A Theory of Correlated-Demand Driven Liquidity

Commonality*

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Abstract

Many empirical studies suggest that correlated demand is important in driving liquidity commonality among stocks. However, there are still no theoretical studies on how demand-side factors cause and affect liquidity and return commonality. We propose a tractable equilibrium model with asymmetric information and imperfect competition among market makers to study the effect of correlated demand on the commonality in liquidity, in liquidity risks and in returns. We solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed-form. Our model can help explain why market liquidity can be worse and liquidity commonality can be greater in significantly declined markets and in more volatile markets. Our model also generates some new empirically testable implications.

JEL Classification Codes: D42, D53, D82, G12, G18.

Keywords: Commonality in Liquidity, Bid-Ask Spread, Liquidity Risk, Asymmetric Information.
I. Introduction

It has been widely documented that liquidity strongly covaries across stocks even under normal economic conditions.\(^1\) The recent financial crisis highlighted the importance of understanding the mechanism through which assets exhibit illiquidity commonality and how this commonality is impacted by various factors such as information asymmetry and trading correlation. Although many empirical studies suggest that correlated trading by liquidity demanders (e.g., most mutual funds) can be the main source of illiquidity commonality (e.g., Koch, Ruenzi and Starks (2010), Corwin and Lipson (2011), Karolyi, Lee, and van Dijk (2011)), the existing theories have focused on the liquidity supply side sources. In this paper, we develop an equilibrium model to study how correlated demands affect the commonality of liquidity across different markets in the presence of asymmetric information, imperfect competition among market makers and risk aversion.

We model the correlated liquidity demands for different assets as the need to trade these assets to hedge against certain non-traded assets.\(^2\) Unlike Cespa and Foucault (2011), to focus on the correlated demands as the single source of liquidity commonality, we assume all stocks have independent payoffs. In contrast to Kyle and Xiong (2001), there is no wealth effect for any trader including the market makers. We solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed-form. Our model can help explain why both liquidity and a stock price can crash even when there is no fundamental news about the stock and why market liquidity can be worse and liquidity commonality can be greater in significantly declined markets and in more volatile markets. In the absence of asymmetric information, liquidity (measured by the bid-ask spread) is positively correlated whether liquidity demands are positively


\(^2\)One example of correlated liquidity trading is portfolio rebalancing of a mutual fund that acts upon private information or upon liquidity shocks such as fund-flow shocks and index reconstitution (e.g., Da, Gao and Jagannathan (2008)).
or negatively correlated and price correlation has the same sign as the liquidity demand correlation. In the presence of asymmetric information, however, price and liquidity correlations can be negative even when hedging demand correlation is positive. In the absence of demand correlation, information about one stock’s payoff does not affect the price or the liquidity of other stocks. With correlated demand, however, information about one asset’s payoff can affect both the price and the liquidity of another asset even though they have independent payoffs. We find that liquidity correlation may decrease with information asymmetry. Moreover, more precise information about one stock’s payoff reduces the estimation precision of the uninformed about the payoff of another stock with independent payoff. In this sense, correlated demand can also cause commonality in information asymmetry. In addition, we also study the price impact of an informed trader’s trade on stock prices as another measure of illiquidity. We find that the price impact on one stock is dependent on the characteristics of the other stock and can be non-monotonic in the private information quality. Our model produces some new empirically testable implications. For example, stocks subject to more volatile liquidity demands have higher liquidity risks and should command a greater risk premium when such liquidity demands are also correlated across a large set of stocks. Also, liquidity risks of stocks that are affected by the hedging demand increase with the volatility of the hedging demand.

Specifically, we consider a one-period, two trading-dates setting with three types of risk averse investors: informed investors, uninformed investors, and designated market makers who are also uninformed. There are one risk-free asset and two risky securities (“stocks”) with independent payoffs. Both informed and uninformed investors optimally choose how to trade these assets to maximize their expected CARA utility and all are endowed with some shares of the stocks but none of the risk-free asset. Type 1 market makers only make market in stock 1 and Type 2 market makers only make market in stock 2. Informed investors can observe a private signal about each stock’s payoff before the terminal date and thus they have trading demand motivated by the private information. They are
also subject to a liquidity shock modeled as a random endowment of a nontradable asset (e.g., highly illiquid position) whose payoff is correlated with both stocks.\footnote{Alternatively, one can directly model the extra trading needs from a liquidity shock.} Accordingly, informed investors also have trading demand motivated by the needs for hedging. Because both stocks, although independent of each other, are correlated with the non-traded asset, the hedging demands across the two markets are correlated. It is this correlated hedging demands that drive the correlated stock returns and liquidity.

The population of the informed and the uninformed is large and thus neither the informed nor the uninformed trade strategically. Informed and uninformed investors must trade through market makers. Following Kyle (1989), we assume that informed and uninformed investors submit their demand schedules simultaneously before trading. Different from Kyle (1989), however, the demand schedules are dependent on bid and ask prices and are submitted not to an auctioneer but to the designated market makers who then determine how to trade at the bid and ask. This assumption is consistent with a market microstructure where designated market makers observe the order flow before determining bid and ask prices and bid and ask depths (e.g., OTC markets). We allow the competition among market makers to be imperfect. In contrast to the standard literature that implicitly assumes Bertrand competition among market makers, we model the competition among market makers as a Cournot competition: they choose simultaneously how much to buy at the bid and how much to sell at the ask, taking into account the price impact of their trades. The equilibrium bid and ask prices are then determined by the market clearing conditions at the bid and at the ask, i.e., the total amount market makers buy at the bid is equal to the total amount other investors sell, and the total amount market makers sell at the ask is equal to the total amount other investors buy. In equilibrium, both the stock markets and the risk-free asset market clear.

As in Liu and Wang (2012), investors buy (sell) a stock if and only if the market price is lower (higher) than their reservation price, which increases with their conditional
expectation of the stock payoff and decreases with the conditional variance. Relative to the informed, the uninformed may underestimate or overestimate the expected stock payoff. The bid ask spread is equal to the absolute value of the reservation price difference between the informed and the uninformed, divided by the number of market makers plus one. In addition, bid-ask spreads can be lower with asymmetric information.

Our model can help explain why crashes in both liquidity and prices can be caused by fundamental news about other stocks with independent payoffs, can occur simultaneously across these stocks, and why market liquidity can be worse and liquidity commonality can be greater in significantly declined markets and in more volatile markets (e.g., Karolyi, Lee, and van Dijk (2011) and Hameed, Kang, and Viswanathan (2010)). Intuitively, consider two stocks with independent payoffs. After a bad signal about the second stock’s payoff, the informed submit a large sell order of the second stock, the uninformed (including the market makers in the first stock) attribute the resulting drop in the second stock’s price partly to a large negative hedging demand shock. If the hedging demands are negatively correlated across the two stocks, then there is likely a large positive hedging demand component in the first stock’s orders. Thus after seeing a sell order of the first stock, the uninformed infer that there is likely a large negative news about the first stock that makes the informed sell even with a large positive hedging demand. Therefore, the uninformed’s conditional expectation of the first stock’s payoff decreases significantly, which then drives down the first stock’s price, drives up the reservation price difference for the first stock and thus also the first stock’s bid-ask spread. One of the reasons why liquidity commonality can be greater in more volatile markets is that an increase in the liquidity demand volatility can drive up market price volatility, the expected spreads and the spread correlation, which makes market volatility and liquidity commonality positively correlated. Although the commonality channel in Cespa and Foucault (2011) is also through cross market learning, the learning mechanism in our model is more subtle. In contrast to Cespa and Foucault (2011), stock payoffs in our model are independent.
Therefore, the information about the second stock’s payoff contained in the second stock’s price is useless for the market maker in the first stock. Instead, the uninformed extract information about the hedging demand from the second stock’s price, which is then combined with the first stock’s orders to help better estimate the payoff of the first stock.

We find that with information asymmetry, the price correlation can be non-monotonic in the quality of the information about stock payoffs, the volatility of stock payoff, the risk aversion and the volatility of the liquidity shock. In addition, the price correlation can become negative even when the hedging demands are positively correlated. Intuitively, in the presence of asymmetric information, the uninformed investors use the equilibrium prices in both markets to infer about the conditional distribution of a stock’s payoff. When the equilibrium price of a stock increases, it may be due to good information about the stock or due to a large liquidity demand shock. As the equilibrium price of one stock increases, the uninformed attribute some of the increase to an increase in the liquidity demand shock. If the hedging demands for the two stocks are positively correlated, then the uninformed attribute a smaller portion of the informed’s trades in the other stock to good private information and therefore the conditional expected payoff of the other stock estimated by the uninformed decreases, which drives down the equilibrium price of the other stock. Thus, when this information filtering effect dominates the effect of the positively correlated liquidity demands, the equilibrium prices become negatively correlated. All the non-monotonicity mentioned above comes from affecting this balance between the information filtering effect and the correlated demand effect. In addition, the informed can trade in opposite directions in the two stocks even when their hedging demands are positively correlated due to the information filtering effect that may cause the reservation prices of the informed to be below that of the uninformed in one market and to be above that of the uninformed in the other market. Furthermore, we show that as the precision of the private information about a stock improves, the uninformed’s estimation precision (measured by the inverse of the uninformed’s conditional variance) of
another stock’s payoff worsens. The main intuition is that as the precision of the private information about one stock improves, the stock price becomes less informative about the hedging demand because the informed put a greater weight on the private information (rather than hedging demands) in their trades. Therefore, the stock price is less useful for the uninformed to separate the effect of hedging demand and the effect of private information about the other stock on the other stock’s price and thus their estimation of the other stock’s payoff becomes less precise.

In the absence of asymmetric information, illiquidity (measured by stock spreads) is always positively correlated whether hedging demands are positively or negatively correlated. This is because the reservation price difference between the informed and the uninformed is a constant multiple of the hedging demand difference and spreads are proportional to the absolute value of the reservation price difference. In the presence of asymmetric information, however, liquidity can become negatively correlated if hedging demands are negatively correlated, because of the impact of information filtering on the reservation prices. For the main intuition, suppose relative to the informed, the uninformed underestimate the expected payoff of stock 1 and therefore have a lower reservation price. In addition, although the uninformed overestimate the expected payoff of stock 2, the higher estimation risk faced by the uninformed makes the uninformed’s reservation price for stock 2 still lower than that of the informed. Thus the informed buy both stock 1 and stock 2. When a better signal about stock 1 is observed by the informed, the uninformed underestimate even more the expected payoff of asset 1 relative to the informed due to information asymmetry, which implies that reservation price difference between the informed and the uninformed increases and so does the spread of asset 1. With a better signal for stock 1, the uninformed attribute less of the informed’s trade in stock 1 to hedging demand and therefore attribute more of the informed’s trade in stock 2 to the information about stock 2’s payoff. Thus, the uninformed overestimate more the expected payoff of stock 2 which in turn offsets more of the estimation risk premium and drives
down the spread of stock 2. Therefore, the spreads of the two stocks move in the opposite directions. As the quality of the private information about a stock’s payoff increases or the liquidity shock volatility increases, the spread correlation can change from negative to positive. On the other hand, as the payoff of a stock becomes more volatile, the spreads can become more negatively correlated or less positively correlated. These patterns arise because the reservation price difference variations are affected by these changes in the parameter values. In addition, the magnitude of the correlation between the spreads is higher with symmetric information than that with asymmetric information and the magnitude of the correlation between the spreads can decrease with information asymmetry for either stock 1 or stock 2. We also find that the liquidity risk of one stock is affected by the quality of the private information about the other stock. When the correlation between hedging demands for the two stocks is negative, the liquidity risk of stock 1 may decrease when the liquidity risk of stock 2 increases.

We also show that the price impact of an informed investor’s trade on a stock is nonmonotonic in the private signal quality about this stock. The non-monotonicity is driven by two effects of a private signal: an information quality effect and an information asymmetry effect. As the signal becomes more precise, both the information quality and the information asymmetry increase. The price impact for a stock decreases in the information quality and increases in the information asymmetry. When the private signal is less noisy, the information quality effect dominates and thus the price impact for a stock decreases in the precision of the private signal about this stock. When the private signal is noisy, the reverse is true. Interestingly, the price impact of stock 1 decreases in the quality of the private signal about stock 2’s payoff. This is because uninformed investors can estimate the hedging demand better from the equilibrium price of stock 2 when informed investors’ private signal about stock 2’s payoff is less precise and thus uninformed investors’ uncertainty about stock 1’s payoff decreases. This implies that the price impact of stock 1 may increase when the price impact of stock 2 decreases.
Correlated illiquidity has important implications for asset pricing and financial market regulations because correlated liquidity risk can become systemic. Broadly speaking, liquidity commonality can arise from both the supply side and the demand side. The existing theories have focused on the liquidity supply side sources. For example, Kyle and Xiong (2001) show that trading losses in one market of financial intermediaries who supply liquidity in two risky asset markets can cause them to supply less liquidity in both markets, resulting in reduced market liquidity and increased correlation, due to the wealth effect for the intermediaries. Brunnemeier and Pedersen (2009) illustrate that financial constraints of liquidity suppliers may lead to co-variations in liquidity because they can restrict liquidity suppliers in different securities simultaneously. Cespa and Foucault (2011) show that learning across markets by informed dealers may cause liquidity spillovers and thereby co-movements in liquidity for correlated stocks. However, many empirical studies suggest that correlated trading by liquidity demanders (e.g., most mutual funds) can be the main source of illiquidity commonality (e.g., Koch, Ruenzi and Starks (2010), Corwin and Lipson (2011), Karolyi, Lee, and van Dijk (2011)). For example, Corwin and Lipson (2011) suggest that commonality in returns and liquidity is driven by the correlated trading decisions of institutional traders. In studying cross-country liquidity commonality, Karolyi, Lee, and van Dijk (2011) conclude that the commonality is more reliably consistent with demand-side explanations than supply-side ones like the funding liquidity hypothesis and the support for the potential role of funding constraints of financial intermediaries is weak.

The remainder of the paper proceeds as follows. In Section II we present the model. In Section III we solve the case with symmetric information, and in Section IV we derive the equilibrium under asymmetric information and provide some comparative statics on

\footnote{For example, Pastor and Stambaugh (2003) and Acharya and Pedersen (2005) argue that returns of financial assets should depend on commonality in liquidity.}

\footnote{Different from our paper, Cespa and Foucault (2011) assume market makers are informed, liquidity demands are independent, and stock payoffs are correlated.}
asset prices and illiquidity. We conclude in Section V. All proofs are in the Appendix.

II. The model

We consider a one period setting with trading dates 0 and 1. There are \( N \) investors: \( N_I \) informed investors (\( I \)), \( N_U \) uninformed investors (\( U \)), and \( N_M \equiv N - (N_I + N_U) \) market makers who are also uninformed. Both \( I \) and \( U \) investors can trade one risk-free asset and two risky securities ("stocks") on date 0 and date 1 to maximize their expected constant absolute risk aversion (CARA) utility from the terminal wealth on date 1. Among the \( N_M \) market makers, \( N_{Mk} \) \( k = 1, 2 \) of them (\( M_k \)) only trade and make the market of stock \( k \).\(^6\) There is a zero net supply for the risk-free asset, which also serves as the numeraire and thus the risk-free interest rate is normalized to 0. The total supply of stock \( k \) is \( N_k \bar{\theta}_k \geq 0 \) \( N_k = N_I + N_U + N_{Mk} \) shares and the date 1 payoff of each share of stock \( k \) is \( \tilde{V}_k \sim N(\bar{V}_k, \sigma^2_{\tilde{V}_k}) \), where \( \bar{V}_k \) is a constant, \( \sigma_{\tilde{V}_k} > 0 \), and \( N(\cdot) \) denotes the normal distribution. The payoffs of stock 1 and stock 2 are independent.\(^7\) Each \( I \) or \( U \) investor is endowed with \( \bar{\theta}_1 \) shares of stock 1 and \( \bar{\theta}_2 \) shares of stock 2. Each market maker of stock \( k \) is only endowed with \( \bar{\theta}_k \) shares of stock \( k \). No one is endowed with the risk-free security.

On date 0, informed investors observe a private signal \( \hat{s}_k = \tilde{V}_k - \bar{V}_k + \hat{\varepsilon}_k \) about the payoff \( \tilde{V}_k \), where \( \hat{\varepsilon}_k \) is independently normally distributed with mean zero and variance \( \sigma^2_{\hat{\varepsilon}_k} \). A type \( I \) investor is also subject to a liquidity shock that is modeled as a random endowment of \( \hat{X}_I \sim N(0, \sigma^2_{X}) \) units of a non-traded risky asset on date 0, with \( \hat{X}_I \) realized and only directly known to the investor on date 0. The non-traded asset has a per-unit payoff of \( \hat{N} \sim N(0, \sigma^2_{\hat{N}}) \) that has a covariance of \( \sigma_{\hat{N}N} \) with \( \hat{V}_k \) and is realized and becomes

\(^6\)Our results do not change if we assume all market makers trade and make the market of both securities. We divide the market makers into two groups to rule out commonality in liquidity arises simply because the same market makers are making the market of both securities.

\(^7\)We assume the payoffs of the two securities are independent to rule out the commonality in illiquidity arises simply because their payoffs are correlated. Our main results do not change if we assume the payoffs of the two securities are correlated.
public on date 1. The correlations between the nontraded asset and asset \( k \) result in a liquidity demand for hedging the nontraded asset payoff.

In this model, all trades and thus liquidity demands are caused by either the private information or the liquidity shock of informed investors. Therefore, informed investors are liquidity demanders and both market makers and uninformed investors are liquidity suppliers. Da, Peng and Jagannathan (2008) find that index funds are almost always liquidity demanders and active funds who either have private information or experience fund flow are also liquidity demanders, while other funds provide liquidity. The informed investors can be viewed as active funds which may have both information and liquidity shocks or index funds that only have liquidity shock when they do not have private information, while the uninformed are similar to funds that do not have information or liquidity shock.

All trades must go through market makers whose market making cost is assumed to be 0. Specifically, given market bid price \( B_k \) and ask price \( A_k \), \( I \) and \( U \) investors sell to market makers at the bid or buy from them at the ask or do not trade at all. We assume that both \( N_I \) and \( N_U \) are relatively large such that all \( I \) and \( U \) investors are price takers and there are no strategic interactions among them or with market makers.

For each \( i \in \{ I, U, M_1, M_2 \} \), investors of type \( i \) are ex ante identical. Accordingly, we restrict our analysis to symmetric equilibria where all type \( i \) investors adopt the same trading strategy. Let \( \mathcal{I}_i \) represent a type \( i \) investor’s information set on date 0 for \( i \in \{ I, U, M_1, M_2 \} \). Given \( B_k \) and \( A_k \), for \( i \in \{ I, U \} \), a type \( i \) investor’s problem is

\[
\max_{\theta_1, \theta_2} E[-e^{-\delta \tilde{W}_i} | \mathcal{I}_i],
\]

subject to the budget constraint

\[
\tilde{W}_i = \sum_{k=1,2} \left( \theta_{ik} B_k - \theta_{ik} A_k + (\bar{\theta}_k + \theta_{ik})\tilde{V}_k \right) + \tilde{X}_i \tilde{N},
\]
\[ \hat{X}_U = 0, \delta > 0 \] is the absolute risk-aversion parameter, \( \theta_{ik} \) is the signed order size of investor \( i \), \( x^+ \equiv \max(0, x) \), and \( x^- \equiv \max(0, -x) \).

Since all trades in asset \( k \) must go through market makers \( M_k \), market makers can have market powers especially when the number of market makers \( M_k \) is small. To model the oligopolistic competition among the market makers \( M_j \), we use the notion of the Cournot competition.\(^8\) Specifically, we assume that market makers \( M_j \) simultaneously choose the optimal number of shares to sell at ask \( (A_k) \) and to buy at bid \( (B_k) \), taking into account the price impact of their trades.

For \( k = 1, 2, \ldots \), let \( \alpha_k = (\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{N_Mk})^\top \) and \( \beta_k = (\beta_{1k}, \beta_{2k}, \ldots, \beta_{N_Mk})^\top \) be the vector of the number of shares market makers \( M_k \) sell at ask (i.e., ask depth) and buy at bid (i.e., bid depth) respectively. Given the demand schedules of the informed and the uninformed \( (\theta^*_I(A_k, B_k) \text{ and } \theta^*_U(A_k, B_k)) \), the bid price \( B_k(\beta_k) \) (i.e., the inverse supply function) and the ask price \( A_k(\alpha_k) \) (i.e., the inverse demand function) can be determined by the following stock market clearing conditions at the bid and ask prices.

\[
\sum_{j=1}^{N_Mk} \alpha_{jk} = \sum_{i=I, U} N_i \theta^*_I(A_k, B_k) , \quad \sum_{j=1}^{N_Mk} \beta_{jk} = \sum_{i=I, U} N_i \theta^*_U(A_k, B_k) , \quad (3)
\]

where the left-hand sides represent the total sales and purchases by market makers \( M_k \) respectively and the right-hand sides represent the total purchases and sales by other investors respectively.

Then for \( j = 1, 2, \ldots, N_{Mk} \) and \( k = 1, 2 \), the market maker \( M_{jk} \)'s problem is

\[
\max_{\alpha_{jk} \geq 0, \beta_{jk} \geq 0} E \left[ -e^{-\delta W_{Mjk}} | I_{M_k} \right] , \quad (4)
\]

\(^8\)As is well-known, it takes only two Bertrand competitors to reach the perfect competition equilibrium prices. However, market prices can be far from the perfect competition ones (e.g., Christie and Schultz (1994), Chen and Ritter (2000), and Biais, Bisière and Spatt (2010)).
subject to the budget constraint

\[
\tilde{W}_{M,jk} = \alpha_{jk} A_k(\alpha_k) - \beta_{jk} B_k(\beta_k) + (\theta_k + \beta_{jk} - \alpha_{jk}) \tilde{V}_k.
\]

(5)

Note that different from other investors, a market maker takes into account the price impact of her own trades, i.e., recognizing both \(A_k\) and \(B_k\) will be affected by her trades. This leads to our definition of the Nash equilibrium of the Cournot competition.\(^9\)

**Definition 1** An equilibrium \((\theta_{I,k}^*(A_k, B_k), \theta_{U,k}^*(A_k, B_k), A_k^*, B_k^*, \alpha_k^*, \beta_k^*)\) is such that

1. given any \(A_k\) and \(B_k\), \(\theta_{I,k}^*(A_k, B_k)\) solves a type \(i\) investor’s Problem (1) for \(i \in \{I, U\}\);

2. given \(\theta_{I,k}^*(A_k, B_k)\) and \(\theta_{U,k}^*(A_k, B_k)\), \(\alpha_{jk}^*\) and \(\beta_{jk}^*\) solve potential market maker \(M_{jk}\)’s Problem (4), for \(j = 1, 2, \ldots, N_{Mk}\);

3. \(A_k^* := A_k(\alpha_k^*)\) and \(B_k^* := B_k(\beta_k^*)\) clear both the risky and the risk-free securities markets.

**III. The equilibrium with symmetric information**

For comparison, we first consider the symmetric information case where all agents observe the private signal. In this case, other investors can infer a type \(I\) investor’s liquidity shock from the equilibrium stock price. The equilibrium illiquidity arises from the market power of market makers.

Given \(A_k, B_k\) (\(k = 1, 2\)), and \(i \in \{I, U\}\), it can be shown the optimal demand schedule

\(^9\)Deviations by undercutting prices can be prevented by matching prices by other market makers in subsequent periods in a repeated-game setting. As in standard Cournot competition models, varying prices is not in the strategy space.
for a type $i$ investor is

$$
\theta^*_i(A_k, B_k) = \begin{cases} 
\frac{P^R_{ik} - A_k}{\delta \text{Var}[\tilde{V}_k|\tilde{s}_k]} & A_k < P^R_{ik}, \\
0 & B_k \leq P^R_{ik} \leq A_k, \\
-\frac{B_k - P^R_{ik}}{\delta \text{Var}[\tilde{V}_k|\tilde{s}_k]} & B_k > P^R_{ik},
\end{cases}
$$

(6)

where

$$
P^R_{ik} \equiv E[\tilde{V}_k|\tilde{s}_k] - \delta \text{Cov}(\tilde{V}_k, \tilde{N}|\tilde{s}_k)\tilde{X}_i - \delta \text{Var}[\tilde{V}_k|\tilde{s}_k]\tilde{\theta}_k, \ i = I, U, \ k = 1, 2
$$

(7)

is the reservation price of stock $k$ for a type-$i$ investor (i.e., the critical price such that non-market-makers buy (sell, respectively) security $k$ if and only if the ask price is lower (the bid price is higher, respectively) than this critical price). Under symmetric information, investors’ information sets are such that $\mathcal{I}_I = \mathcal{I}_U = \mathcal{I}_{M_k} = \{\tilde{X}_I, \tilde{s}_1, \tilde{s}_2\}$. Given the normality and independence of $\tilde{V}_k$ and $\tilde{\varepsilon}_k$, we have

$$
E[\tilde{V}_k|\tilde{s}_k] = \tilde{V}_k + \rho_{I_k} \tilde{s}_k, \ \text{Var}[\tilde{V}_k|\tilde{s}_k] = \rho_{I_k}^2 \sigma_{\tilde{\varepsilon}_k}^2, \ \text{Cov}(\tilde{V}_k, \tilde{N}|\tilde{s}_k) = (1 - \rho_{I_k})\sigma_{kN}, \ (8)
$$

where

$$
\rho_{I_k} = \frac{\sigma_{\tilde{V}_k}^2}{\sigma_{\tilde{V}_k}^2 + \sigma_{\tilde{\varepsilon}_k}^2}. \ (9)
$$

Let $\Delta_k$ denote the difference in the reservation prices of the $I$ and $U$ investors in the symmetric information case, i.e.,

$$
\Delta_k \equiv P^R_{I_k} - P^R_{U_k} = h_k\tilde{X}_I, \ (10)
$$

where $h_k = -\delta(1 - \rho_{I_k})\sigma_{kN}$. The following theorem provides the equilibrium prices and equilibrium stock demand.
Theorem 1

1. The equilibrium ask and bid prices are

\[ A^*_k := A(\alpha^*_k) = P^R_{Uk} + \frac{N_{Mk} N_I}{(N_k + 1)(N_{Mk} + 1)} \Delta_k + \frac{\Delta^+_k}{N_{Mk} + 1}, \]

(11)

\[ B^*_k := B(\beta^*_k) = P^R_{Uk} + \frac{N_{Mk} N_I}{(N_k + 1)(N_{Mk} + 1)} \Delta_k - \frac{\Delta^-_k}{N_{Mk} + 1}, \]

(12)

and we have \( A^*_k > P^*_k > B^*_k \), where

\[ P^*_k = \frac{N_I}{N_k} P^R_{Ik} + \frac{N_U}{N_k} P^R_{Uk} + \frac{N_{Mk}}{N_k} P^R_{Mk} = P^R_{Uk} + \frac{N_I}{N_k} \Delta_k \]

(13)

is the equilibrium price of a perfect competition equilibrium where market makers are also price takers. The bid-ask spread of stock \( k \) is

\[ A^*_k - B^*_k = \frac{|\Delta_k|}{N_{Mk} + 1}. \]

2. The equilibrium stock quantities demanded for stock \( k \) are

\[ \theta^*_{Ik} = \frac{N_{Mk}(N_U + N_{Mk} + 1)}{(N_k + 1)(N_{Mk} + 1)} \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]} \right), \quad \theta^*_{Uk} = -\frac{N_I N_{Mk}}{(N_k + 1)(N_{Mk} + 1)} \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]} \right), \]

(14)

\[ \theta^*_M = -\frac{N_I}{N_k + 1} \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]} \right); \text{ and} \]

the equilibrium quote depths are

\[ \alpha^*_k = \frac{N_I(N_{Mk} + N_U + 1)}{(N_k + 1)(N_{Mk} + 1)} \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]} \right)^-, \quad \frac{N_I N_U}{(N_k + 1)(N_{Mk} + 1)} \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]} \right)^+, \]

\[ \beta^*_k = \frac{N_I(N_{Mk} + N_U + 1)}{(N_k + 1)(N_{Mk} + 1)} \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]} \right)^+, \quad \frac{N_I N_U}{(N_k + 1)(N_{Mk} + 1)} \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]} \right)^-. \]
which implies that the equilibrium trading volume in stock $k$ is

$$N_{M_k}(a_k^* + \beta_k^*) = \frac{N_I N_{M_k} (N_{M_k} + 2 N_U + 1)}{(N_{M_k} + 1)(N_k + 1)} \left( \frac{|\Delta_k|}{\delta \text{Var}[\hat{V}_k|\hat{s}_k]} \right).$$

(15)

Theorem 1 implies that the bid and ask spread of stock $k$ is equal to the absolute value of stock $k$’s reservation price difference $|\Delta_k|$, divided by the number of market makers in stock $k$ plus one. This implies that the spread increases with $|\Delta_k|$ and decreases in the competition among market makers of stock $k$. However, because of the absence of the wealth effect due to the CARA preferences, competition among market makers in one stock market does not affect the prices or the spread of the other stock. Because the reservation price difference is proportional to the liquidity shock $\tilde{X}_I$, our model implies that after a large liquidity shock, spreads in both markets increase, even though there is no information asymmetry or any news about the fundamentals. In addition, if the large liquidity shock results in large sale pressure, both bid and ask prices go down, but the bid price goes down more. On date 1, stock price reverts to the fundamental payoffs. This pattern of prices and spreads resembles what happens in the so called “flash crash,” where there does not appear to be any information events. Our model shows that “flash crashes” can be caused by the limited risk bearing capacity of the liquidity suppliers.

The following proposition provides the correlation between the equilibrium stock returns, bid-ask spreads, and the correlation between the trading volume in both stocks.

**Proposition 1**  
1. The correlations between stock returns on both date 0 and date 1

$$\text{Corr}(P_1^*, P_2^*) = \frac{N_I^2 \sigma_\hat{X}^2 \prod_{k=1,2} \frac{h_k}{N_k}}{\prod_{k=1,2} \sqrt{\frac{\sigma_{V_k}^2}{\sigma_{V_k}^2 + \sigma_{\epsilon_k}^2} + \frac{N_I^2}{N_k} h_k^2 \sigma_\hat{X}^2}};$$

$$\text{Corr}(\tilde{V}_1 - P_1^*, \tilde{V}_2 - P_2^*) = \frac{N_I^2 \sigma_\hat{X}^2 \prod_{k=1,2} \frac{h_k}{N_k}}{\prod_{k=1,2} \sqrt{\frac{\sigma_{V_k}^2}{\sigma_{V_k}^2 + \sigma_{\epsilon_k}^2} + \frac{N_I^2}{N_k} h_k^2 \sigma_\hat{X}^2}};$$

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2. \( \text{Corr}(A_1^* - B_1^*, A_2^* - B_2^*) = \text{Corr}(N_{M_1}(\alpha_1^* + \beta_1^*), N_{M_2}(\alpha_2^* + \beta_2^*)) = 1; \)

3. For \( k = 1, 2 \), \( \text{Var}(A_k^* - B_k^*) = \frac{1 - 2/\pi}{(N_{M_k} + 1)^2} h_k^2 \sigma_X^2. \)

Part 1 of Proposition 1 implies that if the correlation between stock 1 and the non-traded asset has the same (opposite) sign as the correlation between stock 2 and the non-traded asset, then for any investor, the trading directions in the two stocks are the same (opposite). Proposition 1 also implies that it is investors’ correlated hedging trades in two stocks that lead to comovements in stocks’ equilibrium prices and comovements in stocks’ returns. For example, if one stock is not correlated with the non-traded asset, then the correlation between the two stocks reduces to zero. In addition, both the price correlation \( \text{Corr}(P_1^*, P_2^*) \) and the return correlation \( \text{Corr}(\tilde{V}_1 - P_1^*, \tilde{V}_2 - P_2^*) \) have the same sign as the liquidity demands correlation. As shown later, the presence of information asymmetry can make price correlation have the opposite sign to that of the liquidity demands correlation. Moreover, the magnitude of the correlation between the equilibrium stock prices, \( |\text{Corr}(P_1^*, P_2^*)| \), increases in \( \delta, \sigma_X, |\sigma_{kN}| \) (due to the increase of the covariances) and \( \sigma_{s_k} \), and decreases in \( \sigma_{\bar{V}_k} \). The magnitude of the correlation between the equilibrium stock returns on date 1, \( |\text{Corr}(\tilde{V}_1 - P_1^*, \tilde{V}_2 - P_2^*)| \), increases in \( \delta, \sigma_X, \) and \( |\sigma_{kN}| \), and decreases in \( \sigma_{\bar{V}_k} \) and \( \sigma_{s_k} \). This implies that the magnitude of the correlation between the stock prices on date 0 and the correlation between the stock returns on date 1 increases with the volatility of investor \( I \)’s hedging premium. Part 2 of Proposition 1 implies that, in the absence of asymmetric information, investors’ correlated trading in two stocks leads to positively correlated bid-ask spreads and trading volumes of stock 1 and stock 2, that is, the correlation between the liquidity is always positive under symmetric information, even when the trading demands are negatively correlated. We show later that these results no longer hold in the presence of asymmetric information. In addition, Part 3 of Proposition 1 implies that investors’ correlated trading in two stocks also leads to co-movements in liquidity risk (measured by the volatility of the spreads) of the
two stocks. For example, liquidity risks of both stocks increase in the uncertainty of the liquidity shock, $\sigma_X^2$. This provides an empirically testable implication: stocks subject to more volatile liquidity demands have higher liquidity risks and should command a greater risk premium when such liquidity demands are also correlated across a large set of stocks.

Another measure of illiquidity is the price impact of the informed investors’ trading in equilibrium, that is, Kyle’s lambda. As in Vayanos and Wang (2011), we measure the price impact using the regression coefficient of the equilibrium price on the informed investors’ trade, \( \lambda_k \),
\[
\lambda_k = \frac{Cov\left(\frac{A_k^* + B_k^*}{2}, N_I^*\theta_{ik}^*\right)}{Var\left(N_I^*\theta_{ik}^*\right)}.
\]
(16)

We have

**Proposition 2** The price impact is
\[
\lambda_k = \frac{\delta \rho_{1k}\sigma_{2k}^2 \left(N_{MK}N_I + \frac{1}{2}(N_k + 1)\right)}{N_I^*N_{MK}(N_U + N_{MK} + 1)}.
\]
(17)

Proposition 2 implies that, in the absence of asymmetric information, the price impact of stock \( k \) increases in \( \delta, \sigma_{zk} \) and \( \sigma_{V_k} \). In addition, the price impact of stock \( k \) does not depend on the information quality and payoff volatility of the other stock. We will show later the price impact of stock \( k \) changes if the information quality or the payoff volatility of the other stock changes in the presence of asymmetric information.
IV. The equilibrium with asymmetric information

We now assume that only informed investors observe the private signals \( \hat{s}_1 \) and \( \hat{s}_2 \). Therefore, informed investors’ trades can be motivated by both liquidity shock and private information.

From (7), the reservation price of stock \( k \) for \( I \) investors is

\[
P_{Ik}^R = \bar{V}_k + \hat{S}_k - \rho_{Ik} \delta \sigma_{\epsilon_k}^2 \theta_k,
\]

where \( \hat{S}_k = \rho_{Ik} \hat{s}_k + h_k \bar{X}_I \) and \( \rho_{Ik} \) is the weight on the private signal, as defined in (9).

Since the informed investor’s demand of stock \( k \) is a monotonically increasing function of \( \hat{S}_k \), his order reveals the value of \( \hat{S}_k \) to market makers \( M_k \). Thus we conjecture that the equilibrium prices depend on \( \hat{S}_k \). Let \( B_k^* \) and \( A_k^* \) be the equilibrium bid price and ask price for stock \( k \) in the presence of asymmetric information. Since the uninformed investors can then infer the value of \( \hat{S}_k \) from the market prices, the information set for uninformed investors and market makers is \( \mathcal{I}_U = \mathcal{I}_M = \{A_1^*, A_2^*, B_1^*, B_2^*\} = \{\hat{S}_1, \hat{S}_2\} \). Let \( \sigma_{H_k} \equiv -h_k \sigma_X \) be non-zero, with \( \sigma_{H_k}^2 \) being the variance of the premium of stock \( k \) for hedging (i.e., \(-\delta \text{Cov}(\hat{V}_k, \bar{N}[\hat{s}_k] \bar{X}_I)\)). Then the conditional expectation and variance of \( \hat{V}_k \) are

\[
E[\hat{V}_k | \hat{S}_1, \hat{S}_2] = \bar{V}_k + \frac{\rho_{Ik} \sigma_{H_k}^2 \left( \sigma_{\bar{S}_1}^2 \hat{S}_k - \text{Cov}(\hat{S}_1, \hat{S}_2) \hat{S}_k \right)}{\sigma_{\bar{S}_1}^2 \sigma_{\bar{S}_2}^2 - \text{Cov}^2(\hat{S}_1, \hat{S}_2)}.
\]

\(^{10}\)If the informed could only observe a private signal about stock 1 but no private signal about stock 2, then the uninformed would be able to figure out the liquidity shock from the equilibrium price of stock 2 and then figure out the informed’s private signal about stock 1 from the equilibrium price of stock 1. Therefore, the asymmetric information case for stock 1 is reduced to the symmetric information case. The assumption that the investors with liquidity shocks observe private signals about both stocks is a simple way to keep the private information from fully revealing. For example, if those without liquidity shocks observe a private signal about one of the two stocks, then the asymmetric information case for this stock is reduced to symmetric information case. For the more general case where both investors are endowed with some non-traded risky asset and some investors observe a signal about the payoff of stock 1 while others observe a signal about the payoff of stock 2, there are 63 subcases. We can still obtain closed-form solutions. However, the general case does not add new economics and therefore we focus on the simplest case where informed investors observe signals about both stocks to get clear intuitions.
\[
V \text{ar} [\hat{V}_k | \hat{S}_1, \hat{S}_2] = \sigma_{V_k}^2 \left( 1 - \frac{\rho_{\hat{S}_k}^2 \sigma_{\hat{S}_k}^2}{\sigma_{\hat{S}_1}^2 \sigma_{\hat{S}_2}^2 - \text{Cov}(\hat{S}_1, \hat{S}_2)} \right),
\]

where \( \sigma_{\hat{S}_k}^2 = \rho_{\hat{S}_k}^2 (\sigma_{V_k}^2 + \sigma_{\varepsilon_k}^2) + h_{k}^2 \sigma_{\varepsilon_k}^2 \) and \( \text{Cov}(\hat{S}_1, \hat{S}_2) = h_{1} h_{2} \sigma_{\varepsilon_k}^2 \).

It can be shown that
\[
V \text{ar} [\hat{V}_k | \hat{S}_1, \hat{S}_2] = \frac{\sigma_{V_k}^2 \sigma_{\varepsilon_k}^2}{\sigma_{V_k}^2 + \sigma_{\varepsilon_k}^2} + \frac{\sigma_{\hat{S}_k}^2}{1 + \sum_{j=1,2} \frac{\sigma_{\hat{S}_j}^2}{\rho_{\varepsilon_j} \sigma_{\varepsilon_j}^2}}.
\]

Therefore, \( V \text{ar} [\hat{V}_k | \hat{S}_1, \hat{S}_2] \) decreases in \( \sigma_{\varepsilon, -k} \) (\( -k \) indicates the other stock). Intuitively, as \( \sigma_{\varepsilon, -k} \) increases, the informed put less weight on the private signal about the other stock in their trades in the other stock and thus these trades convey more information about the hedging demand, which helps the uninformed increase the precision of the estimation of the stock \( k \) payoff. Because a change in \( \sigma_{\varepsilon, -k} \) does not affect the quality of aggregate information about stock \( k \) (i.e., conditional on all the information in the economy about stock \( k \)). Therefore, \( \tau_{\varepsilon, -k} = \frac{1}{\sigma_{\varepsilon, -k}^2} \) is a measure of information asymmetry of stock \( k \).

Note that in the presence of asymmetric information, the conditional distribution of the stock payoff now depends on the characteristics of the other market, including the information about the other stock, although these two stocks are independent. For example, suppose \( \text{Cov}(\hat{S}_1, \hat{S}_2) > 0 \), then the conditional expected payoff of stock 1 decreases with the signal \( \hat{S}_2 \) because the higher \( \hat{S}_2 \) is, the more likely \( \hat{X}_f \) is high and thus the contribution of \( \hat{V}_1 \) in driving \( \hat{S}_1 \) is more likely to be small. In addition, \( V \text{ar}[\hat{V}_k | \hat{S}_1, \hat{S}_2] > V \text{ar}[\hat{V}_k | \hat{s}_k] \). Let
\[
\nu_k = \frac{V \text{ar}[\hat{V}_k | \hat{S}_1, \hat{S}_2]}{V \text{ar}[\hat{V}_k | \hat{s}_k]} > 1
\]
be the ratio of the conditional variance of stock \( k \) payoff of the uninformed to that of the informed. Then, the reservation price of stock \( k \) for a \( U \) investor and an \( M_k \) investor is
\[
P_{U_k}^R = P_{M_k}^R = E[\hat{V}_k | \hat{S}_1, \hat{S}_2] - \delta V \text{ar}[\hat{V}_k | \hat{S}_1, \hat{S}_2] \theta_k.
\]
Thus the difference in the reservation prices for stock $k$ is:

$$
\Delta_k = P_{Ik}^R - P_{Uk}^R
= h_k \hat{X}_I + (E[\tilde{V}_k|\hat{s}_k] - E[\tilde{V}_k|\hat{S}_1, \hat{S}_2]) + \delta \hat{\theta}_k (Var[\tilde{V}_k|\hat{S}_1, \hat{S}_2] - Var[\tilde{V}_k|\hat{s}_k])
= h_k E[\tilde{X}_I|\hat{S}_1, \hat{S}_2] + \delta h_k^2 Var[\tilde{X}_I|\hat{S}_1, \hat{S}_2] \hat{\theta}_k, \tag{24}
$$

which can be simplified into

$$
\Delta_k = h_k (\hat{z} + \delta h_k C \hat{\theta}_k), \tag{25}
$$

where

$$
E[\tilde{X}_I|\hat{S}_1, \hat{S}_2] = \hat{z} = C \sum_{j=1,2} \frac{h_j}{\rho_{lj} \sigma_{V_j}^2} \hat{S}_j \sim N(0, \sigma_z^2), \tag{26}
$$

$$
\sigma_z^2 = \sigma_X^2 - C, \tag{27}
$$

$$
Var[\tilde{X}_I|\hat{S}_1, \hat{S}_2] = C = \left(1 + \sum_{j=1,2} \frac{\sigma_{H_j}^2}{\rho_{lj} \sigma_{V_j}^2} \right)^{-1} \sigma_X^2, \tag{28}
$$

(24) shows that the reservation price difference is equal to the uninformed’s conditional expectation of the hedging premium of the informed plus the risk premium from the estimation of this hedging premium. Intuitively, if the uninformed had the same hedging demand as the informed, then the uninformed would simply mimic what the informed do in trading, and thus their reservation price would be exactly the same as the informed even though they do not observe either the liquidity shock $\hat{X}_I$ or the private information $\hat{s}_k$. Therefore the reservation price of the uninformed when they do not have any hedging demand must be equal to the reservation price of the informed minus the conditional expectation of the hedging premium of the informed minus the risk premium for this estimation. Because the reservation price differences in both markets are proportional to the same conditional expectation of the liquidity shock $E[\tilde{X}_I|\hat{S}_1, \hat{S}_2]$, the reservation price differences in the two markets are correlated and this drives the liquidity commonality,
as we will analyze in details later.

Define
\[ C_{I_k} \equiv \frac{N_{M_k}(N_U + N_{M_k} + 1)}{(N_{M_k} + 1)(N_k + 1)}, \quad C_{U_k} \equiv \frac{\nu_k N_{M_k} N_I}{(N_{M_k} + 1)(N_k + 1)}. \] (29)

The following theorem provides the equilibrium bid and ask prices and equilibrium stock holdings in the presence of asymmetric information.

**Theorem 2**

1. The equilibrium bid and ask prices are
\[ A^*_k = P^R_{U_k} + C_{U_k} \Delta_k + \frac{\Delta^+_k}{N_{M_k} + 1}, \]
\[ B^*_k = P^R_{U_k} + C_{U_k} \Delta_k - \frac{\Delta^-_k}{N_{M_k} + 1}, \]
and we have \( A^*_k > P^*_k > B^*_k \), where
\[ P^*_k = P^R_{U_k} + \frac{\nu_k N_I}{N_k} \Delta_k \] (30)
is the equilibrium price of a perfect competition equilibrium where market makers are also price takers. The bid and ask spread of stock \( k \) is
\[ A^*_k - B^*_k = \frac{|\Delta_k|}{N_{M_k} + 1}; \] (31)

2. The equilibrium stock quantities demanded are
\[ \theta^*_{I_k} = C_{I_k} \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]}, \quad \theta^*_{U_k} = -C_{U_k} \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{S}_1, \hat{S}_2]}, \quad \theta^*_{M_k} = \frac{(N_{M_k} + 1)}{N_{M_k}} \theta^*_{U_k}; \] (32)
the equilibrium quote depths are
\[ \alpha^*_k = \frac{N_I C_{I_k}}{N_{M_k}} \left( \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{s}_k]} \right)^+ + \frac{\nu_k N_U}{N_U + N_{M_k} + 1} \left( \frac{\Delta_k}{\delta \text{Var}[V_k|\hat{S}_1, \hat{S}_2]} \right)^- \right), \] (33)
and

\[ \beta_k^* = \frac{N_I C_{l_k}}{N_{M_k}} \left( \frac{\nu_k N_U}{N_U + N_{M_k} + 1} \left( \frac{\Delta_k}{\delta \text{Var}[\hat{V}_k|\hat{S}_1, \hat{S}_2]} \right)^+ + \left( \frac{\Delta_k}{\delta \text{Var}[\hat{V}_k|\hat{S}_k]} \right)^- \right), \tag{34} \]

which implies that the equilibrium trading volume is

\[ N_{M_k} (\alpha_k^* + \beta_k^*) = \frac{N_{M_k} N_I (N_{M_k} + 2 N_U + 1)}{(N_{M_k} + 1)(N_k + 1)} \left( \frac{|\Delta_k|}{\delta \text{Var}[\hat{V}_k|\hat{S}_k]} \right). \tag{35} \]

Theorem 2 implies that, as in the symmetric information case, the bid-ask spread of stock \( k \) is equal to the absolute value of the reservation price difference for stock \( k \), divided by \( N_{M_k} + 1 \). Different from the symmetric information case, the equilibrium prices of stock \( k \) depend on both signals \( \hat{S}_1 \) and \( \hat{S}_2 \) even though stocks are independent. This is because the signal about the other stock affects the equilibrium prices of the other stock which in turn affects the inference about this stock due to the correlated liquidity demand component. This finding can help explain why private information about an irrelevant stock can affect a stock’s price and liquidity. Since \( |\Delta_k| \) depends on both \( \hat{S}_1 \) and \( \hat{S}_2 \), the bid-ask spread of stock \( k \) depends on both \( \hat{S}_1 \) and \( \hat{S}_2 \). From (25), (26), and (31), we get

**Proposition 3** The bid-ask spread of stock \( k \) increases linearly in \( -\hat{S}_{-k} \) if

1. \( \Delta_k < 0 \) and \( \sigma_{1N} \sigma_{2N} > 0 \); or
2. \( \Delta_k > 0 \) and \( \sigma_{1N} \sigma_{2N} < 0 \).

Proposition 3 implies that stock \( k \)’s bid-ask spread may significantly increase if there is a large negative signal \( \hat{S}_{-k} \) about stock \( -k \)’s payoff even though there is no bad news about the stock \( k \)’s payoff and no changes in liquidity shocks. Therefore, in our model, liquidity crash can be triggered by fundamental news about an independent stock. The channel is through the correlated demand.
Because correlated hedging demands lead to correlated $\hat{S}_1$ and $\hat{S}_2$, Theorem 2 implies that, similar to the symmetric information case, investors’ correlated trading in two stocks leads to correlated bid-ask spreads of stock 1 and stock 2. More specifically, we have

$$\text{Corr}(A_1^* - B_1^*, A_2^* - B_2^*) = \text{Corr}(|\Delta_1|, |\Delta_2|),$$

(36)

and

$$\text{Corr}(N_{M_1}(\alpha_1^* + \beta_1^*), N_{M_2}(\alpha_2^* + \beta_2^*)) = \text{Corr}(|\Delta_1|, |\Delta_2|).$$

(37)

Then we have

**Proposition 4** Let $P_{1s}^*$ and $P_{2s}^*$ be the prices for the symmetric information case.

1. With asymmetric information,

$$\text{Corr}(A_1^* - B_1^*, A_2^* - B_2^*) < 0,$$

if

$$-4 \frac{1-\rho_{2j}}{1-\rho_{1j}} \sigma_{2N} \widetilde{\theta}_2 < \sigma_{1N} \widetilde{\theta}_1 < -\frac{1-\rho_{2j}}{1-\rho_{1j}} \sigma_{2N} \widetilde{\theta}_2,$$

and

$$0 < \sigma_X^2 < \min \left\{ \frac{\delta^2 \sigma_{1N}^2 (1 - \rho_{1j})^2 \widetilde{\theta}_2^2 - 4C_3}{4 \delta^2 C_3^2}, \frac{\delta^2 \sigma_{2N}^2 (1 - \rho_{2j})^2 \widetilde{\theta}_2^2 - 4C_3}{4 \delta^2 C_3^2} \right\},$$

(38)

where

$$C_3 = \sum_{j=1,2} \frac{\sigma_{2N}^2 (1 - \rho_{1j})^2 (\sigma_{V,j}^2 + \sigma_{\delta,j}^2)}{\sigma_{V,j}^4}. $$

(39)

2. (a) $\text{Corr}(P_{1s}^*, P_{2s}^*) < 0$, if and only if $0 < \delta^2 \sigma_X^2 < C_2$, where $C_2$ is defined in (53) in the Appendix;

(b) If $\delta^2 \sigma_X^2 > C_2$ and $\sigma_{1N} \sigma_{2N} < 0$, then $\text{Corr}(P_{1s}^*, P_{2s}^*) > 0 > \text{Corr}(P_{1s}^*, P_{2s}^*)$;

(c) If $0 < \delta^2 \sigma_X^2 < C_2$ and $\sigma_{1N} \sigma_{2N} > 0$, then $\text{Corr}(P_{1s}^*, P_{2s}^*) > 0 > \text{Corr}(P_{1s}^*, P_{2s}^*)$;
Different from the symmetric information case, Proposition 4 implies that, as illustrated in Figure 1, the correlation between the liquidity of two stocks can be negative, i.e., the liquidity of stock 1 may improve when the liquidity of stock 2 deteriorates and vice versa. This happens usually when the correlation between hedging demands in the two stocks is negative. To understand the intuition, note that by (25), we have for \( k = 1, 2 \),

\[
|\Delta_k| = |h_k||\hat{z} + C\delta h_k \tilde{\theta}_k|.
\]

First, we provide a mathematical intuition. For

\[
\hat{z} \in \left( C\delta \min(-h_1 \tilde{\theta}_1, -h_2 \tilde{\theta}_2), C\delta \max(-h_1 \tilde{\theta}_1, -h_2 \tilde{\theta}_2) \right),
\]

\( |\Delta_1| \) and \( |\Delta_2| \) move in the opposite directions as \( \hat{z} \) changes, and move in the same directions outside this range. This implies that if \( \sigma_{1N}\sigma_{2N} < 0 \), the spreads move in the opposite directions around \( \hat{z} = 0 \). Since the expected value of \( \hat{z} \) is zero, if the volatility of \( \hat{z} \) is small, then most of the states concentrate around 0 and therefore the spreads are overall negatively correlated. For the main economic intuition, suppose \( \sigma_{1N} < 0 \) and \( \sigma_{2N} > 0 \) and the informed have a positive liquidity shock \( \hat{X}_I \) and small signal \( |\hat{s}_1| \) and \( \hat{s}_2 = 0 \) so that the informed buy stock 1 and also buy stock 2 due to the estimation risk premium of the uninformed that lowered the uninformed’s reservation price, i.e., \( \Delta_1 > 0 \) and \( \Delta_2 > 0 \) (or equivalently, \( -\delta h_1 C\tilde{\theta}_1 < \hat{z} < -\delta h_2 C\tilde{\theta}_2 \)). When \( \hat{s}_1 \) increases, relative to the informed, the uninformed attribute less to an increase in \( \hat{s}_1 \), which implies that the reservation price difference for stock 1 increases and so does the spread of stock 1. When \( \hat{s}_1 \) increases, the uninformed also attribute a portion of the increase to an increase in the hedging demand for stock 1. Accordingly, they increase the estimation of the hedging demand component in the informed’s trade in stock 2. Due to the negatively correlated hedging demands, this implies that for a given signal \( \hat{S}_2 \), they increase the estimated expected payoff of stock 2,
Figure 1: The correlation coefficient between the equilibrium bid and ask spreads for stock 1 and stock 2 against $\sigma_{e1}$, $\sigma_{V1}$, $\sigma_{1N}$, $\delta$ and $\sigma_{X}$. The figures of the correlation against $\sigma_{e2}$, $\sigma_{V2}$, and $\sigma_{2N}$ are similar to those against $\sigma_{e1}$, $\sigma_{V1}$, and $\sigma_{1N}$, respectively. The default parameter values are: $\bar{\theta}_1 = \bar{\theta}_2 = 2$, $\delta = 2$, $\sigma_{V1} = 0.4$, $\sigma_{V2} = 0.4$, $\sigma_{e1} = \sigma_{e2} = 0.4$, and $\sigma_{1N} = -0.8$, $\sigma_{2N} = 0.4$, $\sigma_{X} = 0.1$, $N_I = 100$, $N_{M1} = 10$, $N_{M2} = 10$, $N_U = 1000$. 
and thus the reservation price difference for stock 2 and hence the spread of stock 2 goes down. Therefore, the spreads for the two stocks move in the opposite directions. If the variance of \( \hat{z} \) is small, then this opposite comovement dominates and the spreads become negatively correlated.

Consistent with the above intuition, Figure 1 shows that if \( \sigma_{1N}\sigma_{2N} < 0 \), the correlation between the spreads can be negative when \( \sigma_X \) is small or when \( \delta \) is large. A decrease in \( \sigma_X \) reduces \( \sigma_2^2 \), as implied by (27) and an increase in \( \delta \) widens the range within which the two spreads move in the opposite directions. Proposition 4 provides an explicit sufficient condition for the correlation between the spreads to be negative and for the magnitude of the correlation between the spreads to be higher with symmetric information than with asymmetric information. The first subfigure of Figure 1 shows that the correlation between the spreads can decrease with information asymmetry (\( \frac{1}{\sigma_{11}} \) is a measure of information asymmetry of stock 2). The nonmonotonicity of the spreads correlation in \( \sigma_{1N} \) is because \( \sigma_{1N} \) increases both the hedging effect and the opposing information filtering effect. For large or small \( \sigma_{1N}^2 \), the hedging effect dominates and thus spreads correlation is positive as in the symmetric information case. For medium \( \sigma_{1N}^2 \), the information filtering effect can dominate if \( \sigma_{1N}\sigma_{2N} < 0 \) and thus spreads correlation becomes negative.

Part 2 of Proposition 4 implies that the presence of asymmetric information can change the sign of the correlation between the prices. Consider, for example, the case where the correlation of hedging trades is positive (i.e., \( \sigma_{1N}\sigma_{2N} > 0 \)). The price correlation in the absence of asymmetric information is positive because the positively correlated demands drive the prices up or down in the same directions, as shown before. In the presence of asymmetric information, as the signal \( \hat{s}_2 \) increases, the uninformed’s conditional expectation of stock 2’s payoff increases because the uninformed attributes a portion of the increase in \( \hat{s}_2 \) to an increase in the informed’s private signal \( \hat{s}_2 \) about stock 2’s payoff. However, the uninformed’s conditional expectation of stock 1’s payoff decreases because the uninformed also attribute a portion of the increase in \( \hat{s}_2 \) to an increase in the ini-
Figure 2: The correlation coefficient between the equilibrium price for stock 1 and stock 2 against $\sigma_{e1}, \sigma_{V1}, \sigma_{1N}, \delta$ and $\sigma_X$. The figures of the correlation against $\sigma_{e2}, \sigma_{V2}$, and $\sigma_{2N}$ are similar to those against $\sigma_{e1}, \sigma_{V1}$, and $\sigma_{1N}$, respectively. The default parameter values are: $\bar{\theta}_1 = \bar{\theta}_2 = 2$, $\delta = 2, \sigma_{V_1} = \sigma_{V_2} = 0.4, \sigma_{e1} = \sigma_{e2} = 0.4$, and $\sigma_{1N} = -0.8, \sigma_{2N} = 0.4, \sigma_X = 0.4, N_I = 100, N_{M1} = 10, N_{M2} = 10, N_U = 1000$. 

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formed’s hedging demand for stock 2, in turn the uninformed increase the estimate of the hedging demand component in the informed’s trade in stock 1 due to the positive correlation of the hedging demands, which lowers the estimate of the expected payoff of stock 1. Therefore, this information filtering effect introduces a negative correlation component of the conditional expected payoff of the two stocks. When this filtering effect dominates (e.g., when \( \delta^2 \sigma_X^2 < C_2 \)), the price correlation becomes negative. More generally, as \( \sigma_{hk}^2 \) or \( \sigma_{V_k}^2 \) increases or as \( \sigma_{s_k}^2 \) decreases, the uninformed attribute a greater proportion of a change in \( \hat{s}_{-k} \) to the hedging demand component and thus the filtering effect increases.

Figure 2 illustrates some cases where this reversal occurs. For example, the right subfigure on \( \sigma_{1N} \) in Figure 2 shows that, in contrast to the symmetric information case, even when the correlation of the hedging demands in the two markets is positive (\( \sigma_{1N} \sigma_{2N} > 0 \)), the correlation of prices can be negative. Similarly, for small \( \sigma_{\epsilon 1} \) and large \( \sigma_X^2 \) or \( \delta \), the price correlation becomes positive even though the hedging demands are negatively correlated.

The non-monotonicity of the price correlation in the parameters considered in Figure 2 comes from the fact that these parameters affect both the effect of correlated hedging and the effect of the information filtering.

Figure 3: The spreads against \( \hat{X}_I \). The default parameter values are: \( \bar{\theta}_1 = \bar{\theta}_2 = 2 \), \( \delta = 2 \), \( \sigma_{V_1} = \sigma_{V_2} = 0.4 \), \( \sigma_{\epsilon 1} = \sigma_{\epsilon 2} = 0.4 \), \( \hat{s}_1 = -0.1 \), \( \hat{s}_2 = 0.2 \), and \( \sigma_{1N} = 0.2 \), \( \sigma_{2N} = 0.4 \), \( \sigma_X = 0.4 \), \( \hat{N}_{M1} = 10 \), \( \hat{N}_{M2} = 10 \).
Brunnermeier and Pedersen (2009) and some other supply-side models imply that large market declines or high volatility adversely affect the funding liquidity of financial intermediaries that act as liquidity suppliers in financial markets. As a consequence, these intermediaries reduce the provision of liquidity across many securities, which results in a decrease in market liquidity and an increase in commonality in liquidity. Thus these supply-side theories predict that market liquidity is worse and liquidity commonality is greater in more volatile markets and in significantly declined markets. Some empirical studies use this prediction as a test of these supply side theories (e.g., Karolyi, Lee, and van Dijk (2011)). We show next that these liquidity and liquidity commonality patterns can also be consistent with our demand side theory. Therefore, finding that market liquidity is worse and liquidity commonality is greater in more volatile markets and in significantly declined markets does not imply that the empirical evidence supports the supply-side funding liquidity theory.

With large liquidity shocks, the relative importance of information asymmetry in determining spreads decreases and accordingly the correlation gets closer to that of the symmetric information case which is 1. Thus our model predicts that if a large decline in market prices is driven by a large liquidity shock, the liquidity correlation should go up. Therefore, a finding that liquidity correlation goes up when market goes down can also be consistent with our demand side theory. To confirm this, in Figure 4 we plot the ratio of spread correlation conditional on $\hat{X}_I < -\sigma_X$ (CC) to the unconditional spread correlation (UC) against the covariance $\sigma_{1N}$ between stock 1 payoff and the nontraded asset payoff. When $\hat{X}_I < 0$ and $\sigma_{1N} < 0$, the hedging demand is negative. Therefore, with more negative $\hat{X}_I$, the market price is lower. Figure 4 then suggests that indeed when market price is low, liquidity correlation can be significantly greater, as high as more than 4 times that of the unconditional correlation.

As noted after Theorem 1, a large negative liquidity demand shock can cause the spreads of both stocks to go up because it drives up the magnitude of reservation price
Figure 4: The conditional correlation to unconditional correlation ratio against $\sigma_{1N}$. The default parameter values are: $\theta_1 = \theta_2 = 2$, $\delta = 2$, $\sigma_{V_1} = \sigma_{V_2} = 0.4$, $\sigma_{e1} = \sigma_{e2} = 0.4$, $\sigma_{2N} = 0.4$, and $\sigma_X = 0.4$.

difference and can also drive down market prices because of the large selling pressure, as also illustrated in Figure 3. This suggests that liquidity can also get worse in declined markets in our model. We plot the expected spread against the volatility of market prices (measured by the volatility of mid-point price) when we change the liquidity shock volatility $\sigma_X$. As the liquidity shock volatility increases, both the price volatilities and the expected spreads increase and as a result, Figure 5 shows that average spreads can be higher when market volatilities are higher.

In Figure 6 we plot the spread correlation against the volatility of equilibrium prices for stocks 1 and 2 as we change the volatility of liquidity shock $\sigma_X$. Figure 6 shows that if changes in spread correlation and market volatility are caused by changes in liquidity shock volatility, then our model also predicts that spread correlation increases as market volatility increases, consistent with the empirical finding that commonality in liquidity is greater in countries with and during times of high market volatility (e.g., Karolyi, Lee, and van Dijk (2011)).
Figure 5: The expected spreads against price volatility. The default parameter values are: \( \tilde{\theta}_1 = \tilde{\theta}_2 = 2, \delta = 2, \sigma_{V_1} = \sigma_{V_2} = 0.4, \sigma_{e1} = \sigma_{e2} = 0.4, \sigma_{1N} = -0.8, \sigma_{2N} = 0.4, N_I = 100, \ N_{M1} = 10, \ N_{M2} = 10, \) and \( N_U = 1000. \)

Figure 6: Spread correlation against price volatility. The default parameter values are: \( \tilde{\theta}_1 = \tilde{\theta}_2 = 2, \delta = 2, \sigma_{V_1} = \sigma_{V_2} = 0.4, \sigma_{e1} = \sigma_{e2} = 0.4, \sigma_{1N} = -0.8, \sigma_{2N} = 0.4, N_I = 100, \ N_{M1} = 10, \ N_{M2} = 10, \) and \( N_U = 1000. \)
Figure 7: The volatility of bid-ask spreads of stock 1 and stock 2 against $\sigma_X$ and $\delta$. The default parameter values are: $\bar{\theta}_1 = \bar{\theta}_2 = 2, \delta = 2, \sigma_{v_1} = \sigma_{v_2} = 0.4, \sigma_{\epsilon_1} = \sigma_{\epsilon_2} = 0.4$, and $\sigma_{1N} = -0.8, \sigma_{2N} = 0.4, \sigma_X = 0.4, N_I = 100, N_{M1} = 10, N_{M2} = 10, \hat{N}_U = 1000.$
Figure 8: The volatility of bid-ask spreads of stock 1 and stock 2 against $\sigma_{\epsilon_1}$, and $\sigma_{\epsilon_2}$. The default parameter values are: $\hat{\theta}_1 = \hat{\theta}_2 = 2$, $\delta = 2$, $\sigma_{V_1} = \sigma_{V_2} = 0.4$, $\sigma_{\epsilon_1} = \sigma_{\epsilon_2} = 0.4$, and $\sigma_{1N} = -0.8$, $\sigma_{2N} = 0.4$, $\sigma_X = 0.4$, $N_I = 100$, $N_M1 = 10$, $N_M2 = 10$, $N_U = 1000$. 
We now study the co-movements in liquidity risks of the two stocks. We have

\[ Var(A_k^* - B_k^*) = \frac{h_k^2 \sigma_z^2}{(N_{M_k} + 1)^2} \left( 1 + z_k^2 - z_k^2 \left( \frac{2n(z_k)}{z_k} + 2N(z_k) - 1 \right)^2 \right), \]  

(40)

where \( n(\cdot) \) and \( N(\cdot) \) are respectively the pdf and cdf of the standard normal distribution and \( \sigma_z \) is as defined in (27), \( z_k = -\frac{C_{\delta_h k}}{\sigma_z} \), and \( C \) is as defined in (28). As in the symmetric information case, investors’ correlated trading may lead to co-movements in liquidity risks measured by volatilities of bid-ask spreads. As illustrated in Figure 7, for example, the volatilities of the bid-ask spreads of both stocks increase in \( \sigma_X^2 \) and \( \delta \). This provides another empirically testable implication: liquidity risks of stocks that are affected by a liquidity shock increase with the volatility of the liquidity shock.

In addition, in contrast to the symmetric information case, because \( \sigma_z^2 \) depends on the parameter values of both stocks, the liquidity risk of one stock is affected by the characteristics of the other stock. For example, Figure 8 shows that the liquidity risk of stock 2 can be affected nonmonotonically by the quality of the private information about stock 1’s final payoff, through the information filtering channel.

Next we examine the price impact \( \lambda_k \) of the informed investors’ trading in equilibrium. We have

**Proposition 5** The price impact

\[ \lambda_k = \frac{\delta_1 \rho_{1k} \sigma_{\varepsilon_k}^2}{N_1 C_{1k}} \left( \frac{\rho_{11} \rho_{12} \sigma_{V_1}^2 \sigma_{V_2}^2}{\rho_{11} \sigma_{V_1}^2 + \rho_{12} \sigma_{V_2}^2} + C_{Uk} + \frac{1}{2(N_{M_k} + 1)} \right), \]  

and \( \frac{\partial \lambda_k}{\partial \sigma_{\varepsilon,-k}^2} < 0 \)  

(41)

As illustrated in Figures 9 and 10, in contrast to the symmetric information case, the price impact for stock \( k \) may be nonmonotonic in \( \sigma_{\varepsilon_k} \). In addition, the price impact for stock \( k \) decreases in \( \sigma_{\varepsilon,-k}, \delta, \) and \( \sigma_X \). The non-monotonicity of the price impact of stock \( k \) in \( \sigma_{\varepsilon_k}^2 \) is driven by two effects: an information quality effect and an information filtering...
Figure 9: The price impact of stock 1 and stock 2 against $\sigma_{V1}$, $\sigma_{\epsilon1}$, and $\sigma_{\epsilon2}$. The default parameter values are: $\bar{\theta}_1 = \bar{\theta}_2 = 2$, $\delta = 2$, $\sigma_{V1} = \sigma_{V2} = 0.4$, $\sigma_{\epsilon1} = \sigma_{\epsilon2} = 0.4$, and $\sigma_{1N} = -0.8$, $\sigma_{2N} = 0.4$, $\sigma_X = 0.4$, $N_I = 100$, $N_{M1} = 10$, $N_{M2} = 10$, $N_U = 1000$. 
Figure 10: The price impact of stock 1 and stock 2 against $\sigma_X$, and $\delta$. The default parameter values are: $\bar{\theta}_1 = \bar{\theta}_2 = 2$, $\delta = 2$, $\sigma_{V_1} = \sigma_{V_2} = 0.4$, $\sigma_{e1} = \sigma_{e2} = 0.4$, and $\sigma_{1N} = -0.8$, $\sigma_{2N} = 0.4$, $\sigma_X = 0.4$, $N_I = 100$, $N_{M1} = 10$, $N_{M2} = 10$, $N_U = 1000$. 
effect. The price impact of stock $k$ decreases in the quality of the information about stock $k$’s payoff and increases in the asymmetric information filtering effect (across the informed and the uninformed). As $\sigma_{ek}$ increases, the information quality decrease and the asymmetric information filtering effect also decreases. Therefore, as $\sigma_{ek}^2$ increases, the price impact of stock $k$ increases due to the information quality effect and decreases due to the asymmetric information filtering effect. When $\sigma_{ek}^2$ is small, the information quality effect dominates and thus the price impact of stock $k$ increases in $\sigma_{ek}^2$. When $\sigma_{ek}^2$ is large, the asymmetric information filtering effect dominates and thus the price impact of stock $k$ decreases in $\sigma_{ek}^2$. Interestingly, the price impact of stock $k$ decreases in the quality of the information about the payoff of the other stock. This is because uninformed investors can estimate the hedging demand better from the equilibrium price when informed investors’ private signal about the other stock’s payoff is less precise and thus uninformed investors’ uncertainty about stock $k$’s payoff decreases. This implies that the price impact of stock 1 may increase when the price impact of stock 2 decreases.

V. Conclusions

In this paper, we develop the first demand-driven liquidity commonality theory. We solve equilibrium bid and ask prices, bid and ask depths, and inventory levels in closed-form. In the absence of asymmetric information, illiquidity correlation is always positive even when liquidity demands are negatively correlated. In addition, price correlations always have the same sign as the liquidity demand correlation. In the presence of asymmetric information, however, both illiquidity correlation (measured by the correlation of bid-ask spreads) and price correlation can become negative due to information filtering effect. In addition, information about one security can affect the price and the liquidity of another security even though they have independent payoffs. Our model can also help explain why “flash crashes” in liquidity and prices can be caused by fundamental news about
another stock that has independent payoff and why market liquidity can be worse and liquidity commonality can be greater in significantly declined markets and in more volatile markets.
Appendix

Proof of Theorem 1: This proof follows from the proof of Theorem 2, where \( E[\hat{\tilde{V}}_k|\hat{\tilde{S}}_1, \hat{\tilde{S}}_2] \) is replaced by \( E[\hat{V}_k|\hat{s}_k] \) and \( \text{Var}[\hat{\tilde{V}}_k|\hat{\tilde{S}}_1, \hat{\tilde{S}}_2] \) is replaced by \( \text{Var}[\hat{V}_k|\hat{s}_k] \). Q.E.D.

Proof of Proposition 1:

\[
\text{Cov}(P^*_1, P^*_2) = \text{Cov}(\hat{V}_1 - P^*_1, \hat{V}_2 - P^*_2) = N_I^2 \delta^2 \sigma^2_X \prod_{k=1,2} \frac{\sigma_{kN}}{N_k} (1 - \rho_{ik});
\]

\[
\text{Cov}(\theta^*_{11}, \theta^*_{12}) = \sigma^2_X \prod_{k=1,2} \frac{\sigma_{kN}(1 - \rho_{ik})(N_k - N_I)}{N_k \text{Var}[\hat{V}_k|\hat{s}_k]};
\]

\[
\text{Cov}(\theta^*_{1i}, \theta^*_{2i}) = N_I^2 \sigma^2_X \prod_{k=1,2} \frac{\sigma_{kN}(1 - \rho_{ik})}{N_k \text{Var}[\hat{V}_k|\hat{s}_k]}, i = U, M_k;
\]

\[
\text{Var}(P^*_k) = \frac{\sigma^4_{V_k}}{\sigma^2_{V_k} + \sigma^2_{\varepsilon_k}} + \frac{N_I^2}{N^2_k} \delta^2 \sigma^2_{kN}(1 - \rho_{ik})^2 \sigma^2_X;
\]

\[
\text{Var}(\hat{V}_k - P^*_k) = (1 - \rho_{ik})^2 \sigma^2_{V_k} + \rho_{ik}^2 \sigma^2_{\varepsilon_k} + \frac{N_I^2}{N^2_k} \delta^2 \sigma^2_{kN}(1 - \rho_{ik})^2 \sigma^2_X.
\]

\[
\text{Cov}(A^*_1 - B^*_1, A^*_2 - B^*_2) = \delta^2 \sigma^2_X \left(1 - \frac{2}{\pi}\right) \prod_{k=1,2} \frac{\sigma_{kN}}{N_{MK} + 1} (1 - \rho_{ik}).
\]

It follows that \( \text{Corr}(A^*_1 - B^*_1, A^*_2 - B^*_2) = 1 \). Similarly we can show that \( \text{Corr}(N_{M_1}(\alpha^*_1 + \beta^*_1), N_{M_2}(\alpha^*_2 + \beta^*_2)) = 1 \). For Part 3, \( \text{Var}(A^*_k - B^*_k) = E(A^*_k - B^*_k)^2 - E^2(A^*_k - B^*_k) = \frac{1-2/\pi}{(N_{MK}+1)^2} \delta^2 \sigma^2_{kN}(1 - \rho_{ik})^2 \sigma^2_X. \) Q.E.D.

Proof of Proposition 2: From Theorem 1 and (16), we get the price impact is \( \lambda_k = \frac{\text{Cov}(A^*_k + B^*_k, N_I \theta^*_{ik})}{\text{Var}(N_I \theta^*_{ik})} = \frac{\delta \rho_{ik} \sigma^2_{\varepsilon_k} (N_{MK}N_L + 2(N_{K+1}))}{N_IN_{MK}(N_U + N_{MK} + 1)}. \) Q.E.D.

Proof of Theorem 2: We prove the case when \( \Delta_k < 0 \). In this case, we conjecture that
I investors sell stock \( k \) at the bid and \( U \) investors buy stock \( k \) at the ask. Given bid price \( B_k \) and ask price \( A_k \), the optimal demand of \( I \) and \( U \) are:

\[
\theta^*_I = \frac{E[\tilde{V}_k | \tilde{S}_k] + \tilde{H}_k - B_k}{\delta \text{Var}[\tilde{V}_k | \tilde{S}_k]} - \tilde{\theta}_k \quad \text{and} \quad \theta^*_U = \frac{E[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2] - A_k}{\delta \text{Var}[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2]} - \tilde{\theta}_k. \quad (42)
\]

Substituting (42) into the market clearing conditions (3), we get that the market clearing bid and ask prices are:

\[
A_k = E[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2] - \delta \text{Var}[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2] \tilde{\theta}_k - \frac{\delta \text{Var}[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2]}{N_U} \sum_{j=1}^{N_{Mk}} \alpha_{jk},
\]

\[
B_k = E[\tilde{V}_k | \tilde{S}_k] + \tilde{H}_k - \delta \text{Var}[\tilde{V}_k | \tilde{S}_k] \tilde{\theta}_k + \frac{\delta \text{Var}[\tilde{V}_k | \tilde{S}_k]}{N_I} \sum_{j=1}^{N_{Mk}} \beta_{jk}, \quad (43)
\]

where \( \beta_{jk} \) and \( \alpha_{jk} \) are the optimal shares of stock \( M_{jk} \) choose to buy from \( I \) investors and sell to \( U \) investors respectively. Market maker \( M_{jk} \)'s problem is:

\[
\min_{\alpha_{jk}, \beta_{jk}} -\delta(\alpha_{jk}A_k - \beta_{jk}B_k) - \delta(\tilde{\theta}_k + \beta_{jk} - \alpha_{jk})E[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2] + \frac{1}{2} \delta^2 \text{Var}[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2](\tilde{\theta}_k + \beta_{jk} - \alpha_{jk})^2,
\]

where \( A_k \) and \( B_k \) are the market clearing prices given in (43). F.O.C with respect to \( \beta_{jk} \) gives us:

\[
E[\tilde{V}_k | \tilde{S}_k] + \tilde{H}_k - E[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2] + \delta \left( \text{Var}[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2] - \text{Var}[\tilde{V}_k | \tilde{S}_k] \right) \tilde{\theta}_k
\]

\[
+ \frac{\delta \text{Var}[\tilde{V}_k | \tilde{S}_k]}{N_I} \sum_{j=1}^{N_{Mk}} \beta_{jk} + \left( \frac{\text{Var}[\tilde{V}_k | \tilde{S}_k]}{N_I} + \text{Var}[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2] \right) \delta \beta_{jk} - \delta \text{Var}[\tilde{V}_k | \tilde{S}_1, \tilde{S}_2] \alpha_{jk} = 0.
\]

(45)
Summing all, we get:

\[ N_{Mk} \left( E[\tilde{V}_k|\hat{s}_k] + \hat{H}_k - E[\tilde{V}_k|\hat{S}_1, \hat{S}_2] + \delta \left( \text{Var}[\tilde{V}_k|\hat{S}_1, \hat{S}_2] - \text{Var}[\tilde{V}_k|\hat{s}_k] \right) \hat{\theta}_k \right) \]

\[ + \frac{\delta \text{Var}[\tilde{V}_k|\hat{s}_k]}{N_I} \sum_{j=1}^{N_{Mk}} \beta_{jk} + \left( \frac{\text{Var}[\tilde{V}_k|\hat{s}_k]}{N_I} + \text{Var}[\tilde{V}_k|\hat{S}_1, \hat{S}_2] \right) \frac{\delta}{N_I} \sum_{j=1}^{N_M} \beta_{jk} \]

\[ - \delta \text{Var}[\tilde{V}_k|\hat{S}_1, \hat{S}_2] \sum_{j=1}^{N_{Mk}} \alpha_{jk} = 0. \quad (46) \]

Using the F.O.C with respect to \( \alpha_{jk} \), we get:

\[ \frac{\delta}{N_U} \sum_{j=1}^{N_{Mk}} \alpha_{jk} - \delta (\beta_{jk} - \alpha_{jk}) + \frac{\delta}{N_U} \alpha_{jk} = 0. \quad (47) \]

Summing all, we get:

\[ \sum_{j=1}^{N_{Mk}} \beta_j = \frac{N_U + N_{Mk} + 1}{N_U} \sum_{j=1}^{N_{Mk}} \alpha_{jk}. \quad (48) \]

Substituting (48) into (46), we get

\[ \sum_{j=1}^{N_{Mk}} \alpha_{jk} = -\frac{N_{Mk}N_IN_U}{(N_{Mk} + 1)(N_k + 1) \delta \text{Var}[\tilde{V}_k|\hat{s}_k]} \Delta_k. \quad (49) \]

Substituting (49) into (43), we can get the equilibrium ask and bid price \( A_k^* \) and \( B_k^* \). And then substituting \( A_k^* \) and \( B_k^* \) into (42), we can get the optimal stock holdings of \( I \) and \( U \) investors as stated in Theorem 2. Similarly, we can prove the other case of this Theorem when \( I \) investors buy and \( U \) investors sell.

\[Q.E.D.\]
Proof of Proposition 4: Part 1: Assuming \( \sigma_{H1} < 0 \) and \( \sigma_{H2} > 0 \), we have

\[
Cov(|z + \delta \sigma_{H1} \bar{\theta}_1|, |z + \delta \sigma_{H2} \bar{\theta}_2|) = E(|z + \delta \sigma_{H1} \bar{\theta}_1||z + \delta \sigma_{H2} \bar{\theta}_2|) - E(|z + \delta \sigma_{H1} \bar{\theta}_1|)E(|z + \delta \sigma_{H2} \bar{\theta}_2|),
\]

where the first term equals to

\[
E(z^2 + \delta^2 \sigma_{H1} \sigma_{H2} \bar{\theta}_1 \bar{\theta}_2) - 2 \int_{-\delta \sigma_{H1} \bar{\theta}_1}^{-\delta \sigma_{H1} \bar{\theta}_1} E(z^2 + \delta(\sigma_{H1} \bar{\theta}_1 + \sigma_{H2} \bar{\theta}_2)) z + \delta^2 \sigma_{H1} \sigma_{H2} \bar{\theta}_1 \bar{\theta}_2) f(x) dx
\]

\[
= \sigma_z^2 + \delta^2 \sigma_{H1} \sigma_{H2} \bar{\theta}_1 \bar{\theta}_2 + \frac{2 \delta \sigma_z \sigma_{H2} \bar{\theta}_2}{\sqrt{2\pi}} e^{-\frac{\delta^2 \sigma_z^2 \bar{\theta}_2^2}{2z^2}} - \frac{2 \delta \sigma_z \sigma_{H1} \bar{\theta}_1}{\sqrt{2\pi}} e^{-\frac{\delta^2 \sigma_z^2 \bar{\theta}_1^2}{2z^2}}
\]

\[-2 \left( \sigma_z^2 + \delta^2 \sigma_{H1} \sigma_{H2} \bar{\theta}_1 \bar{\theta}_2 \right) \left( N \left( \frac{\delta \sigma_{H2} \bar{\theta}_2}{\sigma_z} \right) - N \left( \frac{\delta \sigma_{H1} \bar{\theta}_1}{\sigma_z} \right) \right),
\]

and we have

\[
E|z + \delta \sigma_{Hk} \bar{\theta}_k| = \delta \sigma_{Hk} \bar{\theta}_k \left( 2N \left( \frac{\delta \sigma_{Hk} \bar{\theta}_k}{\sigma_z} \right) - 1 \right) + \frac{2 \sigma_z}{\sqrt{2\pi}} e^{-\frac{\delta^2 \sigma_z^2 \bar{\theta}_k^2}{2z^2}}, \quad k = 1, 2.
\]

It can be shown that

\[
E(|z + \delta \sigma_{H1} \bar{\theta}_1|)E(|z + \delta \sigma_{H2} \bar{\theta}_2|) > -\delta \sigma_{H1} \bar{\theta}_1 \times \delta \sigma_{H2} \bar{\theta}_2 = -\delta^2 \sigma_{H1} \sigma_{H2} \bar{\theta}_1 \bar{\theta}_2.
\]

Therefore, a sufficient condition for \( Cov(|z + \delta \sigma_{H1} \bar{\theta}_1|, |z + \delta \sigma_{H2} \bar{\theta}_2|) < 0 \) is

\[
= \sigma_z^2 + 2\delta^2 \sigma_{H1} \sigma_{H2} \bar{\theta}_1 \bar{\theta}_2 + \frac{2 \delta \sigma_z \sigma_{H2} \bar{\theta}_2}{\sqrt{2\pi}} e^{-\frac{\delta^2 \sigma_z^2 \bar{\theta}_2^2}{2z^2}} - \frac{2 \delta \sigma_z \sigma_{H1} \bar{\theta}_1}{\sqrt{2\pi}} e^{-\frac{\delta^2 \sigma_z^2 \bar{\theta}_1^2}{2z^2}}
\]

\[-2 \left( \sigma_z^2 + \delta^2 \sigma_{H1} \sigma_{H2} \bar{\theta}_1 \bar{\theta}_2 \right) \left( N \left( \frac{\delta \sigma_{H2} \bar{\theta}_2}{\sigma_z} \right) - N \left( \frac{\delta \sigma_{H1} \bar{\theta}_1}{\sigma_z} \right) \right) < 0.
\]

We use the fact that \( \frac{x}{1+x^2} n(x) < 1 - N(x) < \frac{n(x)}{x} \), for \( x \geq 0 \), where \( n(x) \) is the pdf for
standard normal distribution. We have:

\[
\frac{1}{\sqrt{2\pi}} e^{-\frac{\delta^2\sigma_H^2}{2\sigma_z^2}} < N \left( \frac{\delta\sigma_H \tilde{\theta}_1}{\sigma_z} \right) \left( -\frac{\sigma_z}{\delta\sigma_H \tilde{\theta}_1} - \frac{\delta\sigma_H \tilde{\theta}_1}{\sigma_z} \right),
\]

and

\[
\frac{1}{\sqrt{2\pi}} e^{-\frac{\delta^2\sigma_H^2}{2\sigma_z^2}} < N \left( -\frac{\delta\sigma_H \tilde{\theta}_2}{\sigma_z} \right) \left( \frac{\sigma_z}{\delta\sigma_H \tilde{\theta}_2} + \frac{\delta\sigma_H \tilde{\theta}_2}{\sigma_z} \right).
\]

Therefore, a sufficient condition for \( \text{Cov}(|z + \delta\sigma_H \tilde{\theta}_1|, |z + \delta\sigma_H \tilde{\theta}_2|) < 0 \) is

\[
1 - \frac{2\delta\sigma_H \tilde{\theta}_1}{\sigma_H \tilde{\theta}_2} + 2 \left( 1 - \frac{\delta\sigma_H \tilde{\theta}_2}{\sigma_H \tilde{\theta}_1} \right) N \left( \frac{\delta\sigma_H \tilde{\theta}_1}{\sigma_z} \right) - 2 \left( 1 - \frac{\delta\sigma_H \tilde{\theta}_1}{\sigma_H \tilde{\theta}_2} \right) N \left( \frac{\delta\sigma_H \tilde{\theta}_2}{\sigma_z} \right) < 0. \tag{51}
\]

A sufficient condition for inequality (51) to hold is

\[
\frac{\delta\sigma_H \tilde{\theta}_1}{\sigma_z} < -2, \quad \frac{\delta\sigma_H \tilde{\theta}_2}{\sigma_z} > 2, \quad \text{and} \quad \frac{\sigma_1N(1 - \rho_{I_1})\tilde{\theta}_1}{\sigma_2N(1 - \rho_{I_2})\tilde{\theta}_1} + \frac{\sigma_2N(1 - \rho_{I_2})\tilde{\theta}_2}{\sigma_1N(1 - \rho_{I_1})\tilde{\theta}_2} > -8.
\]

Therefore, \( \text{Corr}(A^*_1 - B^*_1, A^*_2 - B^*_2) < 0 \), if \(-4\frac{1 - \rho_{I_2}}{1 - \rho_{I_1}} \sigma_{2N} \tilde{\theta}_2 < \sigma_{1N} \tilde{\theta}_1 < \frac{1 - \rho_{I_2}}{1 - \rho_{I_1}} \sigma_{2N} \tilde{\theta}_2 \), and

\[
0 < \sigma_X^2 < \min \left\{ \frac{\delta^2\sigma_1^2(1 - \rho_{I_1})^2\tilde{\theta}_1^2 - 4C_3}{4\delta^2C_3^2}, \frac{\delta^2\sigma_2^2(1 - \rho_{I_2})^2\tilde{\theta}_2^2 - 4C_3}{4\delta^2C_3^2} \right\}. \tag{52}
\]

Part 2: \( \text{Corr}(P^*_1, P^*_2) = \)

\[
\text{Corr} \left( (N_1\sigma_{V_1}^2\sigma_{S_2}^2\sigma_{H_2} + \nu_1N_1\sigma_{H_1}\sigma_{V_2}\text{Cov}(\hat{S}_1, \hat{S}_2))\hat{S}_1 - (N_M + N_U)\sigma_{H_2}\sigma_{V_1}\text{Cov}(\hat{S}_1, \hat{S}_2)\hat{S}_2, \right.

\[
(N_2\sigma_{V_2}^2\sigma_{S_1}^2\sigma_{H_1} + \nu_2N_1\sigma_{H_2}\sigma_{V_2}\text{Cov}(\hat{S}_1, \hat{S}_2))\hat{S}_2 - (N_M + N_U)\sigma_{H_1}\sigma_{V_2}\text{Cov}(\hat{S}_1, \hat{S}_2)\hat{S}_1 \right)
\]

\[
= \frac{(a_1a_2 + b_1b_2)\text{Cov}(\hat{S}_1, \hat{S}_2) + a_1b_2\sigma_{S_1}^2 + b_1a_2\sigma_{S_2}^2}{\sqrt{a_1^2\sigma_{S_1}^2 + b_1^2\sigma_{S_2}^2 + 2a_1b_1\text{Cov}(\hat{S}_1, \hat{S}_2)\sqrt{a_2^2\sigma_{S_2}^2 + b_2^2\sigma_{S_1}^2 + 2a_2b_2\text{Cov}(\hat{S}_1, \hat{S}_2)}}},
\]

where

\[
a_k = N_k\sigma_{V_k}^2\sigma_{S_{-k}}^2\sigma_{H_{-k}} + \nu_kN_1\sigma_{H_k}\sigma_{V_{-k}}\text{Cov}(\hat{S}_1, \hat{S}_2),
\]

\[
43
\]
\[ b_k = -(N_U + N_{M_k})\sigma_{H,k}\sigma_{V,k}^2 \rho_{l_{H,k}} \text{Cov}(\hat{S}_1, \hat{S}_2). \]

\[
\text{Cov}(\hat{V}_1 - P_1^*, \hat{V}_2 - P_2^*) = -\text{Cov}(\hat{V}_1, P_2^*) - \text{Cov}(\hat{V}_2, P_1^*) + \text{Cov}(P_1^*, P_2^*)
\]
\[
= C \delta^2 \left( \frac{N_U + N_{M_1}}{N_1} + \frac{N_U + N_{M_2}}{N_2} + C \left( \frac{a_1 a_2 + b_1 b_2}{N_1 N_2 \sigma_{V_1}^2 \sigma_{V_2}^2 \rho_{l_1 \rho_{l_2}} \sigma_{\chi}^2 \text{Cov}^2(\hat{S}_1, \hat{S}_2)} \right) \right) \prod_{k=1,2} \sigma_{kN}(1-\rho_{l_k}).
\]

\[
\text{Cov}(P_1^*, P_2^*) < 0 \text{ is equivalent to }
\]
\[
C \delta^4 \sigma_{\chi}^4 + d_1 \delta^2 \sigma_{\chi}^2 + e_1 < 0,
\]

where
\[
C = \prod_{k=1,2} \sigma_{\chi k}^2 \left( \frac{\sigma_{kN}^2 \sigma_{z_k}^2 \sigma_{V,\chi}^4}{\sigma_{z_k}^2 + \sigma_{V,\chi}^2} \right),
\]
\[
d_1 = \left( \sum_{k=1,2} \sigma_{kN}^2 \sigma_{z_k}^2 \sigma_{V,\chi}^4 + \left( 1 - \frac{(N_{M_1} + N_U)(N_{M_2} + N_U)}{N_1^2} \right) \sum_{k=1,2} \frac{\sigma_{kN}^2 \sigma_{z_k}^2 \sigma_{V,\chi}^4}{\sigma_{z_k}^2 + \sigma_{V,\chi}^2} \right) \prod_{k=1,2} \sigma_{V,k}^4 \sigma_{\chi k}^2,
\]
\[
e_1 = \left( 1 - \frac{(N_{M_1} + N_U)(N_{M_2} + N_U)}{N_1^2} \right) \prod_{k=1,2} \sigma_{\chi k}^8 \sigma_{\chi k}^2.
\]

Therefore, \( \text{Cov}(P_1^*, P_2^*) < 0 \) if and only if \( 0 < \delta^2 \sigma_{\chi}^2 < C_2 \), where
\[
C_2 = -d_1 + \sqrt{d_1^2 - 4Ce_1} \quad (53)
\]

\[ Q.E.D. \]

**Proof of Proposition 5:** From Theorem 2 and (16), we get the price impact is
\[
\lambda_k = \frac{\delta \text{Var}[\hat{V}_k|\hat{s}_k] \text{Cov} \left( E[\hat{V}_k|\hat{s}_1, \hat{s}_2] + \left( C_{U_k} + \frac{1}{2(N_{M_k} + 1)} \right) \Delta_k, \Delta_k \right)}{N_I C_{Ik} \text{Var}(\Delta_k)}
\]
\[
\begin{align*}
\frac{\delta \rho_{1k} \sigma_{\hat{e}k}^2}{N_I C_{Ik}} &= \frac{\rho_{11} \rho_{22} \sigma_{V1}^2 \sigma_{V2}^2}{\rho_{11} \sigma_{V1}^2 \sigma_{H2}^2 + \rho_{12} \sigma_{V2}^2 \sigma_{H1}^2} + C_{Uk} + \frac{1}{2(N_{Mk} + 1)}.
\end{align*}
\]  
(54)

Since \( \nu_k \) decreases in \( \sigma_{\hat{e},-k}^2 \), it follows that \( C_{Ik} \) increases in \( \sigma_{\hat{e},-k}^2 \) and \( C_{Uk} \) decreases in \( \sigma_{\hat{e},-k}^2 \). In addition, it can be shown that \( \frac{\rho_{11} \rho_{12} \sigma_{V1}^2 \sigma_{V2}^2}{\rho_{11} \sigma_{V1}^2 \sigma_{H2}^2 + \rho_{12} \sigma_{V2}^2 \sigma_{H1}^2} \) decreases in \( \sigma_{\hat{e},-k}^2 \). Therefore, from (54), we have \( \frac{\partial \lambda_k}{\partial \sigma_{\hat{e},-k}^2} < 0 \).

\[Q.E.D.\]
References


