Linear Programming: Introduction

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Course Schedule

- Session 1: Introduction to optimization.
  Modelling and Solving simple problems.
  Modelling combinatorial problems.

- Session 2: Duality or Assessing the quality of a solution.

- Session 3: Solving problems in practice or using solvers (Glpk or Cplex).

Motivation

Why linear programming is a very important topic?

- A lot of problems can be formulated as linear programmes, and
- There exist efficient methods to solve them
- or at least give good approximations.

- Solve difficult problems: e.g. original example given by the inventor of the theory, Dantzig. Best assignment of 70 people to 70 tasks.

→ Magic algorithmic box.

What is a linear programme?

- Optimization problem consisting in
  - maximizing (or minimizing) a linear objective function
  - of n decision variables
  - subject to a set of constraints expressed by linear equations or inequalities.
- Originally, military context: "programme"="resource planning". Now "programme"="problem"
- Terminology due to George B. Dantzig, inventor of the Simplex Algorithm (1947)

Terminology

x_1, x_2 : Decision variables

$$\begin{array}{rl} \max & 350x_1 + 300x_2 \\ \text{subject to} & \\ & x_1 + x_2 \leq 200 \\ & 9x_1 + 6x_2 \leq 1566 \\ & 12x_1 + 16x_2 \leq 2880 \\ & x_1, x_2 \geq 0 \end{array}$$

Objective function

Constraints

Terminology

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$$
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& x_1, x_2 \geq 0
\end{aligned}
$$

Objective function

Constraints

In linear programme: objective function + constraints are all linear

Typically (not always): variables are non-negative

If variables are integer: system called Integer Programme (IP)

Terminology

Linear programmes can be written under the standard form:

$$\begin{array}{lrcll} \text{Maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for all } 1 \leq i \leq m \\ & x_j & \geq & 0 & \text{for all } 1 \leq j \leq n. \end{array} \quad (1)$$

- the problem is a maximization;
- all constraints are inequalities (and not equations);
- all variables are non-negative.

Example 1: a resource allocation problem

A company produces copper cable of 5 and 10 mm of diameter on a single production line with the following constraints:

- The available copper allows to produces 21000 meters of cable of 5 mm diameter per week.
- A meter of 10 mm diameter copper consumes 4 times more copper than a meter of 5 mm diameter copper.

Due to demand, the weekly production of 5 mm cable is limited to 15000 meters and the production of 10 mm cable should not exceed 40% of the total production. Cable are respectively sold 50 and 200 euros the meter.

What should the company produce in order to maximize its weekly revenue?

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Example 1: a resource allocation problem

Define two decision variables:

- x_1: the number of thousands of meters of 5 mm cables produced every week
- x_2: the number of thousands meters of 10 mm cables produced every week

The revenue associated to a production (x_1, x_2) is

$$z = 50x_1 + 200x_2.$$

The capacity of production cannot be exceeded

$$x_1 + 4x_2 \leq 21.$$

Example 1: a resource allocation problem

The demand constraints have to be satisfied

$$x_2 \leq \frac{4}{10}(x_1 + x_2)$$

$$x_1 \leq 15$$

Negative quantities cannot be produced

$$x1 \geq 0, x2 \geq 0.$$

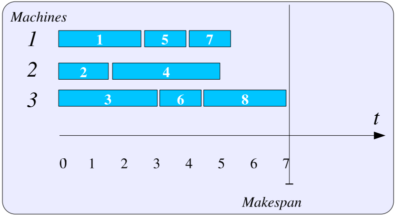
Example 1: a resource allocation problem

The model: To maximize the sell revenue, determine the solutions of the following linear programme x_1 and x_2:

$$
\begin{aligned}
\max \quad & z = 50x_1 + 20x_2 \\
\text{subject to} \quad & \\
& x_1 + 4x_2 \leq 21 \\
& -4x_1 + 6x_2 \leq 0 \\
& x_1 \leq 15 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Example 2: Scheduling

- $m = 3$ machines
- $n = 8$ tasks
- Each task lasts x units of time



Objective: affect the tasks to the machines in order to minimize the duration

- Here, the 8 tasks are finished after 7 units of times on 3 machines.

Example 2: Scheduling

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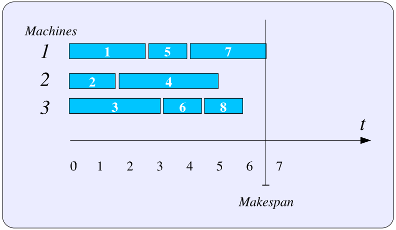
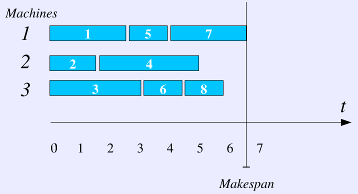


Objective: affect the tasks to the machines in order to minimize the duration

- Now, the 8 tasks are accomplished after 6.5 units of time: OPT?
- m^n possibilities! (Here $3^8 = 6561$)

Example 2: Scheduling

- $m = 3$ machines
- $n = 8$ tasks
- Each task lasts x units of time



Solution: LP model.

$$
\begin{array}{ll}
\min & t \\
\text{subject to} & \\
& \sum_{1 \le i \le n} t_i x_i^j \le t \qquad (\forall j, 1 \le j \le m) \\
& \sum_{1 \le j \le m} x_i^j = 1 \qquad (\forall i, 1 \le i \le n)
\end{array}
$$

with $x_i^j = 1$ if task i is affected to machine j.

Solving Difficult Problems

- Difficulty: Large number of solutions.
  - Choose the best solution among 2^n or $n!$ possibilities: all solutions cannot be enumerated.
  - Complexity of studied problems: often NP-complete.
- Solving methods:
  - Optimal solutions:
    - Graphical method (2 variables only).
    - Simplex method.
  - Approximations:
    - Theory of duality (assert the quality of a solution).
    - Approximation algorithms.

Graphical Method

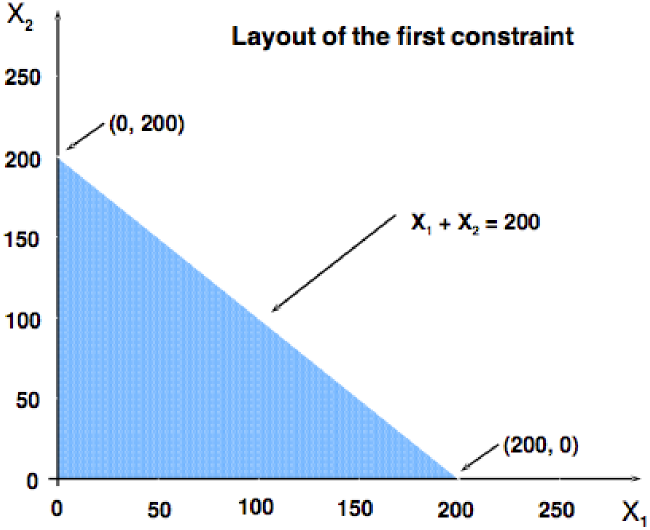
- The constraints of a linear programme define a zone of solutions.
- The best point of the zone corresponds to the optimal solution.
- For problem with 2 variables, easy to draw the zone of solutions and to find the optimal solution graphically.

Graphical Method

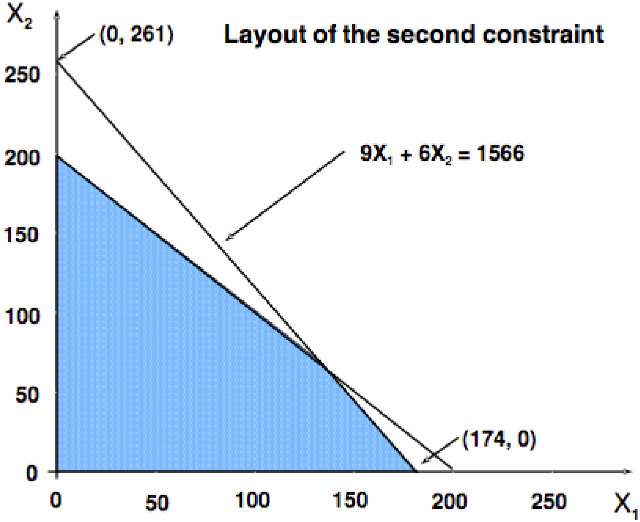
Example:

$$\begin{aligned} \max \quad & 350x_1 + 300x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \leq 200 \\ & 9x_1 + 6x_2 \leq 1566 \\ & 12x_1 + 16x_2 \leq 2880 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Graphical Method



Graphical Method



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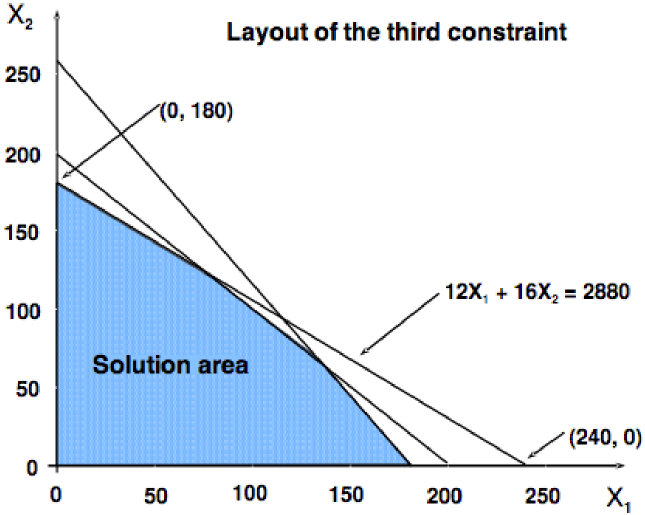

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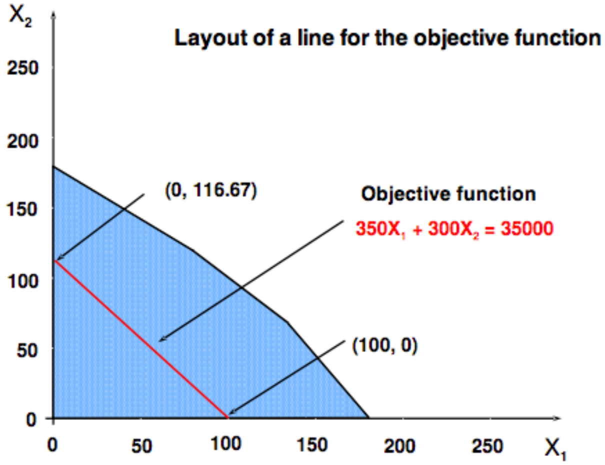
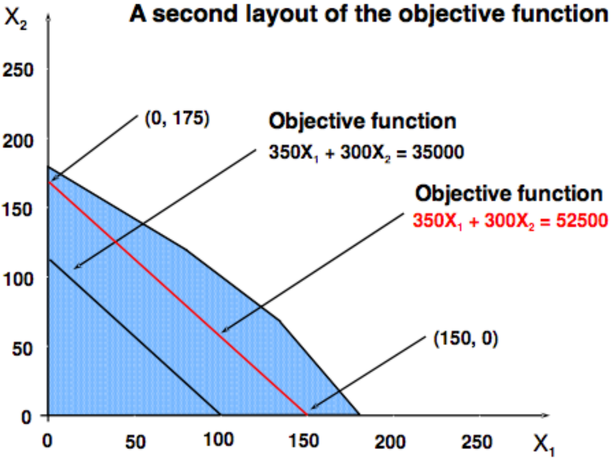
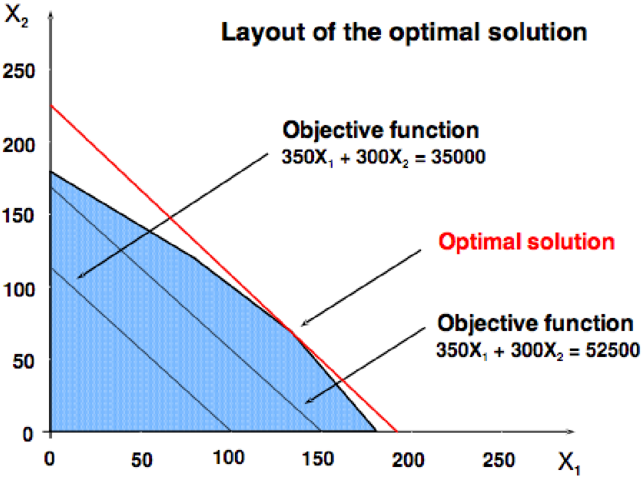


Graphical Method



Graphical Method



Computation of the optimal solution

The optimal solution is at the intersection of the constraints:

$$x_1 + x_2 = 200 \tag{2}$$

$$9x_1 + 6x_2 = 1566 \tag{3}$$

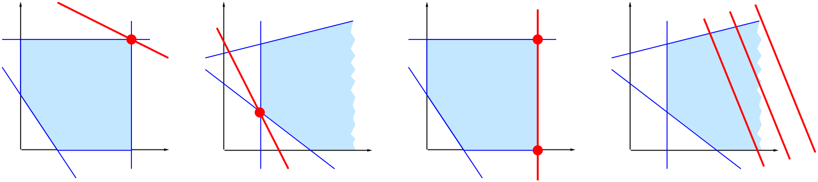
We get:

$$x_1 = 122$$

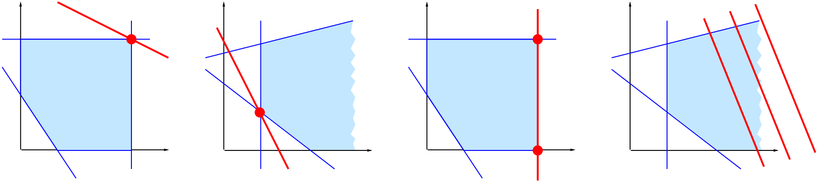
$$x_2 = 78$$

$$\text{Objective} = 66100.$$

Optimal Solutions: Different Cases



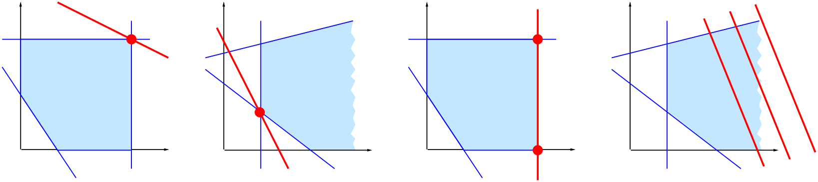
Optimal Solutions: Different Cases



Three different possible cases:

- a single optimal solution,
- an infinite number of optimal solutions, or
- no optimal solutions.

Optimal Solutions: Different Cases

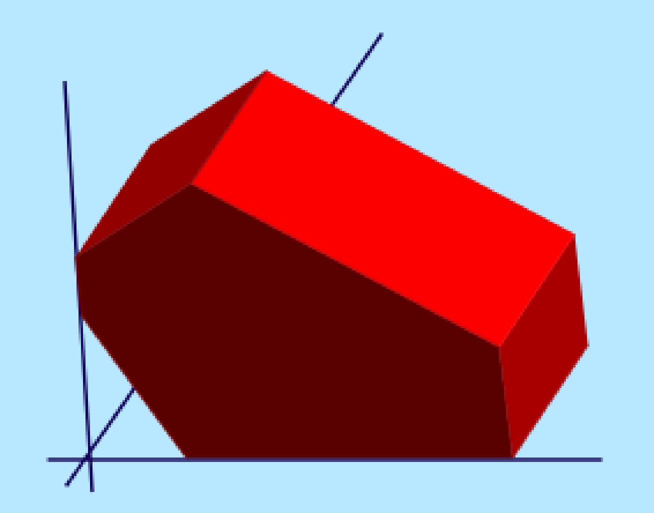


Three different possible cases:

- a single optimal solution,
- an infinite number of optimal solutions, or
- no optimal solutions.

If an optimal solution exists, there is always a corner point optimal solution!

Solving Linear Programmes



Solving Linear Programmes

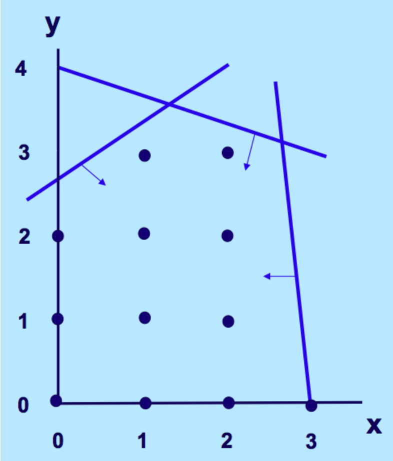
- The constraints of an LP give rise to a geometrical shape: a polyhedron.
- If we can determine all the corner points of the polyhedron, then we calculate the objective function at these points and take the best one as our optimal solution.
- The Simplex Method intelligently moves from corner to corner until it can prove that it has found the optimal solution.

Solving Linear Programmes

- Geometric method impossible in higher dimensions
- Algebraical methods:
  - Simplex method (George B. Dantzig 1949): skim through the feasible solution polytope.
    Similar to a "Gaussian elimination".
    Very good in practice, but can take an exponential time.
  - Polynomial methods exist:
    - Leonid Khachiyan 1979: ellipsoid method. But more theoretical than practical.
    - Narendra Karmarkar 1984: a new interior method. Can be used in practice.

But Integer Programming (IP) is different!

- Feasible region: a set of discrete points.
- Corner point solution not assured.
- No "efficient" way to solve an IP.
- Solving it as an LP provides a relaxation and a bound on the solution.



Summary: To be remembered

- What is a linear programme.
- The graphical method of resolution.
- Linear programs can be solved efficiently (polynomial).
- Integer programs are a lot harder (in general no polynomial algorithms).
  In this case, we look for approximate solutions.