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Increases in risk aversion and the distribution of portfolio payoffs ☆

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Abstract

Oliver Hart proved the impossibility of deriving general comparative static properties in portfolio weights. Instead, we derive new comparative statics for the distribution of payoffs: A is less risk averse than B iff A's payoff is always distributed as B's payoff plus a non-negative random variable plus conditional-mean-zero noise. If either agent has nonincreasing absolute risk aversion, the non-negative part can be chosen to be constant. The main result also holds in some incomplete markets with two assets or two-fund separation, and in multiple periods for a mixture of payoff distributions over time (but not at every point in time).

JEL classification: D33; G11

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1. Introduction

The trade-off between risk and return arises in many portfolio problems in finance. This trade-off is more-or-less assumed in mean-variance optimization, and is also present in the comparative statics for two-asset portfolio problems explored by Arrow [1] and Pratt [15] (for a model with

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a riskless asset) and Kihlstrom, Romer, and Williams [10] and Ross [19] (for models without a riskless asset). However, the trade-off is less clear in portfolio problems with many risky assets, as pointed out by Hart [8]. Assuming a complete market with many states (and therefore many assets), we show that a less risk-averse (in the sense of Arrow and Pratt) agent's portfolio payoff is distributed as the payoff for the more risk-averse agent, plus a non-negative random variable (extra return), plus conditional-mean-zero noise (risk). Therefore, the general complete-markets portfolio problem, which may not be a mean-variance problem, still trades off risk and return.

If either agent has nonincreasing absolute risk aversion, then the non-negative random variable (extra return) can be chosen to be a constant. We also give a counter-example that shows that in general, the non-negative random variable cannot be chosen to be a constant. In this case, the less risk averse agent's payoff can also have a higher mean and a lower variance than the more risk averse agent's payoff. We further prove a converse theorem. Suppose there are two agents, such that in all complete markets, the first agent chooses a payoff that is distributed as the second's payoff, plus a non-negative random variable, plus conditional-mean-zero noise. Then the first agent is less risk averse than the other agent.

Our main result applies directly in a multiple-period setting with consumption only at a terminal date, and perhaps dynamic trading is the most natural motivation for the completeness we are assuming. Our main result can also be extended to a multiple-period model with consumption at many dates, but this is more subtle. Consumption at each date may not be ordered when risk aversion changes, due to shifts in the timing of consumption. However, for agents with the same pure rate of time preference, we show there is a mixture of the payoff distributions across periods that preserves the single-period result.

Our main result also extends to some special settings with incomplete markets, for example, a two-asset world with a risk-free asset. The proof is in two parts. The first part is the standard result: decreasing the risk aversion increases the portfolio allocation to the asset with higher return. The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. However, for a two-asset world without a risk-free asset, both parts of the proof fail in general and we have a counter-example. Therefore, our result is not true in general with incomplete markets. We further provide sufficient conditions under which our results still hold in a two-risky-asset world using Ross's stronger measure of risk aversion. Each result from two assets can be re-interpreted as applying to parallel settings with two-fund separation identifying the two funds with the two assets.

The proofs in the paper make extensive use of results from stochastic dominance, portfolio choice, and Arrow–Pratt and Ross [19] risk aversion. One contribution of the paper is to show how these concepts relate to each other. We use general versions of the stochastic dominance results for L^1 random variables[1] and monotone concave preferences, following Strassen [22] and Ross [17]. To see why our results are related to stochastic dominance, note that if the first agent's payoff equals the second agent's payoff plus a non-negative random variable plus conditional-mean-zero noise, this is equivalent to saying that negative the first agent's payoff is monotone-concave dominated by negative the second agent's payoff.

Section 2 introduces the model setup and provides some preliminary results, Section 3 derives the main results. Section 4 discusses the case with incomplete markets. Section 5 extends

[1] We assume that the consumptions have unbounded distributions instead of compact support (e.g., Rothschild and Stiglitz [20]). Compact support for consumption is not a happy assumption in finance because it is violated by most of our leading models. Unfortunately, as noted by Rothschild and Stiglitz [21], the integral condition is not available in our general setting.

the main results in a multiple-period model. Section 6 illustrates the main results using some examples and Section 7 concludes.

2. Model setup and some standard results

We want to work in a fairly general setting with complete markets and strictly concave increasing von Neumann–Morgenstern preferences. There are two agents A and B with von Neumann–Morgenstern utility functions $U_A(c)$ and $U_B(c)$, respectively. We assume that $U_A(c)$ and $U_B(c)$ are of class C^2, $U'_A(c) > 0$, $U'_B(c) > 0$, $U''_A(c) < 0$ and $U''_B(c) < 0$. Each agent's problem has the form:

Problem 1. Choose random consumption $\tilde{c}$ to

$$\begin{aligned} &\max E\big[U_i(\tilde{c})\big], \\ &s.t. \quad E[\tilde{\rho}\tilde{c}] = w_0. \end{aligned} \tag{1}$$

In Problem 1, $i = A$ or B indexes the agent, w_0 is initial wealth (which is the same for both agents), and $\tilde{\rho} > 0$ is the state price density. We will assume that $\tilde{\rho}$ is in the class $\mathcal{P}$ for which both agents have optimal random consumptions with finite means, denoted $\tilde{c}_A$ and $\tilde{c}_B$.

The first order condition is

$$U'_i(\tilde{c}_i) = \lambda_i \tilde{\rho}, \tag{2}$$

i.e., the marginal utility is proportional to the state price density $\tilde{\rho}$. We have

$$\tilde{c}_i = I_i(\lambda_i \tilde{\rho}), \tag{3}$$

where I_i is the inverse function of $U'_i(\cdot)$. By continuity and negativity of the second order derivative $U''_i(\cdot)$, $\tilde{c}_i$ is a decreasing function of $\tilde{\rho}$.

Our main result will be that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where "$\sim$" denotes "is distributed as", $\tilde{z} \geqslant 0$, and $E[\tilde{\varepsilon}|c_B + z] = 0$.[2] We firstly review and give the proofs in Appendix A of some standard results in the form needed for the proofs of our main results.

Lemma 1. *If B is weakly more risk averse than A ($\forall c$, $-\frac{U''_B(c)}{U'_B(c)} \geqslant -\frac{U''_A(c)}{U'_A(c)}$), then*

1. *for any solution to (2) (which may not satisfy the budget constraint (1)), there exists some critical consumption level c^* (can be $\pm\infty$) such that $\tilde{c}_A \geqslant \tilde{c}_B$ when $\tilde{c}_B \geqslant c^*$, and such that $\tilde{c}_A \leqslant \tilde{c}_B$ when $\tilde{c}_B \leqslant c^*$;*
2. *assuming $\tilde{c}_A$ and $\tilde{c}_B$ have finite means, and A and B have equal initial wealths w_0, then $E[\tilde{c}_A] \geqslant E[\tilde{c}_B] \geqslant \frac{w_0}{E[\tilde{\rho}]}$. Note that $\frac{w_0}{E[\tilde{\rho}]}$ is the payoff to a riskless investment of w_0.*

The first result in Lemma 1 implies that the consumptions function of the less risk averse agent crosses that of the more risk averse agent at most once and from above. This single-crossing result is due to Pratt [15], expressed in a slightly different way. Lemma 1 gives us a sense in which decreasing the agent's risk aversion takes us further from the riskless asset. In fact, we can

[2] Throughout this paper, the letters with "tilde" denote random variables, and the corresponding letters without "tilde" denote particular values of these variables.

obtain a more explicit description (our main result) of how decreasing the agent's risk aversion changes the optimal portfolio choice. The description and proof are both related to monotone concave stochastic dominance.[3] The following theorem gives a distributional characterization of stochastic dominance for all monotone and concave functions of one random variable over another. The form of this result is from Ross [17] and is a special case of a result of Strassen [22] which generalizes a traditional result for bounded random variables to possibly unbounded random variables with finite means.

Theorem 1 *(Monotone concave stochastic dominance). (See Strassen [22] and Ross [17].) Let $\tilde{X}$ and $\tilde{Y}$ be two random variables defined in R^1 with finite means; then $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$, for all concave nondecreasing functions $V(\cdot)$, i.e., $\tilde{X}$ monotone-concave stochastically dominates $\tilde{Y}$, if and only if $\tilde{Y} \sim \tilde{X} - \tilde{Z} + \tilde{\varepsilon}$, where $\tilde{Z} \geqslant 0$, and $E[\tilde{\varepsilon}|X - Z] = 0$.*

Rothschild and Stiglitz [20,21] popularized a similar characterization of stochastic dominance for all concave functions (which implies equal means) that is a special case of another result of Strassen's.

Theorem 2 *(Concave stochastic dominance). (See Strassen [22], and Rothschild and Stiglitz [20,21].) Let $\tilde{X}$ and $\tilde{Y}$ be two random variables defined in R^1 with finite means; then $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$, for all concave functions $V(\cdot)$, i.e., $\tilde{X}$ concave stochastically dominates $\tilde{Y}$, if and only if $\tilde{Y} \sim \tilde{X} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|X] = 0$.*

Rothschild and Stiglitz [20] also offered an integral condition for concave stochastic dominance, which unfortunately does not generalize to all random variables with finite mean, as they note in Rothschild and Stiglitz [21].[4]

3. Main results

Suppose agent A with utility function U_A and agent B with utility function U_B have identical initial wealth w_0 and solve Problem 1. Recall that we assume that $U_A(c)$ and $U_B(c)$ are of class C^2, $U_A'(c) > 0$, $U_B'(c) > 0$, $U_A''(c) < 0$ and $U_B''(c) < 0$. We have

Theorem 3. *Assume a single-period setting with complete markets in which agents A and B solve Problem 1. If B is weakly more risk averse than A in the sense of Arrow and Pratt ($\forall c$, $-\frac{U_B''(c)}{U_B'(c)} \geqslant -\frac{U_A''(c)}{U_A'(c)}$), then for every $\tilde{\rho} \in \mathcal{P}$, $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. Furthermore, if $\tilde{c}_A \neq \tilde{c}_B$, neither $\tilde{z}$ nor $\tilde{\varepsilon}$ is identically zero.*

[3] We avoid using the term "second order stochastic dominance" in this paper because different papers use different definitions. In this paper, we follow unambiguous terminology from Ross [17]: (1) if $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$ for all nondecreasing functions, then $\tilde{X}$ monotone stochastically dominates $\tilde{Y}$; (2) if $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$ for all concave functions, then $\tilde{X}$ concave stochastically dominates $\tilde{Y}$; (3) if $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$ for all concave nondecreasing functions, then $\tilde{X}$ monotone-concave stochastically dominates $\tilde{Y}$.

[4] The integration by parts used to prove the integral condition unfortunately includes a term at the lower endpoint which needs not equal to zero in general. Therefore, the integral condition may not be sufficient or necessary condition for concave stochastic dominance under unbounded distribution. As noted by Rothschild and Stiglitz [21], the integral condition does not appear to have any natural analog in these more general cases. Ross [17] has a sufficient condition for the integral condition to be valid, but unfortunately it is hard to interpret.

Proof. The first step of the proof[5] is to show that $-\tilde{c}_B$ monotone-concave stochastically dominates $-\tilde{c}_A$, *i.e.*, $E[V(-\tilde{c}_B)] \geqslant E[V(-\tilde{c}_A)]$ for any concave nondecreasing function $V(\cdot)$. By Lemma 1, $\tilde{c}_A$ and $\tilde{c}_B$ are monotonely related and there is a critical value c^* above which $\tilde{c}_A$ is weakly larger and below which $\tilde{c}_B$ is weakly larger. Let $V'(\cdot)$ be any selection from the subgradient correspondence $\nabla V(\cdot)$, then $V'(\cdot)$ is positive and nonincreasing and it is the derivative of $V(\cdot)$ whenever it exists. Recall from Rockafellar [16], the subgradient for concave[6] $V(\cdot)$ is $\nabla V(x_1) \equiv \{s \mid (\forall x),\ V(x) \leqslant V(x_1) + s(x - x_1)\}$. By concavity of $V(\cdot)$, $\nabla V(x)$ is nonempty for all x_1. And if $x_2 > x_1$, then $s_2 \leqslant s_1$ for all $s_2 \in \nabla V(x_2)$ and $s_1 \in \nabla V(x_1)$.

The definition of subgradient for concave $V(\cdot)$ implies that

$$V(x + \Delta x) \leqslant V(x) + V'(x)\Delta x. \tag{4}$$

Letting $x = -\tilde{c}_B$ and $\Delta x = -\tilde{c}_A + \tilde{c}_B$ in (4), we have

$$V(-\tilde{c}_A) - V(-\tilde{c}_B) \leqslant V'(-\tilde{c}_B)(-\tilde{c}_A + \tilde{c}_B). \tag{5}$$

If $\tilde{c}_B \geqslant c^*$, then $\tilde{c}_A \geqslant \tilde{c}_B$ (by Lemma 1), and $V'(-\tilde{c}_B) \geqslant V'(-c^*)$, while if $\tilde{c}_B \leqslant c^*$, then $\tilde{c}_A \leqslant \tilde{c}_B$ and $V'(-\tilde{c}_B) \leqslant V'(-c^*)$. In both cases, we always have $(V'(-\tilde{c}_B) - V'(-c^*))(\tilde{c}_A - \tilde{c}_B) \geqslant 0$. Rewriting (5) and substituting in this inequality, we have

$$V(-\tilde{c}_B) - V(-\tilde{c}_A) \geqslant V'(-\tilde{c}_B)(\tilde{c}_A - \tilde{c}_B) \geqslant V'\big(-c^*\big)(\tilde{c}_A - \tilde{c}_B). \tag{6}$$

Since $V(\cdot)$ is nondecreasing and $E[\tilde{c}_A] \geqslant E[\tilde{c}_B]$ (result 2 of Lemma 1), we have

$$E\big[V(-\tilde{c}_B) - V(-\tilde{c}_A)\big] \geqslant E\big[V'\big(-c^*\big)(\tilde{c}_A - \tilde{c}_B)\big] = V'\big(-c^*\big)\big(E[\tilde{c}_A] - E[\tilde{c}_B]\big) \geqslant 0. \tag{7}$$

Therefore, we have that $-\tilde{c}_B$ is preferred to $-\tilde{c}_A$ by all concave nondecreasing $V(\cdot)$, and by Theorem 1, this says that $-\tilde{c}_A$ is distributed as $-\tilde{c}_B - \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}| - c_B - z] = 0$. This is exactly the same as saying that $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + (-\tilde{\varepsilon})$, where $\tilde{z} \geqslant 0$ and $E[-\tilde{\varepsilon}|c_B + z] = 0$. Relabel $-\tilde{\varepsilon}$ as $\tilde{\varepsilon}$, and we have proven the first sentence of the theorem.

To prove the second sentence of the theorem, note that because $\tilde{c}_A$ and $\tilde{c}_B$ are monotonely related, $\tilde{c}_A$ is distributed the same as $\tilde{c}_B$ only if $\tilde{c}_A = \tilde{c}_B$. Therefore, if $\tilde{c}_A \neq \tilde{c}_B$, one or the other of $\tilde{z}$ or $\tilde{\varepsilon}$ is not identically zero. Now, if $\tilde{z}$ is identically zero, then $\tilde{\varepsilon}$ must not be identically zero, and $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{\varepsilon}$, by Jensen's inequality, we have $E[U_A(\tilde{c}_A)] = E[U_A(\tilde{c}_B + \tilde{\varepsilon})] = E[E[U_A(\tilde{c}_B + \tilde{\varepsilon})|\tilde{c}_B]] < E[U_A(E[\tilde{c}_B|\tilde{c}_B] + E[\tilde{\varepsilon}|\tilde{c}_B])] = E[U_A(\tilde{c}_B)]$, which contradicts the optimality of $\tilde{c}_A$ for agent A. If $\tilde{\varepsilon}$ is identically zero, then $\tilde{z}$ must not be, and $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z}$, where $\tilde{z} \geqslant 0$ and is not identically zero. Therefore, $\tilde{c}_A$ strictly monotone stochastically dominates $\tilde{c}_B$, contradicting optimality of $\tilde{c}_B$ for agent B. This completes the proof that if $\tilde{c}_A$ and $\tilde{c}_B$ do not have the same distribution, then neither $\tilde{\varepsilon}$ nor $\tilde{z}$ is identically 0. □

We now prove a converse result of Theorem 3: if in all complete markets, one agent chooses a portfolio whose payoff is distributed as a second agent's payoff plus a non-negative random variable plus conditional-mean-zero noise, then the first agent is less risk averse than the second. Specifically, we have

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[5] As noted in footnote 4, the integral condition does not hold under unbounded distributions, so that a proof using Lemma 1 and the integral condition would be wrong. More specifically, because $\tilde{c}_A$ and $\tilde{c}_B$ might be unbounded, we cannot get that $-\tilde{c}_B$ monotone concave stochastically dominates $-\tilde{c}_A$ directly from $\int_{q=-\infty}^{c}[F_{-c_A}(q) - F_{-c_B}(q)]\,dq \geqslant 0$.

[6] For convex $V(\cdot)$, the inequality is reversed. We follow Rockafellar's convention of calling the derivative correspondence the subgradient in both cases.

Theorem 4. *If for all $\tilde{\rho} \in \mathcal{P}$, $E[\tilde{c}_A] \geqslant E[\tilde{c}_B]$, then B is weakly more risk averse than A ($\forall c$, $-\frac{U_B''(c)}{U_B'(c)} \geqslant -\frac{U_A''(c)}{U_A'(c)}$). This implies a converse result of Theorem 3: if for all $\tilde{\rho} \in \mathcal{P}$, $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$, then B is weakly more risk averse than A.*

Proof. We prove this theorem by contradiction. If B is not weakly more risk averse than A, then there exists a constant $\hat{c}$, such that $-\frac{U_B''(\hat{c})}{U_B'(\hat{c})} < -\frac{U_A''(\hat{c})}{U_A'(\hat{c})}$. Since U_A and U_B are of the class of C^2, from the continuity of $-\frac{U_i''(c)}{U_i'(c)}$, where $i = A, B$, we get that there exists an interval RA containing $\hat{c}$, *s.t.*, $\forall c \in RA$, $-\frac{U_B''(c)}{U_B'(c)} < -\frac{U_A''(c)}{U_A'(c)}$. We pick $c_1, c_2 \in RA$ with $c_1 < c_2$. Now from Lemma 5 in Appendix A, there exist hypothetical agents A_1 and B_1, so that U_{A_1} agrees with U_A and U_{B_1} agrees with U_B on $[c_1, c_2]$, but A_1 is everywhere strictly more risk averse than B_1 (and not just on $[c_1, c_2]$).

Fix any $\lambda_B > 0$ and choose $\tilde{\rho}$ to be any random variable that takes on all the values on $[\frac{U_B'(c_2)}{\lambda_B}, \frac{U_B'(c_1)}{\lambda_B}]$. Then, the corresponding $\tilde{c}_B$ solving the first order condition $U_B'(\tilde{c}_B) = \lambda_B \tilde{\rho}$ takes on all the values on $[c_1, c_2]$. Because $U_B'' < 0$, the $F.O.C$ solution is also sufficient (expected utility exists because $\tilde{\rho}$ and $U_B(\tilde{c}_B)$ are bounded), $\tilde{c}_B$ solves the portfolio problem for utility function U_B, state price density $\tilde{\rho}$ and initial wealth $w_0 = E[\tilde{\rho}\tilde{c}_B]$. Since $U_{B_1} = U_B$ on the support of $\tilde{c}_B$, letting $\tilde{c}_{B_1} = \tilde{c}_B$, then $\tilde{c}_{B_1}$ solves the corresponding optimization for U_{B_1} for $\lambda_{B_1} = \lambda_B$.

We now show that there exists λ_{A_1} such that $\tilde{c}_{A_1} \equiv I_{A_1}(\lambda_{A_1}\tilde{\rho})$ satisfies the budget constraint $E[\tilde{\rho}\tilde{c}_{A_1}] = w_0$. Due to the choice of U_{A_1}, $I_{A_1}(\lambda_{A_1}\tilde{\rho})$ exists and is a bounded random variable for all λ_{A_1}. Letting $\underline{\rho} = \frac{U_B'(c_2)}{\lambda_B}$ and $\overline{\rho} = \frac{U_B'(c_1)}{\lambda_B}$ (so, $\tilde{\rho} \in [\underline{\rho}, \overline{\rho}]$), we define $\lambda_1 = \frac{U_{A_1}'(c_1)}{\underline{\rho}}$ and $\lambda_2 = \frac{U_{A_1}'(c_2)}{\overline{\rho}}$, then we have

$$c_1 = I_{A_1}(\lambda_1\underline{\rho}) > I_{A_1}(\lambda_1\tilde{\rho}), \qquad c_2 = I_{A_1}(\lambda_2\overline{\rho}) < I_{A_1}(\lambda_2\tilde{\rho}). \tag{8}$$

The inequalities follow from $I_{A_1}(\cdot)$ decreasing. From (8) and $c_1 \leqslant \tilde{c}_B \leqslant c_2$, we have

$$\begin{aligned} &E\big[\tilde{\rho} I_{A_1}(\lambda_1\tilde{\rho})\big] < E[\tilde{\rho}c_1] \leqslant E[\tilde{\rho}\tilde{c}_B] = w_0, \\ &E\big[\tilde{\rho} I_{A_1}(\lambda_2\tilde{\rho})\big] > E[\tilde{\rho}c_2] \geqslant E[\tilde{\rho}\tilde{c}_B] = w_0. \end{aligned} \tag{9}$$

Now, $I_{A_1}(\lambda\tilde{\rho})$ is continuous from the assumption that $U_{A_1}(\cdot)$ is in the class of C^2 and $U_{A_1}'' < 0$. By the intermediate value theorem, there exists λ_{A_1}, such that $E[\tilde{\rho} I_{A_1}(\lambda_{A_1}\tilde{\rho})] = w_0$, *i.e.*, $\tilde{c}_{A_1}$ satisfies the budget constraint for $\tilde{\rho}$ and w_0.

From the second result of Lemma 6 in Appendix A, if $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$, then we have that $\tilde{c}_{B_1}$ has a wider range of support than that of $\tilde{c}_{A_1}$. Let the support of A_1's optimal consumption be $[c_3, c_4] \subseteq [c_1, c_2]$. From the construction of U_{A_1}, $U_{A_1} = U_A$ on the support of $\tilde{c}_{A_1}$. Letting $\tilde{c}_A = \tilde{c}_{A_1}$, then $\tilde{c}_A$ solves the corresponding optimization for U_A for $\lambda_A = \lambda_{A_1}$.

Now, since B_1 is strictly less risk averse than A_1, from Theorem 3, $\tilde{c}_{B_1} \sim \tilde{c}_{A_1} + \tilde{z}_1 + \tilde{\varepsilon}_1$, where $\tilde{z}_1 \geqslant 0$ and $E[\tilde{\varepsilon}_1|c_{A_1} + z_1] = 0$. Furthermore, if $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$, then neither $\tilde{z}_1$ nor $\tilde{\varepsilon}_1$ is identically zero. From the first result of Lemma 6 in Appendix A, if A_1 is strictly more risk averse than B_1, then $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$. Thus, by Theorem 3, neither $\tilde{z}_1$ nor $\tilde{\varepsilon}_1$ is identically zero. Therefore, $E[\tilde{c}_{B_1}] > E[\tilde{c}_{A_1}]$, *i.e.* $E[\tilde{c}_B] > E[\tilde{c}_A]$, this contradicts the assumption that, for all $\tilde{\rho} \in \mathcal{P}$, $E[\tilde{c}_A] \geqslant E[\tilde{c}_B]$. This therefore also contradicts the stronger condition: for all $\tilde{\rho} \in \mathcal{P}$, $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. $\square$

Theorem 3 shows that if B is weakly more risk averse than A, then $\tilde{c}_A$ is distributed as $\tilde{c}_B$ plus a risk premium plus random noise. The distributions of the risk premium and the noise term are typically not uniquely determined. Also, it is possible that the weakly less risk averse agent's payoff can have a higher mean and a lower variance than the weakly more risk averse agent's payoff as we will see in Example 6.2. This can happen because although adding condition-mean-zero noise always increases variance, adding the non-negative random variable decreases variance if it is sufficiently negatively correlated with the rest (since $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) + Var(\tilde{z}) + 2\,Cov(\tilde{c}_B, \tilde{z})$, if $Cov(\tilde{c}_B, \tilde{z}) < -\frac{1}{2}(Var(\tilde{z}) + Var(\tilde{\varepsilon}))$, then $Var(\tilde{c}_A) < Var(\tilde{c}_B)$). This should not be too surprising, given that it is well known that in general variance is not a good measure of risk[7] for von Neumann–Morgenstern utility functions,[8] and for general distributions in a complete market, mean-variance preferences are hard to justify.

Our second main result says that when either of the two agents has nonincreasing absolute risk aversion, we can choose $\tilde{z}$ to be nonstochastic, in which case $z = E[\tilde{c}_A - \tilde{c}_B]$. The basic idea is as follows. If either agent has nonincreasing absolute risk aversion, then we can construct a new agent A^* whose consumption equals to A's consumption plus $E[\tilde{c}_A - \tilde{c}_B]$. We can therefore get the distributional results for agent A^* and B since A^* is weakly less risk averse than B.

Theorem 5. *If B is weakly more risk averse than A and either of the two agents has nonincreasing absolute risk aversion, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.*

Proof. Define the utility function $U_{A^*}(\tilde{c}) = U_A(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])$. In the case when A has nonincreasing absolute risk aversion, A^* is weakly less risk averse than B because A is weakly less risk averse than B and nonincreasing risk aversion of A implies that A^* is weakly less risk averse than A. In the case when B has nonincreasing absolute risk aversion, B^* with utility $U_{B^*} = U_B(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])$ is weakly less risk averse than B and A^* is weakly less risk averse than B^*. Therefore, in both cases, we have that A^* is weakly less risk averse than B.

Give agent A^* initial wealth $w_{A^*} = w_0 - E[\tilde{\rho}]E[\tilde{c}_A - \tilde{c}_B]$, where w_0 is the common initial wealth of agent A and B. A^*'s problem is

$$\begin{aligned} &\max_{\tilde{c}} E\big[U_A\big(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B]\big)\big], \\ &s.t. \quad E[\tilde{\rho}\tilde{c}] = w_{A^*}. \end{aligned} \tag{10}$$

The first order conditions are related to the optimality of $\tilde{c}_A$ for agent A. To satisfy the budget constraints, agent A^* will optimally hold $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$.

By Lemma 1, $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ and $\tilde{c}_B$ are monotonely related and there is a critical value c^* above which $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ is weakly larger and below which $\tilde{c}_B$ is weakly larger. This implies that

$$\big(V'(-\tilde{c}_B) - V'\big(-c^*\big)\big)\big(\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] - \tilde{c}_B\big) \geqslant 0, \tag{11}$$

[7] See, for example Hanoch and Levy [9], and the survey of Machina and Rothschild [14].

[8] If von Neumann–Morgenstern utility functions are mean-variance preferences, then they have to be quadratic utility functions, but quadratic preferences are not appealing because they are not increasing everywhere and they have increasing risk aversion where they are increasing. Also, Dybvig and Ingersoll [6] show that if markets are complete, mean-variance pricing of all assets implies there is arbitrage unless the payoff to the market portfolio is bounded above.

where $V(\cdot)$ is an arbitrary concave function and $V'(\cdot)$ is any selection from the subgradient correspondence $\nabla V(\cdot)$. The concavity of $V(\cdot)$ implies that

$$V\left(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]\right) - V(-\tilde{c}_B) \leqslant V'(-\tilde{c}_B)\left(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B] + \tilde{c}_B\right). \tag{12}$$

(11) and (12) imply that

$$V(-\tilde{c}_B) - V\left(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]\right) \geqslant V'\left(-c^*\right)\left(\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] - \tilde{c}_B\right). \tag{13}$$

We have

$$E\left[V(-\tilde{c}_B) - V\left(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]\right)\right] \geqslant E\left[V'\left(-c^*\right)\left(\tilde{c}_A - \tilde{c}_B - E[\tilde{c}_A - \tilde{c}_B]\right)\right] = 0. \tag{14}$$

Therefore, for any concave function $V(\cdot)$, we have

$$E\left[V(-\tilde{c}_B)\right] \geqslant E\left[V\left(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]\right)\right]. \tag{15}$$

By Theorem 2, this says that $-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]$ is distributed as $-\tilde{c}_B + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|-c_B] = 0$. This is exactly the same as saying that $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ is distributed as $\tilde{c}_B + (-\tilde{\varepsilon})$, where $E[-\tilde{\varepsilon}|c_B] = 0$. Relabel $-\tilde{\varepsilon}$ as $\tilde{\varepsilon}$, and we have

$$\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] \sim \tilde{c}_B + \tilde{\varepsilon}, \quad \textit{i.e.}, \quad \tilde{c}_A \sim \tilde{c}_B + E[\tilde{c}_A - \tilde{c}_B] + \tilde{\varepsilon}, \tag{16}$$

where $E[\tilde{\varepsilon}|c_B + z] = 0$. □

The nonincreasing absolute risk aversion condition is sufficient but not necessary. A quadratic utility function has increasing absolute risk aversion. But, as illustrated by Example 6.1, the non-negative random variable can still be chosen to be a constant for quadratic utility functions (which can be viewed as an implication of two-fund separation and Theorem 7). If the non-negative random variable can be chosen to be a constant, then we have the following corollary:

Corollary 1. *If B is weakly more risk averse than A and either of the two agents has nonincreasing absolute risk aversion, then $Var(\tilde{c}_A) \geqslant Var(\tilde{c}_B)$.*

Proof. From Theorem 5, the non-negative random variable $\tilde{z}$ can be chosen to be the constant $E[\tilde{c}_A - \tilde{c}_B]$. Then we have $E(\tilde{\varepsilon}|\tilde{c}_B) = 0$, which implies that $Cov(\tilde{\varepsilon}, \tilde{c}_B) = 0$. Therefore, $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) + 2\,Cov(\tilde{c}_B, \tilde{\varepsilon}) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) \geqslant Var(\tilde{c}_B)$. □

4. Possibly incomplete market case

Our result still holds in a two-asset world with a risk-free asset. For a two-asset world without a risk-free asset, we have a counter-example to our result holding. Therefore, our main result does not hold in general with incomplete markets. However, our result holds in a two-risky-asset world if we make enough assumptions about asset payoffs and the risk-aversion measure. Also, each two-asset result has a natural analog for models with many assets and two-fund separation, since the portfolio payoffs will be the same as in a two-asset model in which only the two funds are traded.[9] Note that while this section is intended to ask to what extent our results can be extended to incomplete markets, the results also apply to complete markets with two-fund separation.

[9] See Cass and Stiglitz [3] and Ross [18] for characterizations of two-fund separation, *i.e.*, for portfolio choice to be equivalent to choice between two mutual funds of assets.

First, we show that our main result still holds in a two-asset world with a risk-free asset. The proof is in two parts. The first part is the standard result: decreasing the risk aversion increases the portfolio allocation to the asset with higher return. The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. To show the second part, we use the following lemma:

Lemma 2.

1. *If* $E[\tilde{q}]=0$ *and* $0\leqslant m_1\leqslant m_2$*, then* $m_2\tilde{q}\sim m_1\tilde{q}+\tilde{\varepsilon}$*, where* $E[\tilde{\varepsilon}|m_1 q]=0$.
2. *Let* $E[\tilde{\chi}]$ *be finite,* $E[\tilde{q}|\chi]\geqslant 0$*, and* $0\leqslant m_1\leqslant m_2$*. Then* $\tilde{\chi}+m_2\tilde{q}\sim\tilde{\chi}+m_1\tilde{q}+\tilde{z}+\tilde{\varepsilon}$*, where* $\tilde{z}=(m_2-m_1)E[\tilde{q}|\chi]\geqslant 0$ *and* $E[\tilde{\varepsilon}|\chi+m_1 q+z]=0$.

Proof. We prove 2, and 1 follows immediately by setting $\tilde{\chi}=0$ and $E[\tilde{q}]=0$. Let $\tilde{z}_0\equiv E[\tilde{q}|\chi]$ and $\tilde{z}\equiv(m_2-m_1)\tilde{z}_0$. By Theorem 2, we only need to show that, for any concave function $V(\cdot)$, $E[V(\tilde{\chi}+m_2\tilde{q})]\leqslant E[V(\tilde{\chi}+m_1\tilde{q}+\tilde{z})]$. Fix $V(\cdot)$ and let $V'(\cdot)$ be any selection from its subgradient correspondence $\nabla V(\cdot)$ (so $V'(\cdot)$ is the derivative of $V(\cdot)$ whenever it exists). The concavity of $V(\cdot)$ and the definitions of $\tilde{z}_0$ and $\tilde{z}$ imply that

$$V(\tilde{\chi}+m_2\tilde{q})-V(\tilde{\chi}+m_1\tilde{q}+\tilde{z})\leqslant V'(\tilde{\chi}+m_1\tilde{q}+\tilde{z})(m_2-m_1)(\tilde{q}-\tilde{z}_0). \tag{17}$$

Furthermore, $V'(\cdot)$ nonincreasing, $m_2\geqslant m_1\geqslant 0$, and the definitions of $\tilde{z}_0$ and $\tilde{z}$ imply

$$\big(V'(\tilde{\chi}+m_1\tilde{q}+\tilde{z})-V'(\tilde{\chi}+m_2\tilde{z}_0)\big)(m_2-m_1)(\tilde{q}-\tilde{z}_0)\leqslant 0. \tag{18}$$

From (17), (18), and the definitions of $\tilde{z}_0$ and $\tilde{z}$, we get

$$\begin{aligned}
E\big[V(\tilde{\chi}+m_2\tilde{q})\big]-E\big[V(\tilde{\chi}+m_1\tilde{q}+\tilde{z})\big] &\leqslant E\big[V'(\tilde{\chi}+m_1\tilde{q}+\tilde{z})(m_2-m_1)(\tilde{q}-\tilde{z}_0)\big]\\
&\leqslant E\big[V'(\tilde{\chi}+m_2\tilde{z}_0)(m_2-m_1)(\tilde{q}-\tilde{z}_0)\big]\\
&= E\big[E\big[V'(\tilde{\chi}+m_2\tilde{z}_0)(m_2-m_1)(\tilde{q}-\tilde{z}_0)|\chi\big]\big]\\
&=0. \qquad \square
\end{aligned}$$

Now, we consider the following portfolio choice problem:

Problem 2 *(Possibly incomplete market with two assets).* Agent i's ($i=A,B$) problem is

$$\max_{\alpha_i\in R} E\big[U_i(w_0\tilde{x}+\alpha_i w_0\tilde{v})\big],$$

where w_0 is the initial wealth, α_i is the proportion invested in the second asset, and $\tilde{v}$ is the excess of the return on the second asset over the first asset, *i.e.*, $\tilde{v}=\tilde{y}-\tilde{x}$, where $\tilde{x}$ and $\tilde{y}$ are the total returns on the two assets. We assume that $E[\tilde{v}]\geqslant 0$, $\tilde{v}$ is nonconstant, and $E[\tilde{v}]$ and $E[\tilde{x}]$ are finite.

Note. Problem 2 is stated in terms of two possibly risky assets, but it is also formally equivalent to a problem with non-traded wealth. Suppose an agent has non-traded wealth $\tilde{N}$ and allocates financial wealth $f_0>0$ between assets paying $\tilde{\chi}_1$ and $\tilde{\chi}_2$ per dollar invested. Then the choice problem is

$$\max_{\alpha_i\in R} E\big[U_i\big(\tilde{N}+f_0\tilde{\chi}_1+\alpha_i f_0(\tilde{\chi}_2-\tilde{\chi}_1)\big)\big]$$

which is equivalent to Problem 2 if we set $\tilde{x}\equiv\tilde{N}/f_0+\tilde{\chi}_1$, $\tilde{v}\equiv\tilde{\chi}_2-\tilde{\chi}_1$, and $w_0\equiv f_0$.

We denote agent A and B's respective optimal investments in the risky asset with payoff $\tilde{y}$ by α_A^* and α_B^*. The payoff for agent A is $\tilde{c}_A = w_0\tilde{x} + \alpha_A^* w_0 \tilde{v}$ and agent B's payoff is $\tilde{c}_B = w_0\tilde{x} + \alpha_B^* w_0 \tilde{v}$. We maintain the utility assumptions made earlier: $U_i'(\cdot) > 0$ and $U_i''(\cdot) < 0$, so $\tilde{v}$ nonconstant implies that α_A^* and α_B^* are unique if they exist. We have the following well-known result.

Lemma 3. *Suppose $\tilde{x}$ is riskless ($\tilde{x}$ nonstochastic), if B is weakly more risk averse than A, then the agents' solutions to Problem 2 satisfy $\alpha_A^* \geqslant \alpha_B^*$.*

Proof. The first order condition of A's problem is:

$$E\big[U_A'\big(xw_0 + \alpha_A^* w_0 \tilde{v}\big) w_0 \tilde{v}\big] = 0. \tag{19}$$

The analogous expression for B is $\varphi(\alpha_B^*) = 0$, where

$$\varphi(\alpha) \equiv E\big[U_B'(xw_0 + \alpha w_0 \tilde{v}) w_0 \tilde{v}\big]. \tag{20}$$

Since $U_B(\cdot) = G(U_A(\cdot))$, where $G'(\cdot) > 0$ and $G''(\cdot) \leqslant 0$, we have:

$$\begin{aligned}
\varphi\big(\alpha_A^*\big) &= E\big[G'\big(U_A\big(xw_0 + \alpha_A^* w_0 \tilde{v}\big)\big) U_A'\big(xw_0 + \alpha_A^* w_0 \tilde{v}\big) w_0 \tilde{v}\big] \\
&= G'\big(U_A(xw_0)\big) E\big[U_A'\big(xw_0 + \alpha_A^* w_0 \tilde{v}\big) w_0 \tilde{v}\big] \\
&\quad + E\big[\big(G'\big(U_A\big(xw_0 + \alpha_A^* w_0 \tilde{v}\big)\big) - G'\big(U_A(xw_0)\big)\big) U_A'\big(xw_0 + \alpha_A^* w_0 \tilde{v}\big) w_0 \tilde{v}\big] \\
&\leqslant 0,
\end{aligned} \tag{21}$$

where the first term in (21) is zero by (19) and the expression inside the expectation in the second term is non-positive because $G''(\cdot) \leqslant 0$ and $U_A'(\cdot) > 0$. Finally, the concavity of $U_B(\cdot)$ implies that $\varphi(\cdot)$ is decreasing, and therefore from (20) and (21), we must have $\alpha_A^* \geqslant \alpha_B^*$. $\square$

Lemma 3 implies that decreasing the risk aversion increases the portfolio allocation to the asset with higher return. Now, we show that our main result still holds in a two-asset world with a risk-free asset. We have

Theorem 6 *(Two-asset world with a riskless asset). Consider the two-asset world with a riskless asset ($\tilde{x}$ nonstochastic) of Problem 2, if B is weakly more risk averse than A in the sense of Arrow and Pratt, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.*

Proof. When the first asset in Problem 2 is riskless, then we have $\tilde{c}_A - E[\tilde{c}_A] = \alpha_A^* w_0(\tilde{y} - E[\tilde{y}])$ and $\tilde{c}_B - E[\tilde{c}_B] = \alpha_B^* w_0(\tilde{y} - E[\tilde{y}])$. From Lemma 3, $\alpha_A^* \geqslant \alpha_B^*$. Let $\tilde{q} \equiv \tilde{y} - E[\tilde{y}]$, $m_1 \equiv \alpha_B^* w_0$ and $m_2 \equiv \alpha_A^* w_0$ in the first part of Lemma 2, we have $\tilde{c}_A - E[\tilde{c}_A] \sim \tilde{c}_B - E[\tilde{c}_B] + \tilde{\varepsilon}$, which implies that $\tilde{c}_A$ is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. $\square$

Theorem 6 generalizes in obvious ways to settings with two-fund separation since optimal consumption is the same as it would be with ordering the two funds as assets. The main requirement is that one of the funds can be chosen to be riskless, for example, in a mean-variance world with a riskless asset and normal returns for risky assets.[10] In this example, if B is weakly

[10] This example is a special case of two-fund separation in mean-variance worlds or the separating distributions of Ross [18].

more risk averse than A, Theorem 6 tells us that $\tilde{c}_A \sim \tilde{c}_B + z + \tilde{\varepsilon}$, where $z \geqslant 0$ is constant and $E[\tilde{\varepsilon}|c_B + z] = 0$. We know that A's optimal portfolio is further up the frontier than B's, *i.e.*, $E[\tilde{c}_A] \geqslant E[\tilde{c}_B]$ and $Var[\tilde{c}_A] \geqslant Var[\tilde{c}_B]$. This result is verified by noting that we can choose $z = E[\tilde{c}_A - \tilde{c}_B]$, $\tilde{\varepsilon} \sim N(0, Var[\tilde{c}_A] - Var[\tilde{c}_B])$, and $\tilde{\varepsilon}$ is drawn independently of $\tilde{c}_B$.

Now, we examine the case with two risky assets in Problem 2. For a two-asset world without a riskless asset, we have a counter-example to our result holding. In the counter-example, $\alpha_A^* > \alpha_B^*$, but the distributional result does not hold.

Example 4.1. We assume that there are two risky assets and four states. The probabilities for the four states are 0.2, 0.3, 0.3 and 0.2 respectively. The payoff of $\tilde{x}$ is $(10\ 8\ 1\ 1)^T$ and the net payoff $\tilde{v}$ is $(-1\ 1\ 1\ -1)^T$. Agent's utility function is $U_i(\tilde{w}_i) = -e^{-\delta_i \tilde{w}_i}$, where $i = A, B$, and $\tilde{w}_i$ is agent i's terminal wealth. We assume that agent B is weakly more risk averse than A with $\delta_A = 1$ and $\delta_B = 1.5$. The agents solve Problem 2 with initial wealth $w_0 = 1$.

The agents' problems are:

$$\max_{\alpha_A} 0.2e^{-(10-\alpha_A)} + 0.3e^{-(8+\alpha_A)} + 0.3e^{-(1+\alpha_A)} + 0.2e^{-(1-\alpha_A)},$$

and

$$\max_{\alpha_B} 0.2e^{-1.5(10-\alpha_B)} + 0.3e^{-1.5(8+\alpha_B)} + 0.3e^{-1.5(1+\alpha_B)} + 0.2e^{-1.5(1-\alpha_B)}.$$

First order conditions give $\alpha_A^* = \frac{1}{2}\log(\frac{3+3e^{-7}}{2+2e^{-9}}) = 0.2$, and $\alpha_B^* = \frac{1}{3}\log(\frac{3+3e^{-10.5}}{2+2e^{-13.5}}) = 0.135$. Therefore, agent A's portfolio payoff is $(9.8\ 8.2\ 1.2\ 0.8)^T$ and agent B's portfolio payoff is $(9.865\ 8.135\ 1.135\ 0.865)^T$. If agent A's payoff $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, then we have $Pr(\tilde{\varepsilon} \geqslant 0|c_B + z) > 0$, therefore, we have $\max \tilde{c}_A \geqslant \max \tilde{c}_B$. However, in this example, we can see that $\max \tilde{c}_A = 9.8$ and $\max \tilde{c}_B = 9.865$, *i.e.*, $\max \tilde{c}_A < \max \tilde{c}_B$. Contradiction! Therefore, in general, our result does not hold in a two-asset world without a riskless asset.

It is a natural question to ask whether our main result holds in a two-risky-asset world if we make enough assumptions about asset payoffs. We can, if we use Ross's stronger measure of risk aversion (see Ross [18]) and his payoff distributional condition. We have

Theorem 7 *(Two risky assets with Ross's measure). Consider the two-risky-asset world of Problem 2 with $E[\tilde{v}|x] \geqslant 0$ for all x. If B is weakly more risk averse than A under Ross's stronger measure of risk aversion, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, and $\tilde{z} \geqslant 0$.*

Proof. Our proof is in two parts. The first part is from Ross [19]: if agent A is weakly less risk averse than B under Ross's stronger measure, then $\alpha_A^* \geqslant \alpha_B^*$. The first order condition of A's problem is

$$E\left[U_A'\left(w_0\tilde{x} + \alpha_A^* w_0 \tilde{v}\right) w_0 \tilde{v}\right] = 0. \tag{22}$$

The analogous expression for B is $\varphi(\alpha_B^*) = 0$, where

$$\varphi\left(\alpha_B^*\right) \equiv E\left[U_B'\left(w_0\tilde{x} + \alpha_B^* w_0 \tilde{v}\right) w_0 \tilde{v}\right]. \tag{23}$$

From Ross [19], if B is weakly more risk averse than A under Ross's stronger measure, then there exist $\lambda > 0$ and a concave decreasing function $G(\cdot)$, such that $U_B(\cdot) = \lambda U_A(\cdot) + G(\cdot)$.

Therefore,

$$\begin{aligned}\varphi(\alpha_A^*) &= E\big[\big(\lambda U_A'\big(w_0\tilde{x}+\alpha_A^* w_0\tilde{v}\big)+G'\big(w_0\tilde{x}+\alpha_A^* w_0\tilde{v}\big)\big)w_0\tilde{v}\big] \\ &= E\big[G'\big(w_0\tilde{x}+\alpha_A^* w_0\tilde{v}\big)w_0\tilde{v}\big] = E\big[E\big[G'\big(w_0\tilde{x}+\alpha_A^* w_0\tilde{v}\big)w_0\tilde{v}|x\big]\big] \leqslant 0, \end{aligned} \tag{24}$$

where the last inequality is a consequence of the fact that $G'(\cdot)$ is negative and decreasing while $E[\tilde{v}|x] \geqslant 0$. The concavity of $U_B(\cdot)$ implies that $\varphi(\cdot)$ is decreasing. Therefore, from (22) and (24), we have $\alpha_A^* \geqslant \alpha_B^*$.

The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. Letting $\tilde{q} \equiv \tilde{v}$, $\tilde{\chi} \equiv w_0\tilde{x}$, $m_1 \equiv \alpha_B^* w_0$ and $m_2 \equiv \alpha_A^* w_0$ in Lemma 2, part 2, we have $w_0\tilde{x}+\alpha_A^* w_0\tilde{v} \sim w_0\tilde{x}+\alpha_B^* w_0\tilde{v}+\tilde{z}+\tilde{\varepsilon}$, *i.e.*, $\tilde{c}_A \sim \tilde{c}_B+\tilde{z}+\tilde{\varepsilon}$, where $\tilde{z} = w_0(\alpha_A^*-\alpha_B^*)E[\tilde{v}|x] \geqslant 0$ and $E[\tilde{\varepsilon}|c_B+z]=0$. □

Theorem 7 implies that our main result holds when we use Ross's stronger measure of risk aversion with the assumption of $E[\tilde{v}|x] \geqslant 0$. If the condition $E[\tilde{v}|x] \geqslant 0$ is not satisfied, then our main result may not hold even when we use Ross's stronger measure of risk aversion as we can see in the following example.

Example 4.2. We assume that there are two risky assets and four states. The probabilities for the four states are 0.3, 0.2, 0.3 and 0.2 respectively. The payoff of $\tilde{x}$ is $(10\ 8\ 1\ 1)^T$ and the net payoff $\tilde{v}$ is $(-1\ 1\ 1\ -1)^T$. Agent A's utility function is $U_A(\tilde{w}_A) = e^6\tilde{w}_A - e^{-\tilde{w}_A}$, and agent B's utility function is $U_B(\tilde{w}_B) = \tilde{w}_B - e^{6-1.5\tilde{w}_B}$, where $\tilde{w}_i$ is the terminal wealth of agent i. The agents solve Problem 2 with initial wealth $w_0 = 1$. We have

$$\frac{U_B''(w)}{U_A''(w)} = \frac{2.25e^{6-1.5w}}{e^{-w}} = 2.25e^{6-0.5w}, \qquad \frac{U_B'(w)}{U_A'(w)} = \frac{1+1.5e^{6-1.5w}}{e^6+e^{-w}}.$$

Therefore, $\inf_w \frac{U_B''(w)}{U_A''(w)} > \sup_w \frac{U_B'(w)}{U_A'(w)}$, for any $0 \leqslant w \leqslant 10$, which implies that agent B is strictly more risk aversion than agent A under Ross's stronger measure of risk aversion.

The agents' problems are:

$$\begin{aligned}\max_{\alpha_A} 0.3&\big(e^6(10-\alpha_A)-e^{-(10-\alpha_A)}\big)+0.2\big(e^6(8+\alpha_A)-e^{-(8+\alpha_A)}\big) \\ &+0.3\big(e^6(1+\alpha_A)-e^{-(1+\alpha_A)}\big)+0.2\big(e^6(1-\alpha_A)-e^{-(1-\alpha_A)}\big),\end{aligned}$$

and

$$\begin{aligned}\max_{\alpha_B} 0.3&\big(10-\alpha_B-e^{6-1.5(10-\alpha_B)}\big)+0.2\big(8+\alpha_B-e^{6-1.5(8+\alpha_B)}\big) \\ &+0.3\big(1+\alpha_B-e^{6-1.5(1+\alpha_B)}\big)+0.2\big(1-\alpha_B-e^{6-1.5(1-\alpha_B)}\big).\end{aligned}$$

From the first order condition, $e^{2\alpha_A^*} = \frac{3+2e^{-7}}{2+3e^{-9}}$, *i.e.*, $\alpha_A^* = \frac{1}{2}\log(\frac{3+2e^{-7}}{2+3e^{-9}}) = 0.2029$, and $e^{3\alpha_B^*} = \frac{3e^{-1.5}+2e^{-12}}{2e^{-1.5}+3e^{-15}}$, *i.e.*, $\alpha_B^* = \frac{1}{3}\log(\frac{3e^{-1.5}+2e^{-12}}{2e^{-1.5}+3e^{-15}}) = 0.1352$. Therefore, agent A's portfolio payoff is

$$(9.7971\ 8.2029\ 1.2029\ 0.7971)^T$$

and agent B's portfolio payoff is

$$(9.8648\ 8.1352\ 1.1352\ 0.8648)^T.$$

If agent A's payoff $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, then we have $Pr(\tilde{\varepsilon} \geqslant 0|c_B + z) > 0$. Therefore, we have $\max \tilde{c}_A \geqslant \max \tilde{c}_B$. However, in this example, we can see that $\max \tilde{c}_A = 9.7971$ and $\max \tilde{c}_B = 9.8648$, *i.e.*, $\max \tilde{c}_A < \max \tilde{c}_B$. Contradiction! Therefore, in a two-risky-asset world, our main result does not hold in general even under Ross's stronger measure of risk aversion if we don't make the assumption that $E[\tilde{v}|x] \geqslant 0$. □

An alternative to the approach following Ross [19] is the approach of Kihlstrom, Romer and Williams [10] for handling random base wealth. They show that the Arrow–Pratt measure works if we restrict attention to comparisons in which (1) at least one of the utility functions has non-increasing absolute risk aversion and (2) base wealth is independent of the other gambles. Here is how their argument works. The independence implies that we can convert a problem with random base wealth x to a problem with nonrandom base wealth by using the indirect utility functions $\hat{U}_i(w) \equiv E[U_i(\tilde{x} + w)]$, and our results for nonrandom base wealth apply directly. For this to work, the indirect utility functions $\hat{U}_A$ and $\hat{U}_B$ must inherit the risk aversion ordering from U_A and U_B, which as they point out, does not happen in general. However, letting $F(\cdot)$ be the distribution function of $\tilde{x}$, simple calculations tell us that provided integrals exist, we can write

$$-\frac{\hat{U}_i''(w)}{\hat{U}_i'(w)} = \int \frac{U_i'(\tilde{x} + w)}{\int U_i'(\tilde{y} + w)\, dF(\tilde{y} + w)} \left(-\frac{U_i''(\tilde{x} + w)}{U_i'(\tilde{x} + w)}\right) dF(\tilde{x}). \tag{25}$$

For both agents, the risk aversion of the indirect utility function is therefore a weighted average of the risk aversion of the direct utility function, but the weights are different so the risk aversion ordering is not preserved in general (since the more risk averse agent may have relatively higher weights from wealth regions where both agents have small risk aversion). However, we do know that the more risk averse agents' weights put relatively higher weight on lower wealth levels (since i's absolute risk aversion is $-d\log(U_i'(w)/dw)$), so if either agent has nonincreasing absolute risk aversion, then the risk aversion ordering of the direct utility function is inherited by the indirect utility function. Subject to existence of some integrals (ensured by compactness in their paper), their results and our Theorem 6 imply that if B is weakly more risk averse than A, at least one of U_A and U_B has nonincreasing absolute risk aversion, and $\tilde{v}$ is independent of $\tilde{x}$, then our main result holds: $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.

As we have shown that our main result does not hold in general in the traditional type of incomplete markets where portfolio payoffs are restricted to a subspace. In addition, as we will see in Example 6.6, our main result does not hold in incomplete markets where agents have non-traded asset. However, it is an open question whether the results extend to more interesting models of incomplete markets in which there is a reason for the incompleteness. For example, a market that is complete over states distinguished by security returns and incomplete over other private states with an insurance need (see Chen and Dybvig [5]). Another type of incompleteness comes from a non-negative wealth constraint (which is an imperfect solution to information problems when investors have private information or choices related to default), which means agents have individual incompleteness and cannot fully hedge future non-traded wealth or else they would violate the non-negative wealth constraint (see Dybvig and Liu [7]).

5. Extension to a multiple-period model

There is a trivial extension of our analysis to multiple-period models with complete markets and consumption only at the end, and indeed this is the normal motivation for Problem 1. See, e.g., Cox and Huang [2] to see how to restate a continuous-time model without consumption withdrawal in the form of Problem 1. This section is concerned with a less trivial extension to consumption at multiple dates. In these models, the result of Theorem 3 typically does not hold at every date because changing risk aversion in effect changes time preference as well, and the resulting shift of consumption over time means that one agent consumes more on average at some dates and the other agent consumes more on average at other dates. However, we can still show that a form of Theorem 3 holds, for a carefully chosen mixture across dates of the distribution of consumption.

We now examine our main results in a multiple-period model. We assume that each agent's problem is:

Problem 3.

$$\max_{\tilde{c}_t} E\left[\sum_{t=1}^{T} D_t U_i(\tilde{c}_t)\right],$$

$$s.t. \quad E\left[\sum_{t=1}^{T} \tilde{\rho}_t \tilde{c}_t\right] = w_0, \tag{26}$$

where $i = A$ or B indexes the agent, $D_t > 0$ is a utility discount factor (e.g., $D_t = e^{-\kappa t}$ if the pure rate of time discount κ is constant), and $\tilde{\rho}_t$ is the state price density in period t.

Again, we will assume that both agents have optimal random consumptions, denoted $\tilde{c}_{At}$ and $\tilde{c}_{Bt}$, and both $\tilde{c}_{At}$ and $\tilde{c}_{Bt}$ have finite means. We also assume that the utility discount factors are the same for both agents. The first order condition gives us

$$U_i'(\tilde{c}_{it}) = \lambda_i \frac{\tilde{\rho}_t}{D_t}, \quad i = A, B,$$

we have

$$\tilde{c}_{it} = I_i\left(\lambda_i \frac{\tilde{\rho}_t}{D_t}\right),$$

where $I_i(\cdot)$ is the inverse function of $U_i'(\cdot)$. By negativity of the second order derivatives, $\tilde{c}_{it}$ is a decreasing function of $\tilde{\rho}_t$.

By similar arguments in the one-period model, we have

Lemma 4. *If B is weakly more risk averse than A, then*

1. *for each t, there exists some critical consumption level c_t^* (can be $\pm\infty$) such that $\tilde{c}_{At} \geqslant \tilde{c}_{Bt}$ when $\tilde{c}_{Bt} \geqslant c_t^*$, and such that $\tilde{c}_{At} \leqslant \tilde{c}_{Bt}$ when $\tilde{c}_{Bt} \leqslant c_t^*$;*
2. *if it happens that the budget shares as a function of time are the same for both agents at some time t, i.e., $E[\tilde{\rho}_t \tilde{c}_{At}] = E[\tilde{\rho}_t \tilde{c}_{Bt}]$, then $E[\tilde{c}_{At}] \geqslant E[\tilde{c}_{Bt}]$, and we have $\tilde{c}_{At} \sim \tilde{c}_{Bt} + \tilde{z}_t + \tilde{\varepsilon}_t$, where $\tilde{z}_t \geqslant 0$ and $E[\tilde{\varepsilon}_t | c_{Bt} + z_t] = 0$. And if $\tilde{c}_{At} \neq \tilde{c}_{Bt}$, then neither $\tilde{z}_t$ nor $\tilde{\varepsilon}_t$ is identically zero. In particular, if the budget shares are the same for all t, then this distributional condition holds for all t.*

The proof of Lemma 4 is essentially the same as the proof of the corresponding parts of Lemma 1, and Theorem 3 in the one-period model. If the D_t is not the same for both agents, or the same for the two agents without any restriction on budget shares, then the distributional condition may not hold in any period. For example, if the weakly more risk averse agent B spends most of the money earlier but the weakly less risk averse agent A spends more later, then the mean payoff could be higher in an earlier period for the weakly more risk averse agent, *i.e.*, $E[\tilde{c}_{Bt}] \geqslant E[\tilde{c}_{At}]$. The following example shows that the result of Theorem 3 typically does not hold at every date because changing risk aversion in effect changes time preference as well.

Example 5.1. An investor chooses $\tilde{c}_t$ to maximize

$$E\left[\int_{t=0}^{\infty} e^{-\beta t} U(\tilde{c}_t)\, dt\right] \quad s.t. \quad w_0 = E\left[\int_{t=0}^{\infty} \tilde{\rho}_t \tilde{c}_t\, dt\right].$$

Investors' utility function $U(\tilde{c}_t) = \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma}$, where γ is investors' risk-aversion. The state price density is $\tilde{\rho}_t = e^{-(r+\frac{1}{2}\kappa^2)t - \kappa Z_t}$, where $\kappa > 0$ is the Sharpe ratio, r is risk-free rate, and Z_t is a standard Wiener process. This is consistent with Merton's model with constant means of returns. A discrete model with finite horizon, more exactly in the form of (26), can also be used to show the same result, but with messier algebra.

The first order condition gives us $e^{-\beta t} c_t^{-\gamma} = \lambda \rho_t$, *i.e.*, $\tilde{c}_t = (\lambda e^{\beta t} \rho_t)^{-\frac{1}{\gamma}}$. Substituting $\tilde{c}_t$ into the budget constraint, we get

$$w_0 = E\left[\int_0^{\infty} e^{-(1-\frac{1}{\gamma})(r+\frac{1}{2}\kappa^2)t - (1-\frac{1}{\gamma})\kappa Z_t - \frac{\beta}{\gamma} t} \lambda^{-\frac{1}{\gamma}}\, dt\right].$$

We get $\lambda = w_0^{-\gamma}((r + \frac{\kappa^2}{2\gamma})(1 - \frac{1}{\gamma}) + \frac{\beta}{\gamma})^{-\gamma}$, and therefore

$$\tilde{c}_t = w_0\left(\left(r + \frac{\kappa^2}{2\gamma}\right)\left(1 - \frac{1}{\gamma}\right) + \frac{\beta}{\gamma}\right) e^{-\frac{\beta}{\gamma} t + (r+\frac{1}{2}\kappa^2)\frac{t}{\gamma} + \frac{\kappa}{\gamma} Z_t}.$$

Therefore, we have $E[\tilde{c}_t]$ increases in γ if and only if

$$t\left(\beta - \frac{\kappa^2}{\gamma} - \left(r + \frac{1}{2}\kappa^2\right)\right)\left(\left(r + \frac{\kappa^2}{2\gamma}\right)\left(1 - \frac{1}{\gamma}\right) + \frac{\beta}{\gamma}\right) > \beta + \frac{\kappa^2}{2} - \frac{\kappa^2}{\gamma} - r.$$

If $\beta < r - \frac{\kappa^2}{2} + \frac{\kappa^2}{\gamma}$,[11] then there exists $\varepsilon > 0$ such that $E[\tilde{c}_t]$ increases in γ for all $t \in [0, \varepsilon]$. Therefore, when there is no restriction on budget shares, the distributional result (of Theorem 3) does not hold for $t \in [0, \varepsilon]$.[12]

Although our main result does not hold period-by-period, it holds "on average" for a mixture of the distributions across periods (with mixing weights proportional to the two agents' shared utility discount factors). Thus, there is a sense in which the overall distribution of payoffs, across time as well as states of nature, represents a trade-off between risk and return, even though no such relationship exists in general period-by-period.

[11] For the choice problem to have a solution, we also need to impose Merton's condition $\beta > (r + \frac{\kappa^2}{2\gamma})(1 - \gamma)$, which is consistent with this restriction.

[12] This example can also be used to show that the wealth processes are not ordered, disproving another possible conjecture.

Theorem 8. *In a multiple-period model, assume agents A and B have the same discount factors D_t and solve Problem 2. Let F_{At} (resp. F_{Bt}) be the cumulative distribution function of $\tilde{c}_{At}$ (resp. $\tilde{c}_{Bt}$). Given any vector $\mu = (\mu_1, \dots, \mu_T)$ of positive weights summing to one, choose $\tilde{c}_A$ (resp. $\tilde{c}_B$) to be a random variable with cumulative distribution function $F_A(c) = \sum_{t=1}^T \mu_t F_{At}(c)$ (resp. $F_B(c) = \sum_{t=1}^T \mu_t F_{Bt}(c)$). Then there exists a vector μ of weights summing to one such that our earlier results hold "on average", i.e. $\tilde{c}_A$ and $\tilde{c}_B$ satisfy*

1. *if B is weakly more risk averse than A, then, $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$, $E[\tilde{\varepsilon}|c_B + z] = 0$;*
2. *if B is weakly more risk averse than A and either of the two agents has nonincreasing absolute risk aversion, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.*

Proof. Suppose the original probability space has probability measure P over states Ω with filtration $\{\mathcal{F}_t\}$. We define the discrete random variable τ on associated probability space $(\Omega^*, \mathcal{F}^*, P^*)$ so that $P^*(\tau = t) = \mu_t \equiv D_t / (\sum_{t=1}^T D_t)$. We then define a single-period problem on a new probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Define the state of nature in the product space $(t, \omega) \in \hat{\Omega} \equiv \Omega^* \times \Omega$ with t and ω drawn independently. Let $\hat{\mathcal{F}}$ be the optional σ-algebra, which is the completion of $\mathcal{F}^* \times \mathcal{F}_\tau$. The synthetic probability measure is the one consistent with independence generated from $\hat{P}(f^*, f) = P^*(f^*) \times P(f)$ for all subsets $f^* \in \mathcal{F}^*$ and subsets $f \in \mathcal{F}_\tau$.

The synthetic probability measure assigns a probability measure that looks like a mixture model, drawing time first assigning probability μ_t to time t, and then drawing from $\tilde{\rho}_t$ using its distribution in the original problem.

Recall that under the original probability measure, each agent's problem is given in (26). We choose $\tilde{c}_A$ (resp. $\tilde{c}_B$) to be a random variable with cumulative distribution function $F_A(c) = \sum_{t=1}^T \mu_t F_{At}(c)$ (resp. $F_B(c) = \sum_{t=1}^T \mu_t F_{Bt}(c)$). Now we want to write down an equivalent problem, in terms of the choice of distribution of each $\tilde{c}_t$, but with the new synthetic probability measure. The consumption $\tilde{c}$ under the new probability space over which synthetic probabilities are defined is a function of $\tilde{\rho}$ and t; we identify $\tilde{c}(\tilde{\rho}, t)$ with what used to be $\tilde{c}_t(\tilde{\rho})$. To write the objective function in terms of the synthetic probabilities, we can write

$$\begin{aligned} E\left[\sum_{t=1}^T D_t U(\tilde{c}_t)\right] &= \sum_{t=1}^T D_t E\big[U(\tilde{c}_t)\big] = \sum_{t=1}^T \left(\sum_{s=1}^T D_s\right) \mu_t \hat{E}\big[U(\tilde{c})|t\big] \\ &= \left(\sum_{s=1}^T D_s\right) \sum_{t=1}^T \mu_t \hat{E}\big[U(\tilde{c})|t\big] = \left(\sum_{s=1}^T D_s\right) \hat{E}\big[U(\tilde{c})\big], \end{aligned} \tag{27}$$

where $\hat{E}$ denotes the expectation under the synthetic probability. $\sum_{s=1}^T D_s$ is a positive constant, so the objective function is equivalent to maximizing $\hat{E}[U(\tilde{c})]$.

Now, we can write the budget constraint in terms of the synthetic probabilities,

$$w_0 = E\left[\sum_{t=1}^T \tilde{\rho}_t \tilde{c}_t\right] = \sum_{t=1}^T \mu_t E\left[\frac{\tilde{\rho}_t}{\mu_t} \tilde{c}_t\right] = \sum_{t=1}^T \mu_t \hat{E}\left[\frac{\tilde{\rho}}{\mu} \tilde{c}|t\right] = \hat{E}\left[\frac{\tilde{\rho}}{\mu} \tilde{c}\right]. \tag{28}$$

Then we can apply our single-period results (Theorem 3, 4 and 5) and the formula for the distribution of the mixture of the $\tilde{c}_{it}$'s to derive that our main results hold on a mixture model of the $\tilde{c}_A$ and $\tilde{c}_B$ over time. □

Therefore, if (as we expect) the budget shares are not the same for both agents at each time period t, then the distributional result may not hold period-by-period in a multiple-period model with time-separable von Neumann–Morgenstern utility having identical weights over time. However, Theorem 8 implies that our main results still hold for a weighted average distribution function in a multiple-period model. These results retain the spirit of our main results while acknowledging that changing risk aversion may cause consumption to shift over time.

6. Examples

In Example 6.1, we illustrate our main result with specific distribution of $\tilde{c}_A$, $\tilde{c}_B$ and $\tilde{\varepsilon}$. In this example, the non-negative random variable $\tilde{z}$ can be chosen to be a constant, and therefore from Corollary 1 in Section 3, the variance of the less risk averse agent's payoff is higher.

Example 6.1. B is weakly more risk averse than A, A and B have the same initial wealth $w_0 = 12$ and the utility functions are as follows

$$U_A(\tilde{c}) = -\frac{1}{2}(40-\tilde{c})^2, \qquad U_B(\tilde{c}) = -\frac{1}{2}(26-\tilde{c})^2,$$

where $\tilde{c} < 40$ for agent A, and $\tilde{c} < 26$ for agent B. We assume that the state price density $\tilde{\rho}$ is uniformly distributed in $[\frac{2}{3}, \frac{4}{3}]$. The first order conditions give us $\tilde{c}_A = 40 - \lambda_A\tilde{\rho}$, and $\tilde{c}_B = 26 - \lambda_B\tilde{\rho}$. Because $E[\tilde{\rho}] = 1$ and $E[\tilde{\rho}^2] = \frac{28}{27}$, the budget constraint $E[\tilde{\rho}\tilde{c}_i] = 12$, $i = A, B$, implies that $\lambda_A = 27$ and $\lambda_B = \frac{27}{2}$. Therefore, $\tilde{c}_A$ is uniformly distributed in $[4, 22]$ and $\tilde{c}_B$ is uniformly distributed in $[8, 17]$. We have $E[\tilde{c}_A] - E[\tilde{c}_B] = \frac{1}{2}$. Let $\tilde{\varepsilon}$ have a Bernoulli distribution drawn independently of $\tilde{c}_B$ with two equally possible outcomes $\frac{9}{2}$ and $-\frac{9}{2}$. It is not difficult to see that $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} = E[\tilde{c}_A] - E[\tilde{c}_B] = \frac{1}{2}$, and $\tilde{\varepsilon}$ is independent of $\tilde{c}_B$, which implies $E[\tilde{\varepsilon}|c_B + z] = 0$.

Next, in Example 6.2, we show that in general $\tilde{z}$ may not be chosen to be a constant. Interestingly, the variance of the weakly less risk averse agent's payoff can be smaller than the variance of the weakly more risk averse agent's payoff.

Example 6.2. B is weakly more risk averse than A, A and B have the same initial wealth $w_0 = 1$ and the utility functions are as follows

$$U_A(\tilde{c}) = -\frac{(8-\tilde{c})^3}{3}, \qquad U_B(\tilde{c}) = -\frac{(8-\tilde{c})^5}{5},$$

where $\tilde{c} < 8$. The first order conditions give us

$$U_A'(\tilde{c}_A) = (8-\tilde{c}_A)^2 = \lambda_A\tilde{\rho}, \qquad U_B'(\tilde{c}_B) = (8-\tilde{c}_B)^4 = \lambda_B\tilde{\rho}. \tag{29}$$

Therefore,

$$\tilde{c}_A = 8 - \sqrt{\lambda_A\tilde{\rho}}, \qquad \tilde{c}_B = 8 - (\lambda_B\tilde{\rho})^{1/4}. \tag{30}$$

From (30), we get

$$\tilde{c}_A = 8 - \sqrt{\frac{\lambda_A}{\lambda_B}}(8-\tilde{c}_B)^2. \tag{31}$$

We have: $\tilde{c}_A \geqslant \tilde{c}_B$ iff $\tilde{c}_B \geqslant 8 - \sqrt{\frac{\lambda_B}{\lambda_A}}$. From Theorem 3, we know that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. To find an example that the variance of the less risk averse agent's payoff can be smaller, we assume that $\tilde{\rho}$ has a discrete distribution, *i.e.*, $\rho_1 = \varepsilon$ with probability $\frac{1}{2}$, $\rho_2 = \frac{1}{4}$ with probability $\frac{1}{4}$, and $\rho_3 = \frac{1}{2}$ with probability $\frac{1}{4}$. If ε is very tiny (close to zero), then from (30) and the budget constraint $E[\tilde{\rho}\tilde{c}_A] = 1$. It is not difficult to compute $\lambda_A \approx 17.5$, $\lambda_B \approx 125.8$, $\tilde{c}_A \approx (8\ 5.91\ 5.045)$ and $\tilde{c}_B \approx (8\ 5.632\ 5.184)$. Therefore, $E[\tilde{c}_A] \approx 6.73$, $E[\tilde{c}_B] \approx 6.70$, and $Var(\tilde{c}_A) \approx 1.684 < Var(\tilde{c}_B) \approx 1.704$, *i.e.*, the variance of the weakly more risk averse agent's payoff is higher. In this example, both agents' utility functions have increasing absolute risk aversion, which gets very high at the shared satiation point $\tilde{c} = 8$. In the high-consumption (low $\tilde{\rho}$), the optimal consumptions of agent A and B are both very close to 8. To have $E[\tilde{c}_A] > E[\tilde{c}_B]$, $\tilde{z}$ is greater in the low-consumption states. Therefore $\tilde{z}$ is large when $\tilde{c}$ is small and small when $\tilde{c}$ is large, and thus $\tilde{z}$ is very negatively correlated with $\tilde{c}_B + \tilde{\varepsilon}$. As noted in Section 3, we know that if the non-negative random variable $\tilde{z}$ can be chosen to be a constant, then $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) \geqslant Var(\tilde{c}_B)$. Therefore, in this example, $\tilde{z}$ cannot be chosen to be a constant.

The next example shows that if the utility functions are not strictly concave, then our main result does not hold.

Example 6.3. B is weakly more risk averse than A, A and B have the same initial wealth $w_0 = 1$ and the utility functions are $U_A(\tilde{c}) = U_B(\tilde{c}) = \tilde{c}$. We assume there are two states with $\rho_1 = \frac{1}{2}$ with probability $\frac{1}{3}$, and $\rho_2 = \frac{1}{2}$ with probability $\frac{2}{3}$. It is not difficult to see that $\tilde{c}_A = (0, 3)$ and $\tilde{c}_B = (4, 1)$ is an optimal consumption for agent A and B for $\lambda_A = \lambda_B = 2$. We have $E[\tilde{c}_A] = E[\tilde{c}_B] = 2$ and $Var[\tilde{c}_A] = Var[\tilde{c}_B] = 2$. If $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$, then $\tilde{z} = 0$ and $\tilde{\varepsilon} = 0$, we get $\tilde{c}_A \sim \tilde{c}_B$. Contradiction! So, we cannot have $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.

Example 6.3 is degenerate with constant $\tilde{\rho}$ and linear utility. It is not difficult to construct a more general example (Example 6.4), where $\tilde{\rho}$ is random and the utility function has two straight segments. The optimal portfolio is not unique on these two straight segments taken together and therefore our payoff distributional result may not hold.

Example 6.4. B is weakly more risk averse than A, A and B have the same initial wealth $w_0 = 2$ and the utility functions are as follows

$$U_A(\tilde{c}) = U_B(\tilde{c}) = \begin{cases} -(\tilde{c}-1)^4 + \tilde{c} & \tilde{c} < 1 \\ \tilde{c} & 1 \leqslant \tilde{c} \leqslant 2 \\ \frac{1}{256}(\tilde{c}^4 - 16\tilde{c}^3 + 72\tilde{c}^2 + 128\tilde{c} + 80) & 2 < \tilde{c} < 6 \\ \frac{1}{2}\tilde{c} + 2 & 6 \leqslant \tilde{c} \leqslant 14 \\ \frac{1}{2}e^{-(\tilde{c}-14)} - 2e^{-(\tilde{c}-14)/2} + 9 & \tilde{c} \geqslant 14. \end{cases}$$

In this example, the utility function has two straight segments and the optimal portfolio is not unique on these two straight segments taken together. We assume that $\rho_1 = \frac{1}{2}$ with probability $\frac{1}{2}$ and $\rho_2 = \frac{1}{4}$ with probability $\frac{1}{2}$. Then, it is not difficult to see that $\tilde{c}_A = (2, 12)$ and $\tilde{c}_B = (1, 14)$ is the optimal consumption for agent A and B for $\lambda_A = \lambda_B = 2$. So, while A is weakly less risk averse than B (their risk aversion is equal everywhere), $\tilde{c}_A$ is not distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$ with $\tilde{z} \geqslant 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.

It is natural to think of the completeness in our model as coming from dynamic trading in a continuous-time model. This is a good setting for seeing that our distributional result holds even if it is hard to interpret what is happening with portfolio weights. In the next example, we consider a continuous-time model with one-year investment horizon. There are two assets: a locally riskless bond and a one-year risky discount bond. We show that a very risk averse agent may invest all of his wealth in the one-year risky discount bond while a less risk averse agent invests part of his wealth in the locally riskless bond. Therefore, the comparative statics results in portfolio weights do not hold in a continuous-time model with two assets.[13] However, our comparative statics results in the distribution of portfolio payoffs still hold.

Example 6.5. There are two assets that trade continuously: a locally riskless bond and a one-year discount bond that is locally risky because the interest rate is random. Agents are endowed with wealth w_0 at time 0 and consume $\tilde{c}$ at time 1. Each investor has constant relative risk aversion $U(\tilde{c}) = \frac{\tilde{c}^{1-\gamma}}{1-\gamma}$ (or $U(\tilde{c}) = \log(\tilde{c})$ if $\gamma = 1$), and chooses a dynamic portfolio strategy to maximize $E[U(\tilde{c})]$, where $\tilde{c}$ equals wealth at time 1. The interest rate follows the absolute[14] Vasicek process $dr_t = \sigma\, dZ_t$, or equivalently $r_t = r_0 + \sigma Z_t$, where Z_t is a standard Wiener process. The state price density is

$$\tilde{\rho}_t = e^{-\int_0^t (r_s + \frac{1}{2}\kappa^2)\, ds - \int_0^t \kappa\, dZ_s} = e^{-r_0 t - \int_0^t (\kappa + \sigma(t-s))\, dZ_s - \frac{\kappa^2}{2} t}, \tag{32}$$

where $\kappa > 0$ is the local Sharpe ratio. We have $\tilde{\rho}_1 = e^{-r_0 - \int_0^1 (\kappa + \sigma(1-s))\, dZ_s - \frac{\kappa^2}{2}}$. Agents' problem is $\max_{\tilde{c}} E[\frac{\tilde{c}^{1-\gamma}}{1-\gamma}]$, subject to the budget constraint $E[\tilde{\rho}_1 \tilde{c}] = w_0$.

The first order condition gives us $\tilde{c}^{-\gamma} = \lambda \tilde{\rho}_1$. Substituting $\tilde{c} = (\lambda \rho_1)^{-\frac{1}{\gamma}}$ into the budget constraint, we get

$$\lambda = e^{-r_0(\gamma-1) - \frac{\kappa^2}{2}(\gamma-1) + \frac{\gamma}{2}(1-\frac{1}{\gamma})^2(\kappa^2 + \frac{\sigma^2}{3} + \kappa\sigma)}.$$

Therefore, we have

$$\tilde{c} = e^{r_0 + \frac{1}{2}\kappa^2 - \frac{1}{2}(1-\frac{1}{\gamma})^2(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma) + \frac{1}{\gamma}\int_0^1 (\kappa + \sigma(1-s))\, dZ_s}. \tag{33}$$

Suppose that there are two agents A and B with risk aversion γ_A and γ_B, with $\gamma_A < \gamma_B$. For $i = A, B$, we have

$$\tilde{c}_i \sim \ln N\left(r_0 + \frac{1}{2}\kappa^2 - \frac{1}{2}\left(1 - \frac{1}{\gamma_i}\right)^2 \left(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma\right), \frac{1}{\gamma_i^2}\left(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma\right)\right). \tag{34}$$

It is not difficult to show that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} = \tilde{c}_B(e^{(\frac{1}{\gamma_A} - \frac{1}{\gamma_B})(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma)} - 1) > 0$, and

$$\tilde{\varepsilon} = \tilde{c}_B e^{(\frac{1}{\gamma_A} - \frac{1}{\gamma_B})(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma)}\left(e^{\eta - \frac{1}{2}(\frac{1}{\gamma_A^2} - \frac{1}{\gamma_B^2})(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma)} - 1\right),$$

where $\eta \sim N(0, (\frac{1}{\gamma_A^2} - \frac{1}{\gamma_B^2})(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma))$ and is drawn independently of $\tilde{c}_B$. This confirms our comparative statics result for the distribution of portfolio payoffs from Theorem 3. However,

[13] See also J. Liu [11], who studies the dynamic asset allocation between a riskless and risky asset with stochastic volatility and shows that a more risk-averse investor may hold more of the risky asset because of hedging demand for the volatility factor.

[14] A more complex Vasicek process with mean reversion gives similar results but the calculations are more complex.

we next show that the comparative static result in portfolio weights does not hold, *i.e.*, the more risk averse agent may invest more in the locally risky bond.

Investor's wealth at time t,

$$W_t = E_t\left[\frac{\tilde{\rho}_1 \tilde{c}}{\tilde{\rho}_t}\right] = f(t) e^{\int_0^t ((\kappa+\sigma(t-s))-(1-\frac{1}{\gamma})(\kappa+\sigma(1-s)))\, dZ_s}, \tag{35}$$

where $f(t) = e^{r_0 t + \frac{1}{2}\kappa^2 t - \frac{1}{2}(1-\frac{1}{\gamma})^2(\frac{1}{3}\sigma^2 t^2 - \sigma(\kappa+\sigma)t + (\kappa+\sigma)^2)t}$. Using Ito's Lemma, we get

$$\begin{aligned}\frac{dW_t}{W_t} &= \left(r_t + \kappa\left(\kappa - \left(1 - \frac{1}{\gamma}\right)\big(\kappa + \sigma(1-t)\big)\right)\right) dt \\ &\quad + \left(\kappa - \left(1 - \frac{1}{\gamma}\right)\big(\kappa + \sigma(1-t)\big)\right) dZ_t.\end{aligned} \tag{36}$$

The discount bond price at time t,

$$B_t = E_t\left[\frac{\tilde{\rho}_1}{\tilde{\rho}_t}\right] = g(t) e^{\int_0^t \sigma(t-1)\, dZ_s}, \tag{37}$$

where $g(t) = e^{-(r_0 + \frac{1}{2}\kappa^2)(1-t) + \frac{1}{6\sigma}((\kappa+\sigma(1-t))^3 - \kappa^3)}$. Using Ito's Lemma, we have

$$\frac{dB_t}{B_t} = \big(r_t + \kappa\sigma(t-1)\big) dt + \sigma(t-1)\, dZ_t. \tag{38}$$

From (36) and (38), we get that the investor with risk aversion γ optimally invests

$$\frac{\kappa - (1-\frac{1}{\gamma})(\kappa + \sigma(1-t))}{\sigma(t-1)} = 1 - \frac{1}{\gamma}\left(1 + \frac{\kappa}{(1-t)\sigma}\right) \tag{39}$$

proportion of wealth in the risky discount bond. Therefore, the proportion of wealth invested in the locally risky bond increases in investors' risk aversion. It is useful to consider the intuition in a limiting case when $\kappa \downarrow 0$ and $\gamma_B \uparrow \infty$, with $\gamma_A = 1$. In this case, agent A with log utility holds approximately the locally riskless asset, because log utility is myopic, and the agent does not invest much in the risky bond when its local risk premium is small. The very risk averse agent B puts approximately 100% in the locally risky bond with a positive risk premium. This generates a nearly riskless payoff at the end, which is what a very risk averse agent wants. This example illustrates that although it is hard to get comparative statics results in portfolio weights, our comparative statics result in the distribution of portfolio payoffs still holds.

We have not of course exhausted all possibilities of generating comparative statics results for portfolio weights. For example, suppose we are willing to restrict the return process drastically to the lognormal case of continuous i.i.d. geometric returns in continuous time with a single risky asset and constant interest rate. We conjecture that in this special case with only terminal consumption, increasing risk aversion in the Ross sense will always reduce the initial optimal portfolio weight on the risky asset, but that increasing risk aversion in the Arrow–Pratt sense will not.

Example 6.6 shows that our main result may not hold for incomplete markets where agents have a non-traded asset.

Example 6.6. We assume that there are one risk-free asset with zero interest rate, one risky stock and one non-traded risky asset. The return of the stock is $\tilde{x} = (u\ d)^T$, where $u = \frac{3}{2}$ and $d = \frac{2}{3}$.

These two states have equal probabilities. Conditional on the stock return being u the probability of the non-traded asset payoff $\tilde{N}$ being 1 (0, resp.) is 0.8 (0.2, resp.). Conditional on the stock return being d the probability of the non-traded asset payoff $\tilde{N}$ being 1 (0, resp.) is 0.4 (0.6, resp.). Agent i's ($i = A, B$) problem is

$$\max_{\alpha_i} E\big[-\exp\big(-\delta_i(w_0 - \alpha_i + \alpha_i\tilde{x} + \tilde{N})\big)\big],$$

where α_i is the dollar amount invested in the stock. We assume that agent B is weakly more risk averse than A with $\delta_A = 2$ and $\delta_B = 3$. Investors' initial wealth $w_0 = 1$. First order conditions give

$$\alpha_A^* = \frac{3}{5}\ln\frac{3(0.2e^{-2}+0.05)}{0.2e^{-2}+0.3} = -0.208, \qquad \alpha_B^* = 0.4\ln\frac{3(0.2e^{-3}+0.05)}{0.2e^{-3}+0.3} = -0.2177.$$

Therefore, agent A's portfolio payoff is $(1.896\ 0.896\ 2.069\ 1.069)^T$ and agent B's portfolio payoff is $(1.891\ 0.891\ 2.073\ 1.073)^T$. If agent A's payoff $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, then we have $\max\tilde{c}_A \geqslant \max\tilde{c}_B$. However, in this example, we can see that $\max\tilde{c}_A = 2.069$ and $\max\tilde{c}_B = 2.073$, *i.e.*, $\max\tilde{c}_A < \max\tilde{c}_B$. Contradiction! Therefore, in general, our result does not hold in incomplete markets with non-traded risky assets.

Example 6.6 shows that the results do not generalize to the case of non-traded wealth, even if markets are complete over states distinguished by the traded securities (as in Dybvig [4]) and if there are two traded assets, one of which is risky. Probably this was to be expected given the Note after Problem 2. That note also tells us that there is a positive result to be had since Theorem 7 can be re-interpreted as a result with non-traded wealth.

7. Concluding remarks

Hart [8] proved the impossibility of deriving general comparative static properties for portfolio weights changing in response to wealth and therefore implicitly in response to risk aversion.[15] We have proven comparative statics results instead in the distribution of portfolio payoffs. Specifically, in a complete market, we show that an agent who is less risk averse than another will choose a portfolio whose payoff is distributed as the other's payoff plus a non-negative random variable plus conditional-mean-zero noise. This result holds for any increasing and strictly concave C^2 utility functions. If either agent has nonincreasing absolute risk aversion, then the non-negative random variable can be chosen to be a constant. The nonincreasing absolute risk aversion condition is sufficient but not necessary. We also provide a counter-example showing that, in general, this non-negative random variable cannot be chosen to be a constant.

We further prove a converse theorem. If in all complete markets the first agent chooses a payoff that is distributed as the second's payoff, plus a non-negative random variable, plus conditional-mean-zero noise, then the first agent is less risk averse than the second agent. We also extend our main results to a multiple-period model. Due to shifts in the timing of consumption, agents' optimal consumption at each date may not be ordered when risk aversion changes. However, for agents with the same pure rate of time preference, there is a natural weighting of probabilities across periods that preserves the single-period result.

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[15] If an agent has monotone risk aversion, a comparative static in risk aversion would imply a comparative static in wealth.

The optimal consumption may not be ordered for agents with different risk aversion when agents' utility functions are concave but not strictly concave as we have shown in Examples 6.3 and 6.4. Intuitively, the problem is that even with identical preferences, two different optimal consumptions may not be ordered. We conjecture that there exists some canonical choice of optimal consumption for each agent that extends our main results to weakly concave preferences. Our paper derives comparative statics results in complete markets for agents with von Neumann–Morgenstern preferences. Machina [13] has shown that many previous comparative statics results generalize to the broader class of Machina preferences (Machina [12]). Our proofs do not generalize obviously to this class, but we conjecture that our results are still true.

We also show that our main result still holds in a two-asset world with a risk-free asset or more generally in a two-fund separation world with a risk-free asset. However, our main result is not true in general with incomplete markets. We further provide sufficient conditions under which our results still hold in a two-risky-asset world or a world with two-fund separation.

Appendix A

Proof of Lemma 1. By Pratt [15], we have the concave transform characterization[16] that there exists $G(\cdot) \in C^2$, such that

$$U_B(c) = G\big(U_A(c)\big), \tag{40}$$

where $G'(\cdot) > 0$ and $G''(\cdot) \leqslant 0$. Using the concave transform characterization of more risk averse in (40), the first order condition (2) becomes

$$U_A'(\tilde{c}_A) = \lambda_A \tilde{\rho} = \frac{\lambda_A}{\lambda_B}\lambda_B \tilde{\rho} = \frac{\lambda_A}{\lambda_B} G'\big(U_A(\tilde{c}_B)\big)U_A'(\tilde{c}_B). \tag{41}$$

Because marginal utility is strictly decreasing, we have: if $G' < \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A > \tilde{c}_B$; if $G' = \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A = \tilde{c}_B$; and if $G' > \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A < \tilde{c}_B$. Choose c^* so that $G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$ if possible, or pick $c^* = -\infty$ if $G' < \frac{\lambda_B}{\lambda_A}$ everywhere or $c^* = +\infty$ if $G' > \frac{\lambda_B}{\lambda_A}$ everywhere. If $\tilde{c}_B \geqslant c^*$, then $G'(U_A(\tilde{c}_B)) \leqslant G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$, *i.e.*, $G' \leqslant \frac{\lambda_B}{\lambda_A}$, therefore, $\tilde{c}_A \geqslant \tilde{c}_B$. If $\tilde{c}_B \leqslant c^*$, then $G'(U_A(\tilde{c}_B)) \geqslant G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$, *i.e.*, $G' \geqslant \frac{\lambda_B}{\lambda_A}$, therefore, $\tilde{c}_A \leqslant \tilde{c}_B$. This proves statement 1.

Now suppose that A and B have equal initial wealths, then the budget constraints for the agents are that

$$E[\tilde{\rho}\tilde{c}_A] = E[\tilde{\rho}\tilde{c}_B] = w_0, \tag{42}$$

therefore, we have $E[\tilde{\rho}(\tilde{c}_A - \tilde{c}_B)] = 0$. Since $\lambda_B \tilde{\rho} = U_B'(\tilde{c}_B)$ and $U_B'' < 0$, $\tilde{\rho}$ and $\tilde{c}_B$ are negatively monotonely related. Let $\rho^* \equiv U_B'(c^*)/\lambda_B > 0$. Then $\tilde{\rho} \geqslant \rho^* \Rightarrow \tilde{c}_A \leqslant \tilde{c}_B$ and $\tilde{\rho} \leqslant \rho^* \Rightarrow \tilde{c}_A \geqslant \tilde{c}_B$. Therefore, $(\tilde{\rho} - \rho^*)(\tilde{c}_A - \tilde{c}_B) \leqslant 0$ and we have

$$0 = E\big[\tilde{\rho}(\tilde{c}_A - \tilde{c}_B)\big] = E\big[\rho^*(\tilde{c}_A - \tilde{c}_B)\big] + E\big[\big(\tilde{\rho} - \rho^*\big)(\tilde{c}_A - \tilde{c}_B)\big] \leqslant \rho^* E[\tilde{c}_A - \tilde{c}_B]. \tag{43}$$

Therefore, $E[\tilde{c}_A] \geqslant E[\tilde{c}_B]$. This proves statement 2. □

[16] This result can be obtained by defining $G(\cdot)$ implicitly from (40) and using the implicit function theorem to compute the derivatives of $G(\cdot)$.

Proof of Theorem 1. (*Sufficiency*) The monotonicity and concavity of the function and Jensen's inequality yield $E[V(\tilde{Y})] = E[V(\tilde{X} - \tilde{Z} + \tilde{\varepsilon})] = E[E[V(\tilde{X} - \tilde{Z} + \tilde{\varepsilon})|X, Z]] \leqslant E[V(\tilde{X} - \tilde{Z})] \leqslant E[V(\tilde{X})]$.

(*Necessity*) Let μ_1 be the distribution of $-\tilde{X}$, and let μ_2 be the distribution of $-\tilde{Y}$. From Theorem 9 of Strassen [22],[17] the following two statements are equivalent.

(i) For any concave nondecreasing function $V(s)$, $\int V(-s)\,d\mu_1(s) \geqslant \int V(-s)\,d\mu_2(s)$.
(ii) There exists a submartingale $\tilde{\xi}_n$ $(n = 1, 2)$, *i.e.*, $E[\tilde{\xi}_2|\xi_1] \geqslant \tilde{\xi}_1$, such that the distribution of $\tilde{\xi}_n$ is μ_n.

Let $\tilde{Z} \equiv E[\tilde{\xi}_2|\xi_1] - \tilde{\xi}_1$ and $\tilde{\varepsilon} \equiv -\tilde{\xi}_2 + E[\tilde{\xi}_2|\xi_1]$, then (ii) implies that $\tilde{Z} \geqslant 0$. Since $\tilde{\xi}_1 + \tilde{Z} = E[\tilde{\xi}_2|\xi_1]$, we have $E[\tilde{\varepsilon}|\xi_1 + Z] = E[(-\tilde{\xi}_2 + E[\tilde{\xi}_2|\xi_1])|E[\tilde{\xi}_2|\xi_1]] = 0$. (i) implies $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$, and since $\tilde{\xi}_2 = \tilde{\xi}_1 + (E[\tilde{\xi}_2|\xi_1] - \tilde{\xi}_1) + (\tilde{\xi}_2 - E[\tilde{\xi}_2|\xi_1])$, we have $-\tilde{Y} \sim -\tilde{X} + \tilde{Z} - \tilde{\varepsilon}$, where $\tilde{Z} \sim E[-\tilde{Y}| - X] + \tilde{X} \geqslant 0$ and $\tilde{\varepsilon} \sim \tilde{Y} + E[-\tilde{Y}| - X]$. It follows that $\tilde{Y} \sim \tilde{X} - \tilde{Z} + \tilde{\varepsilon}$, where $\tilde{Z} \geqslant 0$ and $E[\tilde{\varepsilon}|X - Z] = 0$. □

Proof of Theorem 2. The sufficiency follows directly from Jensen's inequality. The necessity can be proved using Theorem 8 in Strassen [22]. We prove it instead using Theorem 1 above. We have $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$ for all concave function, and in particular, $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$ for all concave nondecreasing functions. Therefore, by Theorem 1, $\tilde{Y} \sim \tilde{X} - \tilde{Z}_1 + \tilde{\varepsilon}_1$, where $\tilde{Z}_1 \geqslant 0$ and $E[\tilde{\varepsilon}_1|X - Z_1] = 0$. We have

$$\begin{aligned} E[\tilde{Y}] &= E\big[E[\tilde{Y}|X - Z_1]\big] = E\big[E[\tilde{X} - \tilde{Z}_1 + \tilde{\varepsilon}_1|X - Z_1]\big] \\ &= E[\tilde{X}] - E[\tilde{Z}_1] \leqslant E[\tilde{X}]. \end{aligned} \tag{44}$$

Now $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$ for all concave functions also implies $E[V(\tilde{X})] \geqslant E[V(\tilde{Y})]$ for all concave nonincreasing functions, *i.e.*, $E[V(-\tilde{X})] \geqslant E[V(-\tilde{Y})]$ for all concave nondecreasing functions. From Theorem 1, $-\tilde{Y} \sim -\tilde{X} - \tilde{Z}_2 + \tilde{\varepsilon}_2 \Rightarrow \tilde{Y} \sim \tilde{X} + \tilde{Z}_2 - \tilde{\varepsilon}_2$, where $\tilde{Z}_2 \geqslant 0$, and $E[\tilde{\varepsilon}_2|X + Z_2] = 0$. We have

$$\begin{aligned} E[\tilde{Y}] &= E\big[E[\tilde{Y}|X + Z_2]\big] = E\big[E[\tilde{X} + \tilde{Z}_2 - \tilde{\varepsilon}_2|X + Z_2]\big] \\ &= E[\tilde{X}] + E[\tilde{Z}_2] \geqslant E[\tilde{X}]. \end{aligned} \tag{45}$$

Therefore, $E[\tilde{X}] = E[\tilde{Y}]$, which implies $E[\tilde{Z}_1] = 0$. Since $\tilde{Z}_1 \geqslant 0$, we must have $\tilde{Z}_1 = 0$. It follows that $\tilde{Y} \sim \tilde{X} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|X] = 0$. □

Lemma 5. *Suppose B is not weakly more risk averse than A, then there exists a bounded nondegenerate interval $[c_1, c_2]$ and hypothetical agents A_1 and B_1, such that A_1 is strictly more risk averse than B_1 ($\forall c$, $-\frac{U''_{B_1}(c)}{U'_{B_1}(c)} < -\frac{U''_{A_1}(c)}{U'_{A_1}(c)}$) and $\forall c \in [c_1, c_2]$, $U_{A_1}(c) = U_A(c)$ and $U_{B_1}(c) = U_B(c)$.*

Proof. If B is not weakly more risk averse than A, then there exists a constant $\hat{c}$, such that $-\frac{U''_B(\hat{c})}{U'_B(\hat{c})} < -\frac{U''_A(\hat{c})}{U'_A(\hat{c})}$. Since U_A and U_B are of the class of C^2 (see our assumptions in the beginning

[17] In applying Strassen's result, we ignore ξ_n for $n > 2$. Formally, we set $\xi_n = \xi_2$ and $\mu_n = \mu_2$ for all $n > 2$.

of Section 2), from the continuity of $-\frac{U_i''(c)}{U_i'(c)}$, where $i = A, B$, we get that there exists an interval RA containing $\hat{c}$, *s.t.*, $\forall c \in RA$, $-\frac{U_B''(c)}{U_B'(c)} < -\frac{U_A''(c)}{U_A'(c)}$. We pick $c_1, c_2 \in RA$ with $c_1 < c_2$. Now, let

$$U_{A_1}(c) = \begin{cases} a_1 - m_1 \exp(\frac{U_A''(c_1)}{U_A'(c_1)}c) & c < c_1 \\ U_A(c) & c_1 \leqslant c \leqslant c_2 \\ a_2 - m_2 \exp(\frac{U_A''(c_2)}{U_A'(c_2)}c) & c > c_2, \end{cases}$$

and let

$$U_{B_1}(c) = \begin{cases} b_1 - n_1 \exp(\frac{U_B''(c_1)}{U_B'(c_1)}c) & c < c_1 \\ U_B(c) & c_1 \leqslant c \leqslant c_2 \\ b_2 - n_2 \exp(\frac{U_B''(c_2)}{U_B'(c_2)}c) & c > c_2, \end{cases}$$

where a_j and m_j $(j = 1, 2)$ are determined by the continuity and smoothness of $U_{A_1}(c)$, and b_j and n_j $(j = 1, 2)$ are determined by the continuity and smoothness of $U_{B_1}(c)$. More specifically, for $j = 1, 2$, we have

$$m_j = -\frac{(U_A'(c_j))^2}{U_A''(c_j)} \exp\left(-\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right), \qquad a_j = m_j \exp\left(\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right) + U_A(c_j), \tag{46}$$

and

$$n_j = -\frac{(U_B'(c_j))^2}{U_B''(c_j)} \exp\left(-\frac{U_B''(c_j)}{U_B'(c_j)}c_j\right), \qquad b_j = n_j \exp\left(\frac{U_B''(c_j)}{U_B'(c_j)}c_j\right) + U_B(c_j). \tag{47}$$

Now, $U_{A_1}(c)$ is in the class of C^2 since from (46), we have:

$$-m_j \exp\left(\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right)\left(\frac{U_A''(c_j)}{U_A'(c_j)}\right)^2 = U_A''(c_j),$$

i.e., U_{A_1} is twice differentiable. Similarly, we can show that $U_{B_1}(c)$ is also in the class of C^2. Also, we have $U_{A_1}''(c) < 0$, $U_{B_1}''(c) < 0$, and $\forall c$, $-\frac{U_{B_1}''(c)}{U_{B_1}'(c)} < -\frac{U_{A_1}''(c)}{U_{A_1}'(c)}$, *i.e.*, agent A_1 is more risk averse than B_1. □

Lemma 6. *Suppose B is strictly more risk averse than A* ($\forall c$, $-\frac{U_A''(c)}{U_A'(c)} < -\frac{U_B''(c)}{U_B'(c)}$)*, and A and B have equal initial wealths. A has an optimal choice $\tilde{c}_A$, and B has an optimal choice $\tilde{c}_B$. We assume that the state price density $\tilde{\rho}$ is not a constant. Then, we have*

1. $\tilde{c}_A \neq \tilde{c}_B$;
2. *if $\tilde{c}_A$ has a bounded support* $[c_1, c_2]$*, then we have* $\sup \tilde{c}_A \geqslant \sup \tilde{c}_B$*, and* $\inf \tilde{c}_A \leqslant \inf \tilde{c}_B$.

Proof. We first prove statement 1 by contradiction. If $\tilde{c}_A = \tilde{c}_B$, then we pick any two points, for example, c_3, c_4 $(c_3 < c_4)$ in the support of both $\tilde{c}_A$ and $\tilde{c}_B$. From the first order conditions, we get: $\frac{U_A'(c_3)}{U_A'(c_4)} = \frac{U_B'(c_3)}{U_B'(c_4)}$, *i.e.*, $\frac{U_A'(c_3)}{U_B'(c_3)} = \frac{U_A'(c_4)}{U_B'(c_4)}$. However, from $-\frac{U_A''(c)}{U_A'(c)} < -\frac{U_B''(c)}{U_B'(c)}$, we have: $\frac{d}{dc}(\log \frac{U_B'(c)}{U_A'(c)}) < 0$, *i.e.*, $\frac{U_B'(c)}{U_A'(c)}$ decreases in c. We have: $\frac{U_B'(c_3)}{U_A'(c_3)} > \frac{U_B'(c_4)}{U_A'(c_4)}$. Contradiction! So, $\tilde{c}_A \neq \tilde{c}_B$.

Since B is more risk averse than A, from Lemma 1, we know that there exists c^*, such that $\tilde{c}_A \geqslant \tilde{c}_B$ when $\tilde{c}_B \geqslant c^*$, and $\tilde{c}_A \leqslant \tilde{c}_B$ when $\tilde{c}_B \leqslant c^*$. And we have $c^* \in [c_1, c_2]$, or else either $\tilde{c}_A \leqslant \tilde{c}_B$ but $\tilde{c}_A \neq \tilde{c}_B$ or $\tilde{c}_A \geqslant \tilde{c}_B$ but $\tilde{c}_A \neq \tilde{c}_B$ and both could not satisfy the budget constraint ($E[\tilde{\rho}\tilde{c}_A] = E[\tilde{\rho}\tilde{c}_B] = w_0$). Therefore, $\tilde{c}_A$ has a wider range of support than that of $\tilde{c}_B$. $\square$

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