.35</td>
<td>0.02</td>
<td>6.1</td>
<td>0.51</td>
</tr>
<tr>
<td>( \epsilon^\text{gold}_{(t-\tau)} \times 100 )</td>
<td>1.17</td>
<td>0.00</td>
<td>0.17</td>
<td>0.08</td>
<td>1.5</td>
<td>0.51</td>
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<td>1.19</td>
<td>0.00</td>
<td>0.12</td>
<td>0.06</td>
<td>1.6</td>
<td>0.62</td>
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<tr>
<td>( \epsilon^\text{gold}_{(t-\tau)} \times 100 )</td>
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<td>( \Delta \text{Spread}_{(t-\tau)} )</td>
<td>1.15</td>
<td>0.00</td>
<td>0.12</td>
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<td>0.20</td>
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<tr>
<td>( r^\text{SDF}_{(t-\tau)} \times 100 )</td>
<td>1.17</td>
<td>0.00</td>
<td>0.33</td>
<td>0.13</td>
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<td>0.44</td>
</tr>
<tr>
<td>( \epsilon^\text{gold}_{(t-\tau)} \times 100 )</td>
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<td>0.09</td>
<td>2.0</td>
<td>0.42</td>
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<tr>
<td>( \Delta \text{Spread}_{(t-\tau)} )</td>
<td>1.10</td>
<td>0.00</td>
<td>0.14</td>
<td>0.01</td>
<td>0.6</td>
<td>0.51</td>
</tr>
<tr>
<td>( r^\text{SDF}_{(t-\tau)} \times 100 )</td>
<td>1.12</td>
<td>0.00</td>
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<td>0.09</td>
<td>4.4</td>
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<td>( \epsilon^\text{gold}_{(t-\tau)} \times 100 )</td>
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<td>0.00</td>
<td>0.17</td>
<td>0.15</td>
<td>1.4</td>
<td>0.52</td>
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</table>

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</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \text{Spread}_{(t-\tau)} )</td>
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<td>0.00</td>
<td>0.13</td>
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<td>0.60</td>
</tr>
<tr>
<td>( r^\text{SDF}_{(t-\tau)} \times 100 )</td>
<td>1.10</td>
<td>0.00</td>
<td>0.21</td>
<td>0.11</td>
<td>2.8</td>
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<td>( \epsilon^\text{gold}_{(t-\tau)} \times 100 )</td>
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<td>0.04</td>
<td>0.60</td>
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Table 4  
Multivariate regression analysis of the lower bound on the expected excess return

We perform multivariate regressions of the type: \( \text{LB}_{t+1} = \alpha + X'_{t} \beta + e_{t+1} \), where \( X_{t} = [\Delta \text{Spread}_{t} \ SDF_{t} \ \text{er}^{\text{gold}}_{t}]' \). Each variable is annualized and in percentage terms.

Reported are the coefficient estimates, as well as the two-sided \( p \)-values, denoted by NW\[p\], based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994), reported in the “optimal lag” column. The adjusted \( R^2 \) (in %) is denoted by \( \bar{R}^2 \). Reported are the results from the three bivariate regressions, and all three variables are included.

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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \Delta \text{Spread}_{t+1} )</td>
<td>( SDF_{t+1} )</td>
<td>( \text{er}^{\text{gold}}_{t+1} )</td>
<td>( \bar{R}^2 )</td>
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<td>0.10</td>
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<td>0.35</td>
<td>0.02</td>
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<tr>
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<td>0.00</td>
<td>0.11</td>
<td>0.02</td>
<td>0.16</td>
<td>0.09</td>
<td>2.2</td>
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<td>0.09</td>
<td>0.05</td>
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<tr>
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<td>Panel C: Subsample, 1998 to 2013</td>
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<td>0.35</td>
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<td>8.1</td>
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Table 5
Multivariate regression analysis of the lower bound, with controls

This table shows multivariate regressions of the form: \( \text{LB}_{t-1} = \alpha + X'_{t-1} \beta + \epsilon_{t-1} \), where \( X_{t-1} = [ \Delta \text{Spread}_{t-1} \ r_{t-1}^{SDF} \ \text{er}_{t-1}^{gold} \ \text{Control}_{t-1} ]' \). Reported are the coefficient estimates, as well as the two-sided \( p \)-values for the null hypothesis \( \beta = 0 \), denoted by \( \text{NW}[p] \), based on the procedure in Newey and West (1987) with optimal lag selected as in Newey and West (1994), reported in the “optimal lag” column. The adjusted \( R^2 \) (in \%) is denoted by \( \bar{R}^2 \). We consider three control variables, namely, the industrial production growth, the inflation, and the log change in bond supply. The monthly time series of industrial production growth and inflation are taken from Datastream. Following Greenwood and Vayanos (2014), the bond supply is the long-term debt divided by the nominal GDP. The long-term debt is the entire face value of Treasury bonds with maturity between 10 and 30 years, plus the sum of remaining coupon payments (source: CRSP Treasury data). We do not report the intercept in the regressions to save space.

<table>
<thead>
<tr>
<th></th>
<th>( \Delta \text{Spread}_{t-1} )</th>
<th>( r_{t-1}^{SDF} )</th>
<th>( \text{er}_{t-1}^{gold} )</th>
<th>Control_{t-1}</th>
<th>Coef.</th>
<th>NW[p]</th>
<th>Coef.</th>
<th>NW[p]</th>
<th>Coef.</th>
<th>NW[p]</th>
<th>( R^2 ) (%)</th>
<th>Optimal Lag</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Control is the industrial production growth</td>
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<td></td>
</tr>
<tr>
<td>1982:10-2013:12</td>
<td>0.09</td>
<td>0.03</td>
<td>0.32</td>
<td>0.01</td>
<td>0.16</td>
<td>0.11</td>
<td>-0.18</td>
<td>0.28</td>
<td>10.1</td>
<td>11</td>
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</tr>
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</tr>
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<td>0.01</td>
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<td>Panel B: Control is the inflation</td>
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<td></td>
<td></td>
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<td>0.04</td>
<td>0.31</td>
<td>0.01</td>
<td>0.18</td>
<td>0.13</td>
<td>-0.03</td>
<td>0.13</td>
<td>9.5</td>
<td>11</td>
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<td></td>
</tr>
<tr>
<td>1982:10-1997:12</td>
<td>0.06</td>
<td>0.15</td>
<td>0.39</td>
<td>0.01</td>
<td>0.05</td>
<td>0.55</td>
<td>-0.03</td>
<td>0.14</td>
<td>8.9</td>
<td>7</td>
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</tr>
<tr>
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<td>0.30</td>
<td>0.05</td>
<td>0.30</td>
<td>0.10</td>
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<td>0.08</td>
<td>11.5</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1990:01-2013:12</td>
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<td>0.00</td>
<td>0.26</td>
<td>0.03</td>
<td>0.22</td>
<td>0.17</td>
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<td>0.19</td>
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<td>0.01</td>
<td>0.85</td>
<td>0.00</td>
<td>0.65</td>
<td>3.5</td>
<td>10</td>
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<td>Panel C: Control is the log change in bond supply</td>
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<td>1982:10-2013:12</td>
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<td>0.35</td>
<td>0.02</td>
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<td>0.39</td>
<td>0.07</td>
<td>0.26</td>
<td>0.09</td>
<td>7.82</td>
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<td>0.00</td>
<td>0.33</td>
<td>0.08</td>
<td>0.19</td>
<td>0.17</td>
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<td>0.48</td>
<td>7.0</td>
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<td>1982:10-2008:12</td>
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<td>0.01</td>
<td>0.19</td>
<td>0.12</td>
<td>0.01</td>
<td>0.86</td>
<td>0.33</td>
<td>0.86</td>
<td>3.4</td>
<td>8</td>
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</table>

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Table 6
Lower bound on the expected excess returns (annualized, in percentages) during periods of the highest and lowest returns of the SDF security

We first divide the returns of the SDF security, \( r_{t-1:t}^{\text{SDF}} \), into two groups containing positive and negative returns. Then we divide the sample of positive (negative) returns into two parts: (i) the months with the highest (lowest) 25 monthly returns, and (ii) the remainder of positive (negative) returns. Reported are the properties of the lower bound LB\(_{t \rightarrow t+1}\). Also shown is the difference between the lower bounds during months with the 25 most positive returns and the 25 most negative returns, with the \( p \)-value in parentheses. The highest 25 positive returns to the SDF security correspond to bad times with highest marginal utility of wealth.

<table>
<thead>
<tr>
<th>Positive returns</th>
<th>Negative returns</th>
<th>25 highest minus 25 lowest (p-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Highest</td>
<td>Remaining</td>
</tr>
<tr>
<td></td>
<td>25 positive</td>
<td>positive</td>
</tr>
<tr>
<td>Mean</td>
<td>2.14</td>
<td>1.12</td>
</tr>
<tr>
<td>Std.</td>
<td>1.52</td>
<td>0.52</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.61</td>
<td>0.22</td>
</tr>
<tr>
<td>Maximum</td>
<td>6.07</td>
<td>2.85</td>
</tr>
</tbody>
</table>
Fig. 1. Variation in the lower bound on the expected excess return of a long-term bond

Plotted is the time series of the lower bound on the expected excess return of a long-term bond. The lower bound is expressed in annualized percentage terms and is computed following equation (12) as:

$$\text{LB}_{(t \to t+1)} = \frac{2R_{t+1,f}^2}{f_t^2} \left( \int_{\{K>f_t\}} C_t[K]dK + \int_{\{K<f_t\}} P_t[K]dK \right),$$

where $C_t[K]$ ($P_t[K]$) are the prices of the out-of-the-money calls (puts) written on the 30-year Treasury bond futures at the end of month $t$. The futures price is $f_t$, and $R_{t+1,f}$ is the gross return of the one-month Treasury bill that is known at the end of month $t$. The yellow shaded regions are the recession months as classified by the NBER. The sample period is October 1982 to December 2013.
Fig. 2. Variation in the explanatory variables

We plot the variation in the three featured explanatory variables: (i) changes in yield spread (ΔSpread), (ii) return of the SDF security (rSDF), and (iii) excess returns of gold futures (ergold). The yield spread is the difference between the yield of the 30-year Treasury bond and the yield of the one-month Treasury bill. The return of the SDF security is as described in equation (18), while the excess return of gold futures is the return of a fully collateralized long futures position. The yellow shaded regions are the recession months as classified by the NBER. The three monthly series are all expressed in annualized percentage terms.
An Inquiry into the Nature and Sources of Variation in the Expected Excess Return of a Long-Term Bond

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1. Conversion of quotes to dollar prices on options and futures on the 30-year Treasury bond

In what follows, we provide an example of how we convert quotes to dollar prices in the options and futures markets. On April 30, 2003, we first retain the nearest maturity options and then the five nearest out-of-the-money calls and puts, which all expire on May 23, 2003. The underlying futures is the June 2003 contract.

<table>
<thead>
<tr>
<th>Date</th>
<th>Put/Call</th>
<th>Strike</th>
<th>Options contract</th>
<th>Futures contract</th>
<th>Futures contract</th>
<th>Option expiration</th>
</tr>
</thead>
</table>

The futures prices are quoted in points and 1/32 of a point. Each point is $1,000. Therefore, the June 2003 futures closing quote of 114-01 translates to \((114 + 01/32) \times 1000 = 11,4031.3\).

Next, we consider options on the futures, where the strikes are quoted in points with increment of one point. So a strike price of 110-00 maps to a strike price of \(110 \times 1000 = 110,000\).

The minimum tick size of the settlement price of options is 1/64 of a point ($15.625). If the settlement price quote is lower than 100, then the option price is the quote multiplied by $15.625. For example, if the option settlement quote is 9, then the price is \(9 \times 15.625 = 140.625\). If the settlement price quote is higher than 100, then the option price is one full point plus the remainder multiplied by the minimum tick size. Specifically, for the option settlement quote of 105, the price is \((1 \times 64 + 5) \times 15.625 = 1,078.125\).
2. **Robustness of the results using second-nearest maturity options**

The paper features the results on the lower bound of the expected excess return of the long-term bond based on the nearest-maturity options with an average maturity of 29.16 days. To assess robustness, Table Online I presents the properties of the lower bound using the second-nearest maturity options. These options have an average maturity of 84.31 days, and there are on average, 11.37 calls and 13.42 puts at the end of each month.

We note that the time series of the lower bound, obtained using the second-nearest maturity options, mimics many of the properties reported in Table 2. For instance, the mean (annualized) lower bound on the expected excess return of the long-term bond is 1.12%. Moreover, the two series have a correlation of 0.92.

3. **An illustrative model of the expected excess return of the long-term bond**

The tractability of the lower bound approach is illustrated in the context of a model, in which the expected excess return of the long-term bond can be computed analytically.

We reiterate that the equation (23) of the paper identifies two sources of variation in the expected excess return of the long-term bond. One source can be identified by computing the lower bound (as in our Propositions 1 and 2), whereas the contribution of the covariance source can only be determined through a model parametrization. However, any model development is complicated by the fact that $R_{t+1,\infty} = (m^T_{t+1})^{-1}$, and $m^P_{t+1}$ must be determined through the solution to the eigenfunction problem (Hansen and Scheinkman (2009, Proposition 2), or refer to equation (2) of the paper).

Let $z_t$ be a state variable, and suppose the evolution of the transitory and the permanent components of $m_t$ is governed by:

\[
\log(m^T_{t+1}) = \rho + \eta (\theta - z_t) - \sigma \sqrt{\epsilon_t} e_{t+1} \quad \text{and} \quad \log(m^P_{t+1}) = -\frac{1}{2} \sigma_p^2 z_t - \sigma_p \sqrt{\epsilon_t} e_{t+1}, \tag{A1}
\]
where $\rho < 0$ (i.e., the eigenvalue is negative) and $m_{t+1}^p$ is a martingale. We specify the dynamics of $z_t$ as:

$$z_{t+1} - z_t = (1 - \delta_z)(\theta_z - z_t) + \sigma_z \sqrt{z_t} \epsilon_{t+1}, \quad \text{with } \epsilon_{t+1} \sim \mathcal{N}(0, 1). \quad (A2)$$

The mean is $\theta_z$, and $\sigma_z$ controls the volatility of $z_t$.

The dynamics of $m_{t+1}^p$ and $m_{t+1}^T$ in equation (A1) imply that $m_{t+1} = m_{t+1}^p m_{t+1}^T$ is:

$$\log (m_{t+1}) = \rho + \eta \theta_z - \left( \eta + \frac{1}{2} \sigma_p^2 \right) z_t - (\sigma_p + \sigma_\infty) \sqrt{z_t} \epsilon_{t+1}. \quad (A3)$$

Taking the conditional expectation of $m_{t+1}$ in equation (A3) and using the moment generating function of the standard Normal distribution, we obtain the expression for the risk-free return below:

$$R_{t+1, f} = \exp \left( -\rho - (1 - \delta_z) c \theta_z + \left( (1 - \delta_z) c + \frac{1}{2} \sigma_p^2 - \frac{1}{2} (\sigma_\infty + \sigma_z^2) \right) z_t \right). \quad (A4)$$

The decomposition of $m_t$ in its permanent and transitory components is unique when the solution to the eigenfunction problem

\begin{equation*}
E_t \left( m_{t+1}^p \frac{\phi[z_{t+1}]}{\phi[z_t]} \right) = e^\rho
\end{equation*}

is of the form $\phi[z_t] = \exp(c z_t)$. Inserting the conjectured form of $\phi[z_t]$ and using the method of undetermined coefficients, we obtain:

$$c = \frac{(1 - \delta_z + \sigma_z (\sigma_\infty + \sigma_p)) + \sqrt{(1 - \delta_z + \sigma_z (\sigma_\infty + \sigma_p))^2 - \sigma_z^2 (\sigma_\infty \sigma_p + \sigma_z^2 - 2 \delta_z)}}{\sigma_z^2}, \quad (A5)$$

$$\eta = (1 - \delta_z) c. \quad (A6)$$

The coefficients $c$ and $\eta$ determine the dynamics of the long-term bond, together with $\rho$ and $\sigma_\infty$. Hence,

$$R_{t+1, \infty} = \left( m_{t+1}^T \right)^{-1} = \left( e^{\rho \frac{\phi[z_t]}{\phi[z_{t+1}^p]}} \right)^{-1} = e^{-\rho} \exp \left( c (z_{t+1} - z_t) \right). \quad (A7)$$

Then,

$$E_t (R_{t+1, \infty}) = \exp \left( -\rho + \frac{1}{2} c^2 \sigma_z^2 \theta_z + \left( (1 - \delta_z) c - \frac{1}{2} c^2 \sigma_z^2 \right) (\theta_z - z_t) \right).$$
In addition, we can write $E_t^s(R_{t+1,\infty}^2) = E_t^s\left(\frac{m_{t+1}}{E(m_{t+1})}R_{t+1,\infty}\right)$ as follows:

$$E_t^s(R_{t+1,\infty}^2) = R_{t+1,f} \exp \left( -\rho + \frac{\theta}{2} (c\sigma_z - \sigma_p)^2 - \sigma_p^2 \right) \left( (1 - \delta_c) c + \frac{1}{2} \left( \sigma_p^2 - (c\sigma_z - \sigma_p)^2 \right) \right) \left( \theta - \delta_t \right).$$ (A8)

Equations (A7)–(A8) represent a seven-parameter model, in which the conditional expected return of the long-term bond depends on the state variable $z_t$. The risk-neutral return variance of the long-term bond can then be computed as $E_t^s(R_{t+1,\infty}^2) - R_{t+1,f}^2$.

Recognize that generating the time series of $E_t(R_{t+1,\infty})$ and $E_t^s(R_{t+1,\infty}^2)$ requires some combination of estimation and/or calibration and the identification of $z_t$, whereas the lower bound is amenable to calculation from observed option prices on the Treasury bond futures at the end of each month.
References


Table Online I

**Lower bound extracted from second-nearest maturity options (annualized, in percentages)**

Reported are the lower bounds on the expected excess returns of the long-term bond, computed using second-nearest maturity options:

\[
LB_{t\rightarrow t+1} = \frac{2R_{t+1,f}^2}{f_t^2} \left( \int_{\{K>f_t\}} C_t[K] dK + \int_{\{K<f_t\}} P_t[K] dK \right),
\]

where \(C_t[K] (P_t[K])\) are the prices of the out-of-the-money calls (puts) written on the 30-year Treasury bond futures, reported at the end of month \(t\). The futures price is \(f_t\) and \(R_{t+1,f}\) is the gross return of the one-month Treasury bill that is known at the end of month \(t\). *The average maturity is 84.31 days*, and the sample period is 1982:10 to 2013:12, with a total of 375 monthly observations. On average, there are 11.37 calls and 13.42 puts in each month. The lower and upper 95% confidence intervals are constructed following the stationary bootstrap procedure of Politis, White, and Patton (2009), with optimal block size, and are based on 10,000 bootstrap draws. All the statistics are annualized and expressed in percentages. ACF\(_1\) is the first-order autocorrelation.

<table>
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<th>Sample</th>
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<th>Lower</th>
<th>Upper</th>
<th>Min.</th>
<th>Max.</th>
<th>ACF(_1)</th>
<th>Nobs.</th>
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<td>0.93</td>
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