Efficient Sensitivity Analysis of Mortgage Backed Securities

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Abstract

Because of the complicated behavior of their cash flows, mortgage-backed securities (MBS) are almost always priced using Monte Carlo simulation. Sensitivity analysis is a critical component of these simulations. Risk managers need these numbers to hedge their exposure to interest risk and prepayment risk. In order to carry this out in a computationally efficient manner, we derive perturbation analysis (PA) gradient estimators in a general setting, without restrictions to any specific interest rate model or prepayment model. Then we apply the estimators to the Hull-White interest rate model and a common prepayment model to derive the corresponding specific PA gradient estimators, assuming the shock of interest rate term structure takes the form of a Fourier-like harmonic series. Numerical experiments comparing finite difference (FD) gradient estimators with our PA estimators indicate that the PA estimators can provide better accuracy than FD estimators while using much lower computational cost. In the test case of 5 duration and 25 convexity estimators, the computational time is reduced by 97.2%. Using the estimators, we analyze the impact of term structure shifts on various mortgage products. Based on insights gained from the analysis, we propose a new product that could potentially benefit mortgage borrowers and investors.

Keywords: Mortgage Backed Securities, Risk Management, Monte Carlo simulation, Perturbation Analysis, Gradient Estimation.

1 Introduction

A mortgage-backed security (MBS) is a security collateralized by residential or commercial mortgage loans. An MBS is generally securitized, guaranteed and issued by three major MBS originating agencies: Ginnie Mae, Fannie Mae, and Freddie Mac. The cash flow of an MBS is generally the collected payment from the mortgage borrower, after the deduction of servicing and guaranty fees. However, the cash flows of an MBS are not as stable as that of a government or corporate coupon bond. Because the mortgage borrower has the prepayment option, mainly exercised when moving or refinancing, an MBS investor is actually writing a call option. Furthermore, the mortgage borrower also has the default option, which is likely to be exercised when the property value drops below the mortgage balance, and continuing mortgage payments would not make economical sense. In this case the guarantor is writing the borrower a put option, and the guarantor absorbs the cost. However, the borrower does not always exercise the options whenever it is financially optimal to do so, because there are always non-monetary factors associated with the home, like shelter, sense of stability, etc. And it is also very hard for the borrower to tell whether it is financially optimal to exercise these options because of lack of complete and unbiased information, e.g., they may not be able to obtain an accurate home price, unless they are selling it. And there are also some other fixed/variable costs associated with these options, such as the commission paid to the real estate agent, the cost to initialize another loan, and the negative credit rating impact when the borrower defaults on a mortgage. All these factors contribute to the complexity of MBS cash flows. In practice, the cash flows are generally projected by complicated prepayment models, which are based on statistical estimation on large historical data sets. Because of the complicated behaviors of the MBS cash flow, due to the complex relationships with the underlying interest rate term structures, and path dependencies in prepayment behaviors, Monte Carlo simulation is the generally only applicable method to price MBS.

Associated with the uncertainty of cash flows are different kinds of risks. Treasury bonds only bear interest rate risk, whereas non-callable corporate bonds carry interest rate and credit risk. MBS are further complicated by prepayment risk (resulting from both prepayment and default). Thus risk management is especially critical for portfolios with large holdings in MBS. Duration and convexity are mainly the risk measurements for fixed income portfolio mangers. Many practitioners use either the Macaulay duration, or modified duration (Robert Kopprasch [1987]) to capture the MBS price sensitivity w.r.t. interest rate changes, but these assume a constant yield and known deterministic prepayment pattern, which is rarely the case in practice. So these two approaches to calculate duration can lead to serious errors when used in hedging. Bennett W. Golub [2001] proposed four different approaches to get the duration: Percent of Price (POP), Option-Adjusted Duration (OAD), Implied Duration, and Coupon Curve Duration (CCD). The first two approaches apply parallel shifts in the yield curve, which is not a very realistic assumption. The latter two approaches require large numbers of previous or current accurate MBS prices that are comparable to the MBS whose duration is to be measured. This might not be practical for on-the-fly pricing and sensitivity analysis.

Another drawback of these approaches is that they handle only duration and convexity (delta and gamma), but not sensitivity to interest rate volatility (vega). OAD can estimate the vega using a finite difference method, which requires 3 simulations to estimate one gradient: the base, up and down cases. And non-parallel yield curve shifts require more parameters to characterize the shift. Thus, in the setting we consider, to estimate the duration w.r.t. yield curve shift of 4 summed harmonic functions would require 9 (2n+1, n=4) simulations. To estimate vega requires 2 additional simulations. So estimating the duration and vega roughly increases the computational cost by a factor of 10. Calculating convexity would require 75 duration estimators to calculate 25 convexity estimators, increasing the simulation factor to 225. In other words, if one were to use 10,000 replications to estimate the MBS price, over 2.25 million simulations would be required to estimate the various sensitivities. Our work aims to decrease this computational burden dramatically.

Most literature on MBS has concentrated on prepayment model estimations, although some of the recent work has focused on computational efficiency, e.g., dimensionality reduction via Brownian bridge (Caflisch et al. [1997]), and quasi-Monte Carlo (Åkesson and Lehoczky [2000]). However, there is no work that we are aware of that addresses efficient sensitivity analysis of MBS pricing and hedging. Related work in equities includes Fu and Hu[1995], Broadie and Glasserman [1996], Fu et al. [2000], [2001], and Wu and Fu [2001]. Perhaps the most relevant paper to our work is Glasserman [1999], "Fast Greeks in Forward LIBOR Model", which applied perturbation analysis (PA) method for caplet price sensitivity analysis. Yet most of these models involve only a single exercise decision with a one-time payoff, whereas an MBS is a pool of homogenous mortgages rather than an individual mortgage loan. So the cash flows exist until the maturity of the collateral, and they are highly path dependent, which makes sensitivity analysis of MBS more complicated.

The other relevant body of research literature analyzes the duration of different mortgage products. We know that adjustable rate mortgage (ARM) products will have a different response from fixed rate mortgage (FRM) products, due to ARMs' coupon-reset plan and different prepayment function. In a series of papers, Kau et al.[1990,1992,1993] priced the ARMs and performed some sensitivity analysis. Chiang [1997] applied a simple simulation scheme to estimate the modified duration of ARMs. Stanton [1999] calculated the duration of different indexed ARMs via a scheme like Kau's. However, most of these papers are based on solving the PDE equations, using simplified assumptions that often miss essential features that can be captured by Monte Carlo simulation. The three major drawbacks of these models that make them impractical in the mortgage industry are the following:

- They assume borrowers exercise the prepayment option only when it is financially optimal to do so. This ignores the fact that people routinely prepay even in financially adversary environment, e.g., house sales. Also seasoning and burnout effects are not considered.
- By solving the PDE, one can only obtain a set of present values of the MBS along the interest rate axis. By applying the finite difference method, duration of the MBS could be acquired. However, it provides no information about the discounting factor

and cash flows along the time horizon. So you will have no information about how the interest shift affects different components of the present value.

- The PDE method generally uses the same interest rate both for discounting and for the prepayment model, which ignores the difference between short-term and long-term interest rates.

In this paper, we apply perturbation analysis (PA) to estimate the sensitivities of MBS. Our work possesses the following advantages, compared with previous research:

- We can apply the sensitivity analysis along the simulation path, to obtain sensitivities for whatever measures we are interested, i.e., we not only have the sensitivities of net present value (NPV), but also the sensitivities of unpaid balance (UPB), cash flows, discounting factor, prepayment rates, etc. So we can estimate how the parameter changes in the model would affect each component, and hence affect the price of the MBS.
- We incorporate the whole term structure in our interest rate model, and use the sum of four harmonic functions to model the shift of term structure, which captures more complicated term structure and its move.
- We can estimate all sensitivities w.r.t. any parameters in the interest rate model and prepayment model in one single simulation. In our example, we calculate 5 duration estimators and 16 convexity estimators in our simulation, which would require 155 simulations using a conventional simulation scheme.

The paper is organized in the following manner. Section 2 describes the problem setting. We then derive the framework for PA in a general setting in section 3, without restrictions to any specific interest rate model or prepayment model. Then we consider the well-known Hull-White interest rate model and a common prepayment model to derive the corresponding PA sensitivities for FRM and ARM products in section 4, assuming the shock of interest rate term structure takes the form of a series of harmonic functions. Section 5 presents numerical examples, in which we compare the performance of FD and PA estimators, indicating that the PA estimator is at least as good as the FD estimator, while the computation cost is reduced dramatically. Section 6 gives the insights from our simulation results. Section 7 concludes the paper.

2 Problem Setting

Generally the price of any security can be written as the net present value (NPV) of its discounted cash flows. Specifying the price of an MBS (here we consider only the pass-through MBS[1]) is as follows:

$$P = E[V] = E\left[\sum_{t=0}^{M} PV(t)\right] = E\left[\sum_{t=0}^{M} d(t)c(t)\right], \tag{2.1}$$

where P is the price of the MBS,
V is the value of the MBS, which is not explicitly known,
$PV(t)$ is the present value for cash flow at time t,
$d(t)$ is the discounting factor at time t,
$c(t)$ is the cash flow at time t,
M is the maturity of the MBS.

Monte Carlo simulation is used to generate cash flows on many paths, and by the strong law of large numbers, we have the following:

$$E[V] = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} V_i \ , \tag{2.2}$$

where V_i is the value calculated out in path i.

The calculation of $d(t)$ is found from the short-term (risk-free) interest rate process,

$$d(t) = d(0,1)d(1,2)d(t-1,t) = \prod_{i=0}^{t-1} \exp(-r(i)\Delta t) = \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\}, \tag{2.3}$$

where $d(i, i+1)$ is the discounting factor for the end of period $i+1$ at the end of period i;
$r(i)$ is the short term rate used to generate $d(i, i+1)$, observed at the end of period i;
Δt is the time step in simulation, generally monthly, i.e. Δt= 1 month.

An interest rate model is used to generate the short term-rate $r(i)$; then $d(t)$ is instantly available when the short-term rate path is generated.

The difficult part is to generate $c(t)$, the path dependent cash flow of MBS for month t, which is observed at the end of month t. From chapter 19 of Fabozzi [1993], we have the following formula for $c(t)$:

$$\begin{aligned} c(t) &= MP(t) + PP(t) = TPP(t) + IP(t); \\ MP(t) &= SP(t) + IP(t); \\ TPP(t) &= SP(t) + PP(t); \end{aligned} \tag{2.4}$$

[1]A pass-through MBS is an MBS that passes through the principal and interest payments collected from mortgage borrowers, minus the guaranty fee and servicing fee, to the MBS investor directly. This is in contrast to Collaterized Mortgage Obligations (CMOs), which have multiple tranches and pay the principal payments according to the seniorities of tranches.

where *MP(t)*: Scheduled Mortgage Payment for month *t*;
TPP(t): Total Principal Payment for month *t*;
IP(t): Interest Payment for month *t*;
SP(t): Scheduled Principal Payment for month *t*;
PP(t): Principal Prepayment for month *t*.

These quantities are calculated as follows:

$$MP(t) = B(t-1)\left(\frac{WAC/12}{1-(1+WAC/12)^{-WAM+t}}\right);$$

$$IP(t) = B(t-1)\frac{WAC}{12};$$

$$PP(t) = SMM(t)(B(t-1)-SP(t)); \qquad (2.5)$$

$$B(t) = B(t-1) - TPP(t);$$

$$SMM(t) = 1 - \sqrt[12]{1-CPR(t)};$$

B(t): The principal balance of MBS at end of month *t*;
WAC[2]: Weighted Average Coupon rate for MBS;
WAM[3]: Weighted Average Maturity for MBS;
SMM(t): Single Monthly Mortality for month *t*, observed at the end of month *t*;
CPR(t): Conditional Prepayment Rate for month *t*, observed at the end of month *t*.

In Monte Carlo simulation, along the sample path, the only thing uncertain is *CPR(t)*, and everything else can be calculated out once *CPR(t)* is known. Different prepayment models offer different *CPR(t)*, and it is not our goals to derive or compare prepayment models. Instead, our concern is, given a prepayment model, how can we efficiently estimate the price sensitivities of MBS against parameters of interest? Generally different prepayment models will lead to different sensitivity estimates, so it is at the user's discretion to choose an appointment prepayment function, as our method for calculating the "Greeks" is universally applicable.

[2] WAC is the weighted average mortgage rate for a mortgage pool, weighted by the balance of each mortgage.
[3] WAM is the weighted average maturity in month for a mortgage pool, weighted by the balance of each mortgage.

3 Derivation of General PA Estimators

If *P*, the price of the MBS, is a continuous function of the parameter of interest, say θ, we have the following PA estimator by differentiating both sides of (2.1):

$$\frac{dP(\theta)}{d\theta}=\frac{dE[V(\theta)]}{d\theta}=E\left[\frac{d\sum_{t=1}^{M}PV(t,\theta)}{d\theta}\right]=E\left[\sum_{t=1}^{M}\frac{dPV(t,\theta)}{d\theta}\right],$$

$$\frac{d(PV(t,\theta))}{d\theta}=\frac{\partial d(t,\theta)}{\partial\theta}c(t,\theta)+\frac{\partial c(t,\theta)}{\partial\theta}d(t,\theta). \qquad (3.1)$$

Now we have reduced the original problem from estimating the gradient of a sum function to estimating the sum of a bunch of gradients. To be exact, now we only need to estimate two gradient estimators, $\frac{\partial c(t,\theta)}{\partial\theta}$ and $\frac{\partial d(t,\theta)}{\partial\theta}$, at each time step.

3.1 Gradient Estimator for Cash Flow

We first derive $\frac{\partial c(t,\theta)}{\partial\theta}$. To simplify notation, we write *c(t)* for *c(t, θ)*.

A simplified expression for *c(t)* is derived from (2.4) and (2.5) as follows:

$$\begin{aligned}c(t)&=MP(t)+PP(t)=MP(t)+[B(t-1)-SP(t)]SMM(t)\\&=MP(t)+\{B(t-1)-[MP(t)-IP(t)]\}SMM(t)\\&=MP(t)(1-SMM(t))+B(t-1)(1+\frac{WAC}{12})SMM(t)\\&=B(t-1)\frac{WAC/12}{1-(1+WAC/12)^{-WAM+t}}[1-SMM(t)]+B(t-1)(1+\frac{WAC}{12})SMM(t)\\&=B(t-1)\{A(t)[1-SMM(t)]+g\,SMM(t)\},\end{aligned} \qquad (3.2)$$

where

$$A(t)=\frac{WAC/12}{1-(1+WAC/12)^{-WAM+t}},\quad g=(1+\frac{WAC}{12}). \qquad (3.3)$$

Then we can derive the gradient for *c(t)*, if *WAC* and *t* are independent[1] of θ:

$$\frac{\partial c(t)}{\partial\theta}=\frac{\partial B(t-1)}{\partial\theta}\{A(t)[1-SMM(t)]+g\,SMM(t)\}+\frac{\partial SMM(t)}{\partial\theta}B(t-1)[-A(t)+g] \qquad (3.4)$$

[1] A fixed Rate Mortgage (FRM) would satisfy this assumption; however an Adjustable Rate Mortgage (ARM) will not, so we derive the gradient estimator for ARMs later in section 4.

This leads to recursive equations for calculation of the above gradient estimator from (2.5) and (3.2):

$$TPP(t) = c(t) - IP(t) = c(t) - \frac{WAC}{12} B(t-1);$$
$$B(t) = B(t-1) - TPP(t) = B(t-1)(1+\frac{WAC}{12}) - c(t) = B(t-1)g - c(t); \qquad (3.5)$$
$$\frac{\partial B(t)}{\partial \theta} = \frac{\partial B(t-1)}{\partial \theta} g - \frac{\partial c(t)}{\partial \theta}.$$

Assuming we know that the initial balance is not dependent as θ; we have the initial conditions:

$$\frac{\partial B(0)}{\partial \theta} = 0;$$
$$\frac{\partial c(1)}{\partial \theta} = \frac{\partial SMM(1)}{\partial \theta} B(0)(-A(1)+g). \qquad (3.6)$$

This leads to the following

$$\frac{\partial B(1)}{\partial \theta} = \frac{\partial B(0)}{\partial \theta} g - \frac{\partial c(1)}{\partial \theta}, \qquad (3.7)$$

$$\frac{\partial c(2)}{\partial \theta} = \frac{\partial B(1)}{\partial \theta}\{A(2)(1-SMM(2)+gSMM(2)\} + \frac{\partial SMM(2)}{\partial \theta} B(1)(-A(2)+g), \qquad (3.8)$$

$$\frac{\partial c(t)}{\partial \theta}, t = 1,2,\ldots,M \ .$$

Thus the problem of calculating the gradient estimator of cash flow *c(t)* is reduced to calculating:

$$\frac{\partial SMM(t)}{\partial \theta}, t = 1,\ldots,M,$$

Since

$$SMM(t) = 1 - \sqrt[12]{1-CPR(t)},$$

We have

$$\frac{\partial SMM(t)}{\partial \theta} = \frac{1}{12}(1-CPR(t))^{-\frac{11}{12}} \frac{\partial CPR(t)}{\partial \theta}. \qquad (3.9)$$

As discussed earlier, generally *CPR(t)* is given in the form of a prepayment function, and there are four main types of prepayment functions:

1. Arctangent Model: (An example from the Office of Thrift Supervision (OTS).)

$$CPR(t) = 0.2406 - 0.1389\arctan(5.9518(1.089 - \frac{WAC}{r_{10}(t)})). \quad (3.10)$$

2. *CPR(S,A,B,M)* Model:

$$CPR(t) = RI(t)AGE(t)MM(t)BM(t); \quad (3.11)$$

where *RI(t)* is refinancing incentive;
AGE(t) is the seasoning multiplier;
MM(t) is the monthly multiplier, which is constant for a certain month;
BM(t) is the burnout multiplier.

3. Prepayment models incorporating macroeconomic factors, i.e., the health of economics, housing market activity, etc.
4. Prepayment models for individual mortgages.

For the last two types of prepayment models, we do not have any explicitly stated functional forms, mainly because they are proprietary models in the mortgage industry. But since our approach is general for any type of prepayment function, we can derive the derivatives once we are given an explicit form for the prepayment function.

3.2 Gradient Estimator for Discounting Factor

We have derived the gradient estimator of cash flow w.r.t. parameter θ. Next, we derive the gradient estimator of the discounting factor *d(t)*.

We know that the discounting factor takes the following form from section 2, when the option adjusted spread (OAS) is not considered. For simplification, we write *d(t)* as for *d(t, θ)*:

$$d(t) = \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\}. \quad (3.12)$$

Differentiating *w.r.t.* θ:

$$\frac{\partial d(t)}{\partial \theta} = \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\}\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial \theta})\Delta t = d(t)\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial \theta})\Delta t. \quad (3.13)$$

From the gradient estimators for cash flow and discounting factor, we can easily get the gradient estimator of *PV(t)*:

$$\frac{d(PV(t,\theta))}{d\theta} = \frac{\partial d(t,\theta)}{\partial \theta}c(t,\theta) + \frac{\partial c(t,\theta)}{\partial \theta}d(t,\theta). \quad (3.14)$$

The last step would be to apply a specific prepayment model and interest rate model to arrive at the actual implemented gradient estimators. To illustrate the procedure, we carry out this exercise in its entirety for one setting in the following section.

4 Applying the Gradients

We choose our interest model to be the one-factor Hull-White (Hull and White [1990]) model, for its simplicity and easy calibration to market term structure. For the prepayment model, we consider a *CPR(S,A,B,M)* Model.

4.1 Hull-White Model Setup

In this section, we briefly discuss the model and the simulation scheme.

In the one-factor Hull-White interest rate model, the underlying process for the short-term rate *r(t)* is given by

$$dr(t) = (\varphi(t) - ar(t))dt + \sigma dB(t), \tag{4.1}$$

where *B(t)*: a standard Brownian motion;

a: mean reverting speed, constant;

σ: standard deviation, constant;

φ(t): chosen to fit the initial term structure, which is determined by

$$\varphi(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}), \tag{4.2}$$

f(0,t): the instantaneous forward rate, which is determined by

$$\begin{aligned} R(0,t) &= \frac{1}{t}\int_0^t f(0,u)du, \\ R(0,t)t &= \int_0^t f(0,u)du. \end{aligned} \tag{4.3}$$

Differentiating both sides, *w.r.t. t*, we have

$$f(0,t) = t\frac{\partial R(0,t)}{\partial t} + R(0,t), \tag{4.4}$$

where *R(0,t)*: the continuous compounding interest rate from now to time *t*, i.e. the term structure.

In order to simplify the simulation process, the model can be re-parameterized from its original to the following:

$$dx(t) = -a(t)x(t)dt + \sigma dB(t), x(0) = 0; \tag{4.5}$$

x(t) is determined by

$$a(t) = r(t) - x(t) = f(0,t) + \frac{\sigma^2}{2a}(1 - e^{-at})^2. \tag{4.6}$$

The process *x(t)* is called an Ornstein-Uhlenbeck process, and its solution is given by

$$x(t) = \sigma e^{-at}\int_0^t e^{au}dB(u), \tag{4.7}$$

which is a Gaussian Markov process, and can also be represented as

$$x(t) = \sigma e^{-at}W(\frac{e^{2at} - 1}{2a}), \tag{4.8}$$

where *{W(t), t≥0}* is a new Brownian motion.

In this case, the interest rate *r(t)* can be represented in the following form:

$$r(t) = F(a(t) + g(t)W_{h(t)}), \tag{4.9}$$

where *a, g:* $R_+ \to R$ are continuous functions, and the functions $F{:}R \to R$ and $h{:}\ R_+ \to R$ are strictly increasing and continuous. From above we can see that

$$\begin{aligned} &F(x) = x; \\ &a(t) = f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2; \\ &g(t) = \sigma e^{-at}; \\ &h(t) = \frac{e^{2at} - 1}{2a} \end{aligned} \tag{4.10}$$

To simulate *r(t)* given by above, we will first simulate

$$x(t) = g(t)W_{h(t)},$$

which is a Gaussian Markov process, and then compute the short-term interest rate by

$$r(t) = F(a(t) + x(t))$$

For calculating the price of MBS, the short-term rate is not sufficient; the long-term rate process is also required, especially the 10-year Treasury rate, which is a deterministic function of *r(t)* in the Hull-White model. Generally this is the case for short-term rate models, but not true for more complicated interest rate models, e.g., the HJM (Heath, Jarrow and Merton[1992]) model and the LIBOR forward rate model (Jamshidian[1997]). The long-term rate *R(t,T)* is calculated from the following, :

$$\begin{aligned} &P(t,T) = e^{-R(t,T)(T-t)} = A(t,T)e^{-B(t,T)r(t)}; \\ &B(t,T) = \frac{1 - e^{-a(T-t)}}{a}; \\ &\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} - B(t,T)\frac{\partial \ln P(0,t)}{\partial t} - \frac{\sigma^2}{4a^3}(e^{-aT} - e^{-at})^2(e^{2at} - 1). \end{aligned} \tag{4.11}$$

P(t,T) is the zero coupon bond price at time *t*, with face value $1, matured at *T*. Thus we can derive the *R(t,T)* as following:

$$R(t,T) = -\frac{\ln A(t,T) - B(t,T)r(t)}{(T-t)}. \tag{4.12}$$

The standard (forward) path generation method for generating *x(t)* is given by

$$\begin{aligned} x(t_{i+1}) &= \frac{g(t_{i+1})}{g(t_i)}g(t_i)W(h(t_i)) + g(t_{i+1})[W(h(t_{i+1})) - W(h(t_i))] \\ &= \frac{g(t_{i+1})}{g(t_i)}x(t_i) + g(t_{i+1})\sqrt{W(h(t_{i+1})) - W(h(t_i))}z_{i+1}, \end{aligned} \tag{4.13}$$

where $\{z_i\}$ is a series of independent standard normal random variables. In the special case where *x(t)* is from the Hull-White model, we have

$$x(t_{i+1}) = e^{-a\Delta t_i} x(t_i) + \sigma \sqrt{\frac{1-e^{-2a\Delta t_i}}{2a}} z_{i+1} \ , \qquad (4.14)$$

where $\Delta t_i = t_{i+1} - t_i$.

4.2 Harmonic Shocks

There are multiple factors in the interest rate model that can change and affect the cash flows and discounting factor along the simulation path. The major changes could be the initial term structure $R(0,t)$ and the volatility σ.

The most common assumption for term structure change is a parallel shift on all maturities. However, this is often not an adequate model for the real world, where a shift in the term structure can take any shape. For example, short-term rates and long-term rates may change in opposite directions rather than in parallel. We consider a Fourier series decomposition of the term structure shift.

Our domain of concern is interest rates from time 0 to 30 years, since most mortgages are amortized in a 30-year term. So for example, we could assume the shift of term structure takes the following form:

$$\Delta R(0,t) = \sum_{n=0}^{\infty} \Delta_n \cos(\frac{n\pi t}{30}), \qquad (4.15)$$

where Δ_n is the magnitude for the n^{th} Fourier harmonic function. Figure 4.1 depicts the first four harmonics. (n=0,1,2,3), which is all that we will consider in our model. When n=0, the shift is just like a parallel shift in term structure. When n=1, the short-term and long-term rates move in opposite directions. When n=2, the short-term and long-term rates move in the same direction, while the middle-term rate moves in the opposite direction. Thus we decompose any shift in the term structure into the Fourier harmonic functions by Fourier transform. If we have previously calculated the gradients w.r.t. the magnitude of each harmonic function, we can apply these gradients and get the corresponding changes in the cash flows and discounting factors, and hence the change in MBS prices.

The above mentioned Fourier series have a serious drawback: it treats short-term rates the same as long-term rates. However, from experience, we know that the short-term rates generate change more than long-term rates, both in magnitude and frequency. So we would like to change the shape of the harmonic function, which will concentrate more on the short-term rates, and keep the long-term rates relatively stable. The modified harmonic function that we adopt takes the following form:

$$\Delta R(0,t) = \sum_{n=0}^{\infty} \Delta_n \cos(n\pi(1-e^{-t/T_0})) \ , \qquad (4.16)$$

where T_0 is a user-specified parameter of the harmonic shifts. The smaller T_0 is, the more likely short rates and long rates are going to act differently. See figure 4.2 for the modified harmonic functions, where T_0=10. Comparing figures 4.1 and 4.2, you can see that the modified harmonic series concentrate more on the changes with maturities less than T_0, which is both desirable and easier for analytical purposes.

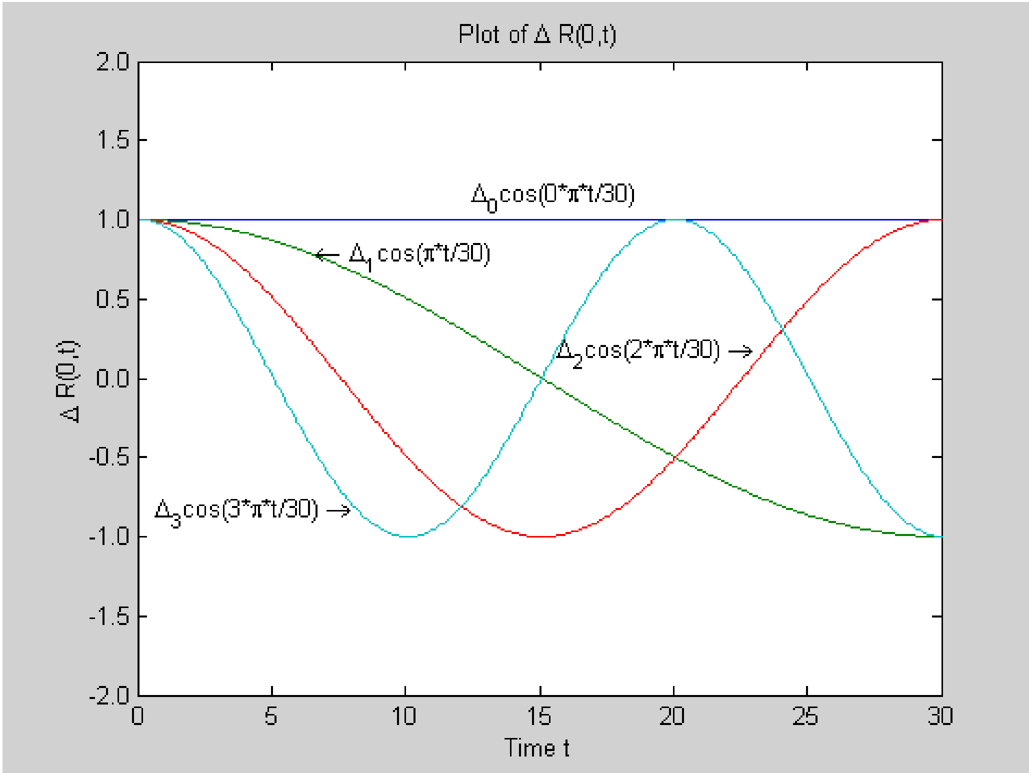


Figure 4.1 $\Delta R(0,t)$ with Fourier Harmonic series

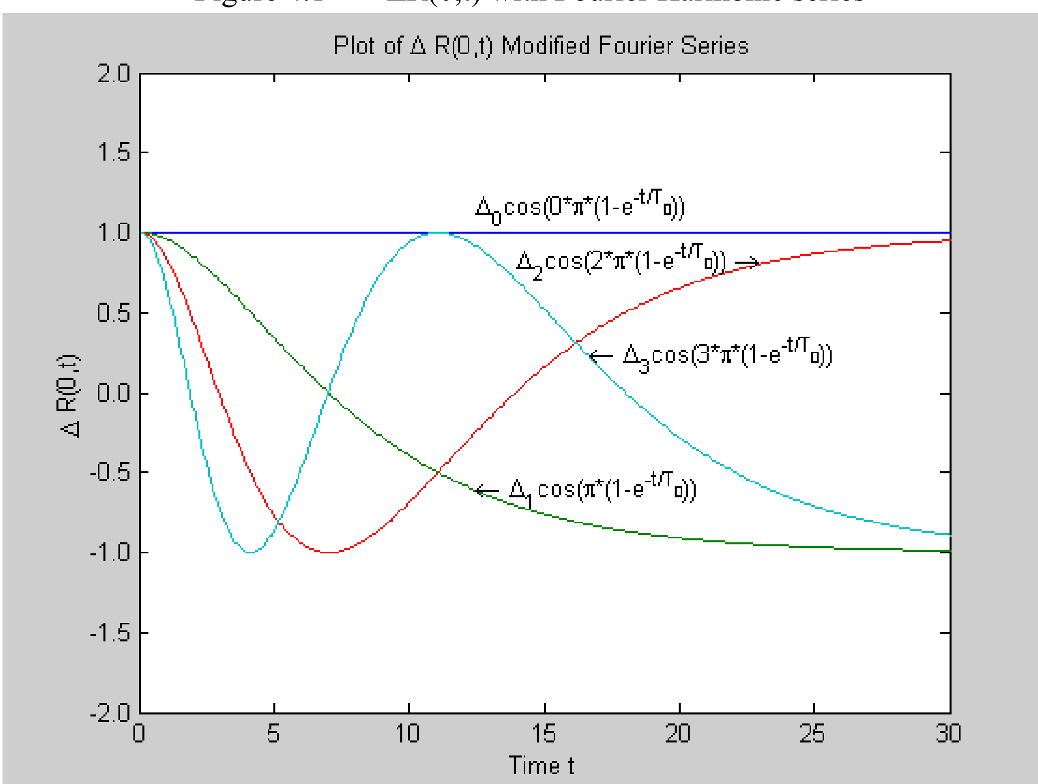


Figure 4.2 $\Delta R(0,t)$ with T_0=10 modified Harmonic series

For a Fourier cosine series that has the following functional form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi t}{T}), \quad (4.17)$$

the coefficients are given by a Fourier cosine transform:

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(\frac{2n\pi t}{T})dt, \; n = 0,1,2,...... \quad (4.18)$$

For our modified harmonic series, perform the following change of variables:

$$\frac{t'}{30} = (1 - e^{-t/T_0})$$

and substitute into the expression of *ΔR(0,t)* to get

$$\Delta R(0,t') = \sum_{n=0}^{\infty} \Delta_n \cos(\frac{2n\pi t'}{60}), \quad (4.19)$$

which is a standard Fourier cosine series, and we can use a Fourier transform to estimate the coefficients. In computer simulation, *t* is a vector of real time points, evenly distributed with sample function value *ΔR(0,t)*, and *t'* is the mapped time point in a new time scale, which is not evenly distributed, with the same sample function value *ΔR(0,t')*. However, in order to utilize the discrete cosine transform function provided in mathematical libraries, we need to resample *ΔR(0,t')* at even time intervals. This is carried out by interpolating the function of *ΔR(0,t')* on the *t* time scale. Figure 4.3 shows a sample of *ΔR(0,t), ΔR(0,t')* resampled on *t*, the coefficients estimated on the resampled *ΔR(0,t')*, and the reconstructed harmonic series of *ΔR(0,t)*.

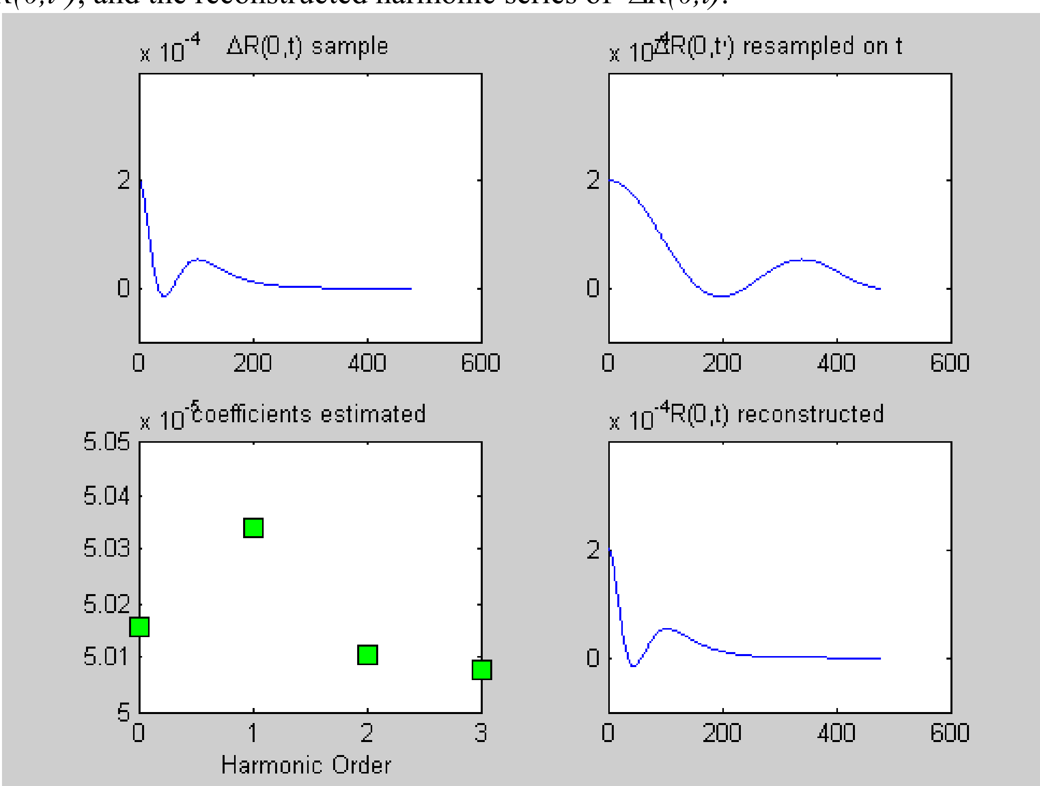


Figure 4.3 Coefficients Estimation for Harmonic series

From the chart, we can see that the reconstructed term structure matches the original sample very well, which validates our method for estimating the coefficients of the modified harmonic series.

4.3 Derivation of Gradients w.r.t. Harmonic Functions

Our major task in this section is to derive the gradient estimator w.r.t. to specific parameters in the interest rate model and prepayment model. Specifically, for the former, we re interested in the parameters of the harmonic functions (Δ_n, $n=0, 1, 2, 3$)

First let us derive the discounting factor gradient estimator:

From (3.12), we know that in order to derive $\frac{\partial d(t)}{\partial \theta}$, we must first derive $\frac{\partial r(i)}{\partial \theta}$, i=0,…, t-1. Let us recall that in section 4.1, we have the following simulation scheme for short term rate r(t):

$r(t) = a(t) + x(t)$

So

$$\frac{\partial r(t)}{\partial \theta} = \frac{\partial a(t)}{\partial \theta} + \frac{\partial x(t)}{\partial \theta} \tag{4.20}$$

where $\frac{\partial a(t)}{\partial \theta}$ and $\frac{\partial x(t)}{\partial \theta}$ are determined as the following in Hull-White model:

$$\begin{aligned} \frac{\partial a(t)}{\partial \theta} &= \frac{\partial f(0,t)}{\partial \theta}, \\ \frac{\partial x(t)}{\partial \theta} &= 0. \end{aligned} \tag{4.21}$$

We also know the relationship between $f(0,t)$ and $R(0,t)$ from (4.4), so $\frac{\partial f(0,t)}{\partial \theta}$ can be derived as:

$$\begin{aligned} \frac{\partial f(0,t)}{\partial \theta} &= \frac{\partial\left(t\frac{\partial R(0,t)}{\partial t}\right)}{\partial \theta} + \frac{\partial R(0,t)}{\partial \theta} \\ &= \frac{\partial t}{\partial \theta}\frac{\partial R(0,t)}{\partial t} + t\frac{\partial R^2(0,t)}{\partial t \partial \theta} + \frac{\partial R(0,t)}{\partial \theta} \\ &= t\frac{\partial R^2(0,t)}{\partial t \partial \theta} + \frac{\partial R(0,t)}{\partial \theta}. \end{aligned} \tag{4.22}$$

Then we plug-in Δ_n's as θ. Considering the changes of $R(0,t)$ which takes the form as in (4.16), we can get the derivatives of $R(0,t)$ straight forward:

$$\begin{aligned} \frac{\partial R(0,t)}{\partial \Delta_n} &= \cos(n\pi(1-e^{-t/T_0})), \\ \frac{\partial R^2(0,t)}{\partial t \partial \theta} &= -\sin(n\pi(1-e^{-t/T_0}))\frac{n\pi(-e^{-t/T_0})}{-T_0} = -\sin(\frac{n\pi t}{t+T_0})\frac{n\pi(e^{-t/T_0})}{T_0} \end{aligned} \tag{4.23}$$

We can get the derivatives of $r(i)$:

$$\frac{\partial r(i)}{\partial \Delta_n} = \frac{\partial f(0,t)}{\partial \Delta_n} = -t\sin(n\pi(1-e^{-t/T_0}))\frac{n\pi(e^{-t/T_0})}{T_0} + \cos(n\pi(1-e^{-t/T_0})) \qquad (4.24)$$

And gradient estimator for discounting factor is also obtained, applying (3.12).

Next, we are going to derive the cash flow gradient estimator *w.r.t.* Δ_n. From our derivation in section 3, we know that in order to get $\frac{\partial c(t)}{\partial \theta}$, we need to derive $\frac{\partial CPR(t)}{\partial \theta}$ first. We use the second type of prepayment function, among the four described in section 3. An example for this type of prepayment model is available from the sample code at the Numerix homepage http://www.numerix.com.

$$CPR(t) = RI(t)AGE(t)MM(t)BM(t); \qquad (4.25)$$

where

$$RI(t) = 0.28 + 0.14\arctan(-8.571 + 430(WAC - r_{10}(t-1)));$$

$$AGE(t) = \min(1, \frac{t}{30});$$

$$MM(t) = [0.94, 0.76, 0.74, 0.95, 0.98, 0.92, 0.98, 1.1, 1.18, 1.22, 1.23, 0.98],$$

starting from January, ending in December;

$$BM(t) = 0.3 + 0.7\frac{B(t-1)}{B(0)};$$

$r_{10}(t)$ is the 10 year rate, observed at the end of period t, which is highly correlated with the prevailing mortgage rate.

From the formulas, only *RI(t)* and *BM(t)* depend on θ, when θ is not time *t*. Thus we have the following formula for $\frac{\partial CPR(t)}{\partial \theta}$:

$$\frac{\partial CPR(t)}{\partial \theta} = \frac{\partial RI(t)}{\partial \theta} AGE(t)MM(t)BM(t) + RI(t)AGE(t)MM(t)\frac{\partial BM(t)}{\partial \theta}; \qquad (4.26)$$

where

$$\frac{\partial RI(t)}{\partial \theta} = 0.14\frac{(-430)}{1+(-8.571+430(WAC - r_{10}(t-1)))^2}\frac{\partial r_{10}(t-1)}{\partial \theta};$$

$$\frac{\partial BM(t)}{\partial \theta} = 0.7\frac{\partial B(t-1)}{\partial \theta}\frac{1}{B(0)} \qquad (4.27)$$

(4.27)

$\frac{\partial B(t)}{\partial \theta}$ is available, when $\frac{\partial c(t)}{\partial \theta}$ is calculated out, so the problem is reduced to calculating $\frac{\partial r_{10}(t)}{\partial \theta}$. In the one-factor Hull-White framework, as we have discussed in section 4.1, the long-term rate is a deterministic function of *r(t)*, so substituting *T=t+10* for (4.11), we have

$$P(t,t+10)=e^{-R(t,t+10)(t+10-t)} = A(t,t+10)e^{-B(t,t+10)r(t)};$$
$$B(t,t+10)=\frac{1-e^{-a(t+10-t)}}{a}=\frac{1-e^{-10a}}{a};$$
$$\begin{aligned}\ln A(t,t+10) &= \ln\frac{P(0,t+10)}{P(0,t)} - B(t,t+10)\frac{\partial \ln P(0,t)}{\partial t} - \frac{\sigma^2}{4a^3}(e^{-a(t+10)}-e^{-at})^2(e^{2at}-1)\\ &= \ln P(0,t+10) - \ln P(0,t) - B(t,t+10)\frac{\partial \ln P(0,t)}{\partial t} - \frac{\sigma^2}{4a^3}(e^{-a(t+10)}-e^{-at})^2(e^{2at}-1)\\ &= -R(0,t+10)(t+10) + R(0,t)t + \frac{1-e^{-10a}}{a}R(0,t) - \frac{\sigma^2}{4a^3}(e^{-a(t+10)}-e^{-at})^2(e^{2at}-1).\end{aligned} \tag{4.28}$$

Since

$$\begin{aligned} r_{10}(t) = R(t,t+10) &= -\frac{\ln A(t,t+10) - B(t,t+10)r(t)}{10}\\ &= -\frac{-R(0,t+10)(t+10) + R(0,t)t + \frac{1-e^{-10a}}{a}R(0,t) - \frac{\sigma^2}{4a^3}(e^{-a(t+10)}-e^{-at})^2(e^{2at}-1) - \frac{1-e^{-10a}}{a}r(t)}{10},\end{aligned} \tag{4.29}$$

$\frac{\partial r_{10}(t)}{\partial \theta}$ takes the following form, when θ is independent of σ and *t*:

$$\frac{\partial r_{10}(t)}{\partial \theta} = -\frac{-(t+10)\frac{\partial R(0,t+10)}{\partial \theta} + (t+\frac{1-e^{-10a}}{a})\frac{\partial R(0,t)}{\partial \theta} - \frac{1-e^{-10a}}{a}\frac{\partial r(t)}{\partial \theta}}{10}. \tag{4.30}$$

Thus we have derived $\frac{\partial r_{10}(t)}{\partial \theta}$ as a function of $\frac{\partial R(t)}{\partial \theta}$ and $\frac{\partial r(t)}{\partial \theta}$ derived earlier.

4.4 Derivation of Gradients w.r.t. Volatility: Vega

The derivation is straightforward as in section 4.3, all we need to do is to substitute θ with σ, instead of Δ_n. In order to get $\frac{\partial d(t)}{\partial \sigma}$, we must first derive $\frac{\partial r(i)}{\partial \sigma}$. Following the same logic in (4.21), we can get the vega of *r(t)*:

$$\begin{aligned}&\frac{\partial a(t)}{\partial \sigma} = \frac{\sigma}{a^2}(1-e^{-at})^2,\\ &\frac{\partial x(t)}{\partial \sigma} = e^{-at}W(\frac{e^{2at}-1}{2a}), \text{so}\\ &\frac{\partial r(t)}{\partial \sigma} = \frac{\sigma}{a^2}(1-e^{-at})^2 + e^{-at}W(\frac{e^{2at}-1}{2a}).\end{aligned} \tag{4.31}$$

And vega of *d(t)* would be:

$$\frac{\partial d(t)}{\partial \sigma} = d(t)\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial \sigma})\Delta t. \tag{4.32}$$

Now we derive $\frac{\partial c(t)}{\partial \sigma}$, which would require us to derive $\frac{\partial CPR(t)}{\partial \sigma}$ first, which has the same form as in (4.27), while $\frac{\partial r_{10}(t)}{\partial \sigma}$ has the form of:

$$\frac{\partial r_{10}(t)}{\partial \sigma} = -\frac{-(t+10)\frac{\partial R(0,t+10)}{\partial \sigma} + (t+\frac{1-e^{-10a}}{a})\frac{\partial R(0,t)}{\partial \sigma} - \frac{\sigma}{2a^3}(e^{-a(t+10)} - e^{-at})^2(e^{2at}-1) - \frac{1-e^{-10a}}{a}\frac{\partial r(t)}{\partial \sigma}}{10}$$
$$= \frac{\frac{\sigma}{2a^3}(e^{-a(t+10)} - e^{-at})^2(e^{2at}-1) + \frac{1-e^{-10a}}{a}\frac{\partial r(t)}{\partial \sigma}}{10}. \quad (4.33)$$

4.5 Derivation of Second Order Gradients w.r.t. Harmonic Functions: Gamma

Another gradient that interests risk mangers is convexity, or the gamma of MBS, which is the second order derivative of price against term structure shifts. Now we derive an estimator for the gamma.

In order to calculate the partial second order derivatives (Hessian matrix), we take θ to be the vector, $\theta=[\Delta_1\ \Delta_2\ \Delta_3\ \Delta_4\ \sigma]'$. Differentiating (2.1), we get

$$\frac{\partial P}{\partial \theta} = \frac{\partial E[V]}{\partial \theta} = E\left[\sum_{t=0}^{M}\frac{\partial PV(t)}{\partial \theta}\right] = E\left[\sum_{t=0}^{M}\frac{\partial d(t)}{\partial \theta}c(t) + \frac{\partial c(t)}{\partial \theta}d(t)\right],$$
$$\frac{\partial^2 P}{\partial \theta^2} = \frac{\partial^2 E[V]}{\partial \theta^2} = E\left[\sum_{t=0}^{M}\frac{\partial PV^2(t)}{\partial \theta^2}\right] = E\left[\sum_{t=0}^{M}\frac{\partial d^2(t)}{\partial \theta^2}c(t) + \frac{\partial d(t)}{\partial \theta}\times\frac{\partial c(t)}{\partial \theta}' + \frac{\partial c(t)}{\partial \theta}\times\frac{\partial d(t)}{\partial \theta}' + \frac{\partial^2 c(t)}{\partial \theta^2}d(t)\right]. \quad (4.35)$$

where $\frac{\partial P}{\partial \theta} = [\frac{\partial P}{\partial \Delta_1}\frac{\partial P}{\partial \Delta_2}\frac{\partial P}{\partial \Delta_3}\frac{\partial P}{\partial \Delta_4}\frac{\partial P}{\partial \sigma}]'$, and $\frac{\partial^2 P}{\partial \theta^2}$ is a 5-by-5 matrix, whose (i, j)th element is determined by $\frac{\partial^2 P}{\partial \theta_i \partial \theta_j}$, where θ_I and θ_I are the ith and jth elements of θ, respectively. The same notation will be used for gradients of other variables, i.e. *c(t), d(t), r(t),* etc.

Since we have calculated $\frac{\partial c(t)}{\partial \theta}$ and $\frac{\partial d(t)}{\partial \theta}$ in previous sections, now the problem is reduced to estimate $\frac{\partial^2 c(t)}{\partial \theta^2}$ and $\frac{\partial^2 d(t)}{\partial \theta^2}$. So we first derive the gamma for the discounting factor *d(t)*. Differentiating (3.12), we get

$$\frac{\partial d^2(t)}{\partial^2 \theta} = d(t)\times\sum_{i=0}^{t-1}(-\frac{\partial r^2(i)}{\partial \theta^2})\times\Delta t + \frac{\partial d(t)}{\partial \theta}\times\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial \theta})'\times\Delta t \quad (4.36)$$

Once we have $\frac{\partial^2 r(i)}{\partial \theta^2}$, the gamma of *d(t)* is easily calculated. Now we derive the gamma for cash flow *c(t)*. From (3.4), we can derive the following gamma equation:

$$\frac{\partial c^2(t)}{\partial \theta^2} = \frac{\partial B^2(t-1)}{\partial \theta^2}\{A(t)[1-SMM(t)]+gSMM(t)\}$$
$$+[\frac{\partial B(t-1)}{\partial \theta}\times\frac{\partial SMM(t)}{\partial \theta}'+\frac{\partial SMM(t)}{\partial \theta}\times\frac{\partial B(t-1)}{\partial \theta}'][-A(t)+g] \quad (4.37)$$
$$+\frac{\partial^2 SMM(t)}{\partial \theta^2}B(t-1)[-A(t)+g].$$

And from (3.5), we can get the gamma of *B(t)*:

$$\frac{\partial B^2(t)}{\partial \theta^2} = \frac{\partial B^2(t-1)}{\partial \theta^2}g - \frac{\partial c^2(t)}{\partial \theta^2}. \quad (4.38)$$

Now we calculate gamma of *SMM(t)*:

$$\frac{\partial^2 SMM(t)}{\partial \theta^2}, t = 1,..., M.$$

As we know from (3.9), we have

$$\frac{\partial^2 SMM(t)}{\partial \theta^2} = \frac{11}{144}(1-CPR(t))^{-\frac{23}{12}}(\frac{\partial CPR(t)}{\partial \theta}\times\frac{\partial CPR(t)}{\partial \theta}') + \frac{1}{12}(1-CPR(t))^{-\frac{11}{12}}\frac{\partial^2 CPR(t)}{\partial \theta^2}. \quad (4.39)$$

$\frac{\partial CPR(t)}{\partial \theta}$ and $\frac{\partial^2 CPR(t)}{\partial \theta^2}$ will be prepayment model specific.

For discounting factors, if we choose the Hull-White one factor model, we have the following:

$$\frac{\partial r(i)}{\partial \theta} = [\frac{\partial r(i)}{\partial \Delta_1} \quad \frac{\partial r(i)}{\partial \Delta_2} \quad \frac{\partial r(i)}{\partial \Delta_3} \quad \frac{\partial r(i)}{\partial \Delta_4} \quad \frac{\partial r(i)}{\partial \sigma}]';$$
$$\frac{\partial r^2(i)}{\partial \theta^2} = \left[\frac{\partial r^2(i)}{\partial \theta_i \partial \theta_j}\right], 1 \le i, j \le 5. \quad (4.40)$$

From and (4.24) and (4.31), we can derive the following:

$$\frac{\partial r^2(i)}{\partial \Delta_i \partial \Delta_j} = 0;$$
$$\frac{\partial r^2(i)}{\partial \Delta_i \partial \sigma} = 0; \quad (4.41)$$
$$\frac{\partial r^2(i)}{\partial \sigma^2} = \frac{(1-e^{-ai\Delta t})^2}{a^2}.$$

And the gamma of *d(t)* would be

$$\frac{\partial d^2(t)}{\partial^2\theta} = d(t)\times\sum_{i=0}^{t-1}(-\frac{\partial r^2(i)}{\partial\theta^2})\times\Delta t + \frac{\partial d(t)}{\partial\theta}\times\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial\theta})'\times\Delta t$$
$$= \frac{\partial d(t)}{\partial\theta}\times\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial\theta})'\times\Delta t \qquad (4.42)$$
$$= \frac{\partial d(t)}{\partial\theta}\times\frac{\partial d(t)}{\partial\theta}'/d(t)$$

For cash flows, based on the equations (4.27) and (4.30) in the *CPR(S, A, B, M)* model, we have:

$$\frac{\partial CPR^2(t)}{\partial\theta^2} = AGE(t)MM(t)[\frac{\partial RI^2(t)}{\partial\theta^2}\times BM(t) + \frac{\partial RI(t)}{\partial\theta}\times\frac{\partial BM(t)}{\partial\theta}'$$
$$+\frac{\partial BM(t)}{\partial\theta}\times\frac{\partial RI(t)}{\partial\theta}' + RI(t)\frac{\partial BM^2(t)}{\partial\theta^2}];$$
$$\frac{\partial RI^2(t)}{\partial\theta^2} = \frac{\partial}{\partial r_{10}}(\frac{301/5}{1+(-8.571+430(WAC-r_{10}(t-1)))^2})(\frac{\partial r_{10}(t-1)}{\partial\theta}\times\frac{\partial r_{10}(t-1)}{\partial\theta}') +$$
$$\frac{301/5}{1+(-8.571+430(WAC-r_{10}(t-1)))^2}\frac{\partial {r_{10}}^2(t-1)}{\partial\theta^2};$$
$$\frac{\partial BM^2(t)}{\partial\theta^2} = 0.7*\frac{\partial B^2(t-1)}{\partial\theta^2}*\frac{1}{B(0)},$$

(4.43)

where we know from (4.30) that

$$\frac{\partial {r_{10}}^2(t)}{\partial\theta_i\partial\theta_j} = -\frac{-(t+10)\frac{\partial R^2(0,t+10)}{\partial\theta_i\partial\theta_j} + (t+\frac{1-e^{-10a}}{a})\frac{\partial R^2(0,t)}{\partial\theta_i\partial\theta_j} - \frac{1-e^{-10a}}{a}\frac{\partial r^2(t)}{\partial\theta_i\partial\theta_j}}{10};$$
$$\frac{\partial R^2(0,t)}{\partial\theta_i\partial\theta_j} = 0; \qquad (4.44)$$
$$\frac{\partial R^2(0,t+10)}{\partial\theta_i\partial\theta_j} = 0;$$
$$\frac{\partial {r_{10}}^2(t)}{\partial\theta_i\partial\theta_j} = \frac{\frac{1-e^{-10a}}{a}\frac{\partial r^2(t)}{\partial\theta_i\partial\theta_j}}{10}.$$

Finally, the gamma of price *P* given by equation (4.35) can be obtained from equations (4.42), (4.43), and (4.44).

4.6 Derivation of ARM PA estimators

In this section, we derive PA estimators for ARMs. We know FRMs only have two sources of uncertainty:

- Short-term rate *r(t)*, which affects the discounting factor *d(t)*, and
- Long-term rate $r_{10}(t)$, which determines the prepayment rate *CPR(t)*, and hence determines the cash flow *C(t)*.

ARMs introduce one more source of uncertainty, the coupon rate *WAC(t)*, which affects both the amortization schedule and the prepayment rate *CPR(t)*, and then affects the cash flow *C(t)*. Coupon rate is determined by many factors:

- The index rate. WAC resets to the index rate plus the margin periodically.
- Margin. The spread between the WAC and the index rate.
- Adjustment period. For fixed period (FP) ARMs, the first adjustment period is different from subsequent adjustment period.
- Period Cap/Floor. The maximum amount the WAC could increase/decrease from previous period.
- Lifetime Cap/Floor. The maximum/minimum coupon rate over the lifetime of the mortgage.

In order to derive the PA gradient estimator of *C(t)* for ARM, we first need to derive the PA gradient estimator for *Index(t)* and *WAC(t)*.

The most commonly used index rate is Treasury 1year rate. In the Hull-White model, it is an explicit function of short-term rate *r(t)* and the term structure *R(0, t)*. As we have derived the function form of $r_{10}(t)$, we can derive the $r_{lag}(t)$ for any lag: (in this case, lag=1)

$$r_{lag}(t) = R(t, t+lag) = -\frac{\ln A(t, t+lag) - B(t, t+lag)r(t)}{lag}$$

$$= -\frac{-R(0, t+lag)(t+lag) - R(0,t)t - \frac{1-e^{-lag*a}}{a}R(0,t) - \frac{1}{4a^3}(e^{-a(t+lag)} - e^{-at})^2(e^{2at}-1) - \frac{1-e^{-lag*a}}{a}r(t)}{lag}. \quad (4.45)$$

Thus we have the PA gradient estimator of $\frac{\partial Index(t)}{\partial \theta}$ in following form:

$$\frac{\partial Index(t)}{\partial \theta} = \frac{\partial r_{lag}(t)}{\partial \theta} = -\frac{-(t+lag)\frac{\partial R(0, t+lag)}{\partial \theta} - (t + \frac{1-e^{-lag*a}}{a})\frac{\partial R(0,t)}{\partial \theta} - \frac{1-e^{-lag*a}}{a}\frac{\partial r(t)}{\partial \theta}}{lag}. \quad (4.46)$$

The hard part is to get the $\frac{\partial WAC(t)}{\partial \theta}$ from $\frac{\partial Index(t)}{\partial \theta}$, because of the complicated rules to determine *WAC(t)*, based on all the factors mentioned above. Given *WAC(t-1)*, *Index(t)*, *Margin*, *Period_Cap*[1], *Period_Floor, Life_Cap, Life_Floor, WAC(t)* is determined as follows:

$WAC(t) = WAC(t-1)$, if t is not an adjustment moment;

Otherwise,

$$Effective\_Floor = \min(Life\_Floor,\ WAC(t\text{-}1)\text{-}Period\_Floor);$$

$$Effective\_Cap = \max(Life\_Cap,\ WAC(t\text{-}1) + Period\_Cap);$$

$$WAC(t) = \begin{cases} Index(t) + \text{Margin, if } Effective\_Floor < Index(t) + \text{Margin} < Effective\_Cap; \\ Effective\_Floor, \text{ if } Effective\_Floor \geq Index(t) + \text{Margin}; \\ Effective\_Cap, \text{ if } Index(t) + \text{Margin} \geq Effective\_Cap; \end{cases} \tag{4.47}$$

Figure 4.4 shows the relationship of WAC with Index.

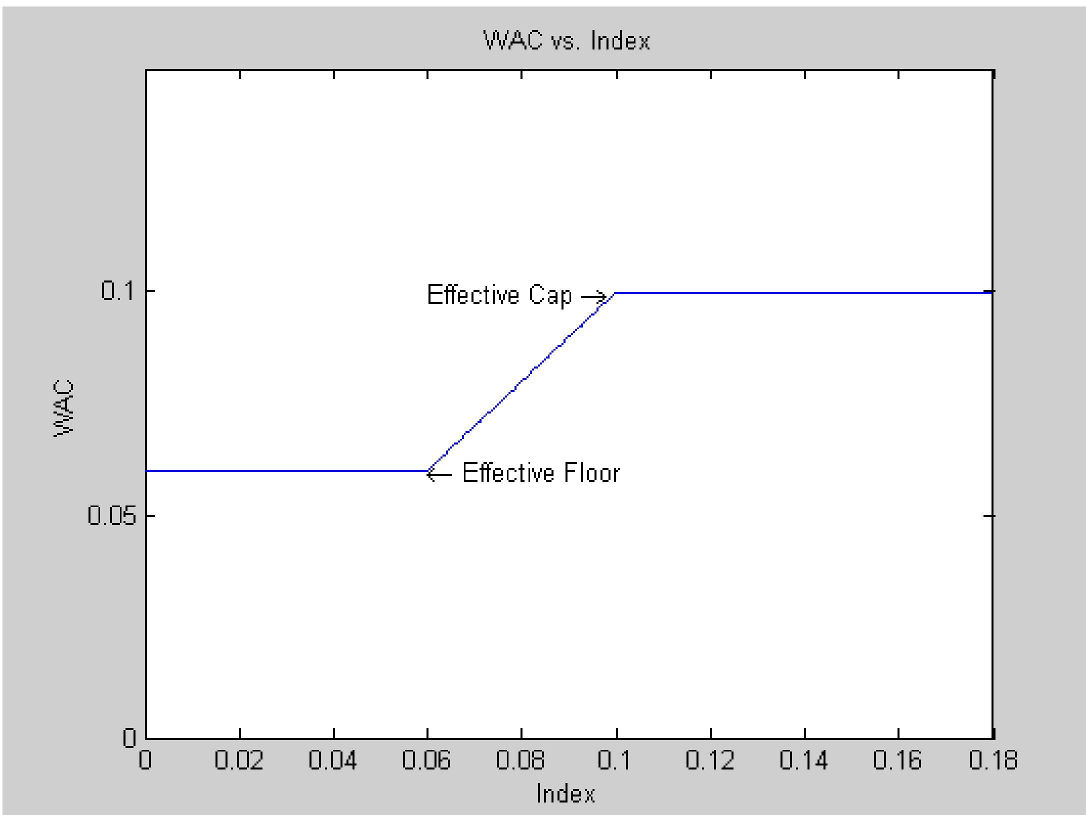


Figure 4.4 WAC as a function of Index

Then we can derive the $\dfrac{\partial WAC(t)}{\partial \theta}$ as following:

---

[1] *Life_Cap/Life_Floor* are absolute numbers, while *Period_Cap/Period_Floor* are relative.

$$\frac{\partial WAC(t)}{\partial \theta} = \frac{\partial WAC(t-1)}{\partial \theta}, \text{if t is not an adjustment moment;}$$

Otherwise,

$Effective\_Floor = \min(Life\_Floor,\ WAC(t\text{-}1)\text{-}Period\_Floor);$

$Effective\_Cap = \max(Life\_Cap,\ WAC(t\text{-}1) + Period\_Cap);$

$$\frac{\partial WAC(t)}{\partial \theta} = \begin{cases} \frac{\partial Index(t)}{\partial \theta}, \text{ if } Effective\_Floor < Index(t) + \text{Margin} < Effective\_Cap; \\ \frac{\partial WAC(t-1)}{\partial \theta} * I\{Effective\_Floor > Life\_Floor\}, \text{ if } Effective\_Floor \geq Index(t) + \text{Margin}; \\ \frac{\partial WAC(t-1)}{\partial \theta} * I\{Effective\_Cap < Life\_Cap\}, \text{ if } Index(t) + \text{Margin} \geq Effective\_Cap; \end{cases}$$

$$\text{where } I\{condition\} = \begin{cases} 1, \text{when condition is true;} \\ 0, \text{when condition is false.} \end{cases}$$

(4.48)

Note that the gradient is 0, when it is bounded by lifetime cap or floor, because a perturbation would not change the *WAC(t)*.

Next, we need to derive $\frac{\partial CPR(t)}{\partial \theta}$ for ARM, assuming ARM borrowers have the same prepayment behavoir as FRM borrowers (which is not necessarily true, but it does not affect our analysis), so we are facing the same prepayment function as FRM30 as in (4.26).

$\frac{\partial CPR(t)}{\partial \theta}$ will be affected because of the uncertainty of *WAC(t)*.

$$\frac{\partial CPR(t)}{\partial \theta} = \frac{\partial RI(t)}{\partial \theta} AGE(t)MM(t)BM(t) + RI(t)AGE(t)MM(t)\frac{\partial BM(t)}{\partial \theta};$$

$$\frac{\partial RI(t)}{\partial \theta} = 0.14\frac{430}{1+(-8.571+430(WAC(t)-r_{10}(t-1)))^2}[\frac{\partial WAC(t)}{\partial \theta} - \frac{\partial r_{10}(t-1)}{\partial \theta}];$$

$$\frac{\partial BM(t)}{\partial \theta} = 0.7\frac{\partial B(t-1)}{\partial \theta}\frac{1}{B(0)}.$$

(4.49)

$$SMM(t) = 1 - \sqrt[12]{1-CPR(t)};$$

$$\frac{\partial SMM(t)}{\partial \theta} = \frac{1}{12}(1-CPR(t))^{-\frac{11}{12}}\frac{\partial CPR(t)}{\partial \theta}.$$

(4.50)

Also *C(t)* will be affected by the introduced uncertainty in *WAC(t)*:

$$c(t) = B(t-1)\{A(t)[1-SMM(t)] + g(t)SMM(t)\}, \quad (4.51)$$

where

$$A(t) = \frac{WAC(t)/12}{1-(1+WAC(t)/12)^{-WAM+t}};$$
$$g(t) = (1+\frac{WAC(t)}{12}); \tag{4.52}$$

And

$$\frac{\partial c(t)}{\partial \theta} = \frac{\partial B(t-1)}{\partial \theta}\{A(t)[1-SMM(t)] + g(t)SMM(t)\} + \frac{\partial SMM(t)}{\partial \theta}B(t-1)[-A(t)+g(t)]$$
$$+B(t-1)\left\{\frac{\partial A(t)}{\partial \theta}[1-SMM(t)] + \frac{\partial g(t)}{\partial \theta}SMM(t)\right\}, \tag{4.53}$$

where

$$\frac{\partial A(t)}{\partial \theta} = [\frac{1}{12}\frac{1}{1-(1+WAC(t)/12)^{-WAM+t}} +$$
$$\frac{1}{144}\frac{WAC(t)}{(1-(1+WAC(t)/12)^{-WAM+t})^2(1+WAC(t)/12)^{-WAM+t}\frac{-WAM+t}{1+WAC(t)/12}}]\frac{\partial WAC(t)}{\partial \theta};$$
$$\frac{g(t)}{\partial \theta} = \frac{1}{12}\frac{\partial WAC(t)}{\partial \theta}; \tag{4.54}$$

And the PA gradient estimator for balance *B(t)* is as the following:

$$B(t) = B(t-1)(1+\frac{WAC(t)}{12}) - c(t);$$
$$\frac{\partial B(t)}{\partial \theta} = \frac{\partial B(t-1)}{\partial \theta}(1+\frac{WAC(t)}{12}) - \frac{\partial c(t)}{\partial \theta} + \frac{B(t-1)}{12}\frac{\partial WAC(t)}{\partial \theta}. \tag{4.55}$$

The PA estimator for discounting factor is unchanged, so we can get the harmonic duration and volatility duration.

5 Numerical Example

5.1 Specification of Numerical Example

We need to specify two sets of data to price the mortgage: the mortgage data and the interest rate data, which includes the initial term structure and parameters for the interest rate model.

We price different mortgages to examine the different impacts that a term structure shift or change in volatility may have on different mortgage products.

The following data are fixed for all products:

Unpaid Balance/UPB =$4,000,000;
WAM =360 months.

Table 5.1 shows the difference between all the products. All the ARM products have the same subsequent adjustment period of 12 months, period cap/floor of 0.02, lifetime cap of initial WAC plus 0.06, and no lifetime floor.

Product	WAC	Index	Adjust First
FRM	0.07425	N/A	N/A
1 Year ARM	0.06425	Treasury 1 Year	12 month
3/1 FP[1]ARM	0.07425	Treasury 1 Year	36 month
5/1 FPARM	0.07425	Treasury 1 Year	60 month
7/1 FPARM	0.07425	Treasury 1 Year	84 month
10/1 FPARM	0.07425	Treasury 1 Year	120 month
1 Year ARM[2]	0.07425	Treasury 10 Year	12 month

Table 5.1 Product Specification for mortgage pricing

We use the same parameters for all the different products in order to have comparable results. Thus we set all the products to have the same coupon rate, except the first 1 year ARM with index of Treasury 1 year rate, which has a 100 basis points (bps) teaser rate. All the ARM products have the same characteristics, except for the Adjust First date, which is the feature that distinguishes these products.

Our initial term structure is the following:

$$f(0,t)=\ln(150+12t)/100, \; t=0,1,\ldots,360.$$

This will produce an upward-sloping curve increasing gradually from 5% to 8.7% along 30 year maturity, and $R(0,t)$ is acquired by calculating the following:

[1] FP ARM refers to Fixed Period ARM, which keep the coupon rate constant for a certain period, and then adjust periodically, generally once a year. So All the FP ARM products are the same, except different Adjust First date, which is the first coupon reset date.

[2] This ARM is not a mortgage product in the market at present, and is constructed for illustration purpose only. The following sections will discuss why we introduce this product, and what nice properties it has.

$$R(0,t) = \frac{\int_0^t f(0,u)du}{t}, R(0,0) = r(0) = f(0,0); \quad (5.1)$$

which increases from 5%, to 7.78% gradually.

Our interest rate model parameters are the following:

a=0.1; σ=0.1; Δ_n=0.00025, n=0,1,2,3 (used in the FD gradient and gamma estimator calculation); $\Delta\sigma$=0.00025, (used in the FD vega estimator calculation).

5.2 Comparison of PA and FD gradient estimators

In order to test whether our PA gradient estimators are accurate, and are within the error tolerance range, we calculate the finite difference (FD) gradient estimators at the same time during our pricing process. This section will demonstrate the accuracy of our PA estimators of delta, vega, and gamma for FRM, as well as the delta and gamma for ARM.

Comparison of Harmonic Gradient Estimators for FRM

Figure 5.1 shows the FD estimator, PA estimator, their difference, and standard deviation of their difference for $\frac{\partial d(t)}{\partial \Delta_n}$. The four curves in each chart are specified as following, which will be the convention for the rest of the paper:

Blue: Harmonic Order 1;
Green: Harmonic Order 2;
Red: Harmonic Order 3;
Cyan: Harmonic Order 4.

We can see that although these two estimators are pretty close, there exists a pattern in the difference of these two estimators. This will be explained later in the error analysis section.

Figure 5.2 shows the gradient estimators for cash flow *c(t)*: they are pretty close, and the difference behaves as random noise. Based on $\frac{\partial c(t,\theta)}{\partial \theta}$ and $\frac{\partial d(t,\theta)}{\partial \theta}$, we can calculate $\frac{dPV(t,\theta)}{d\theta}$, and figure 5.3 shows us the $\frac{dPV(t)}{d\Delta_n}$. Figure 5.4 shows the 95% confidence interval for $\frac{dPV(t)}{d\Delta_n}$, and we can see that 0 is generally contained in the 95% confidence interval.

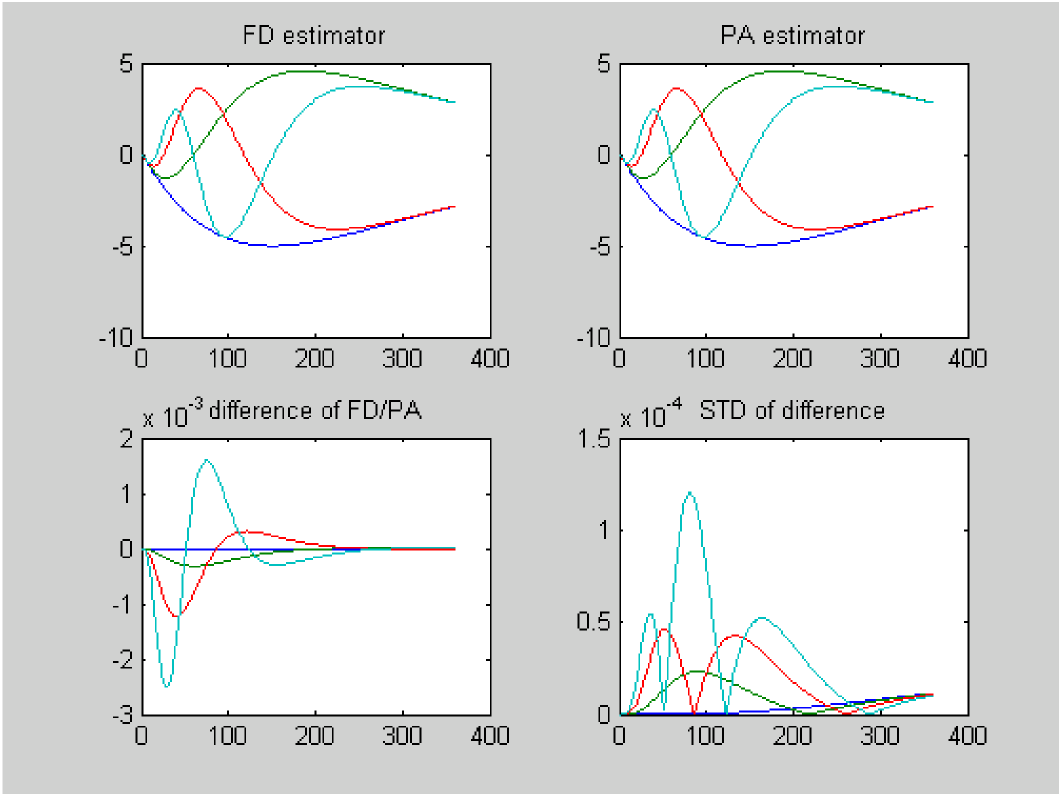


Figure 5.1 Gradient Estimator Comparison for $\partial d(t)/\partial\Delta_n$

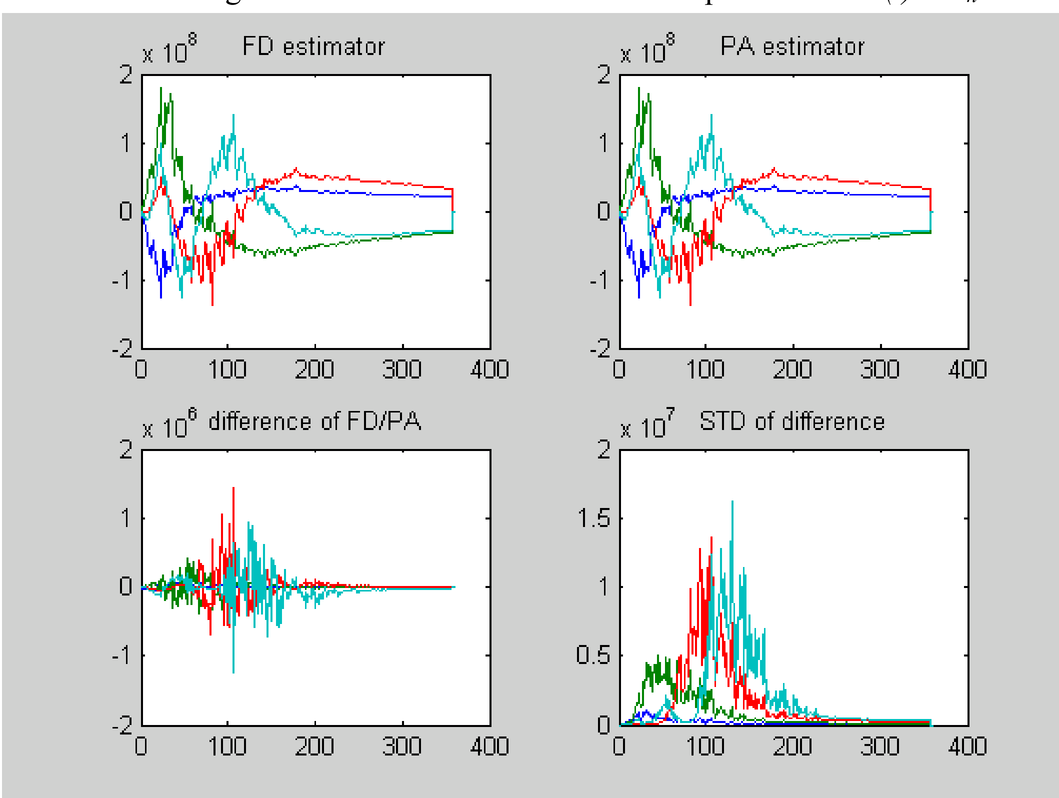


Figure 5.2 Gradient Estimator Comparison for $\partial c(t)/\partial\Delta_n$

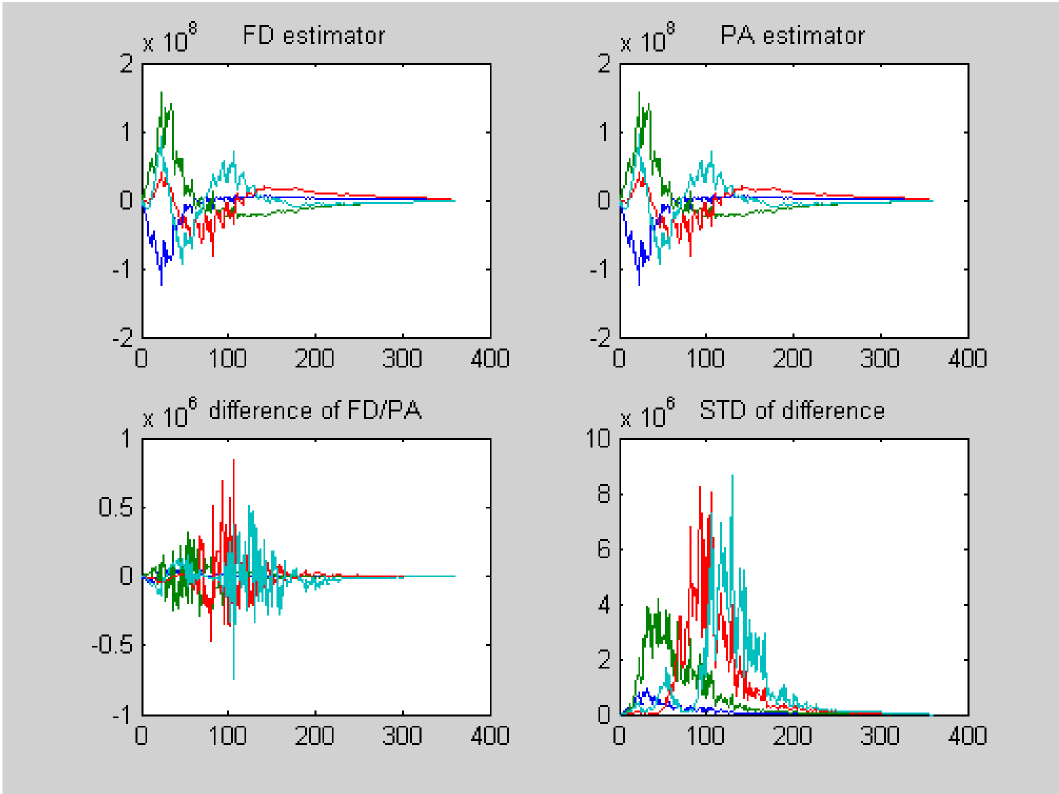


Figure 5.3 Gradient Estimator Comparison for $dPV(t)/ d\Delta_n$

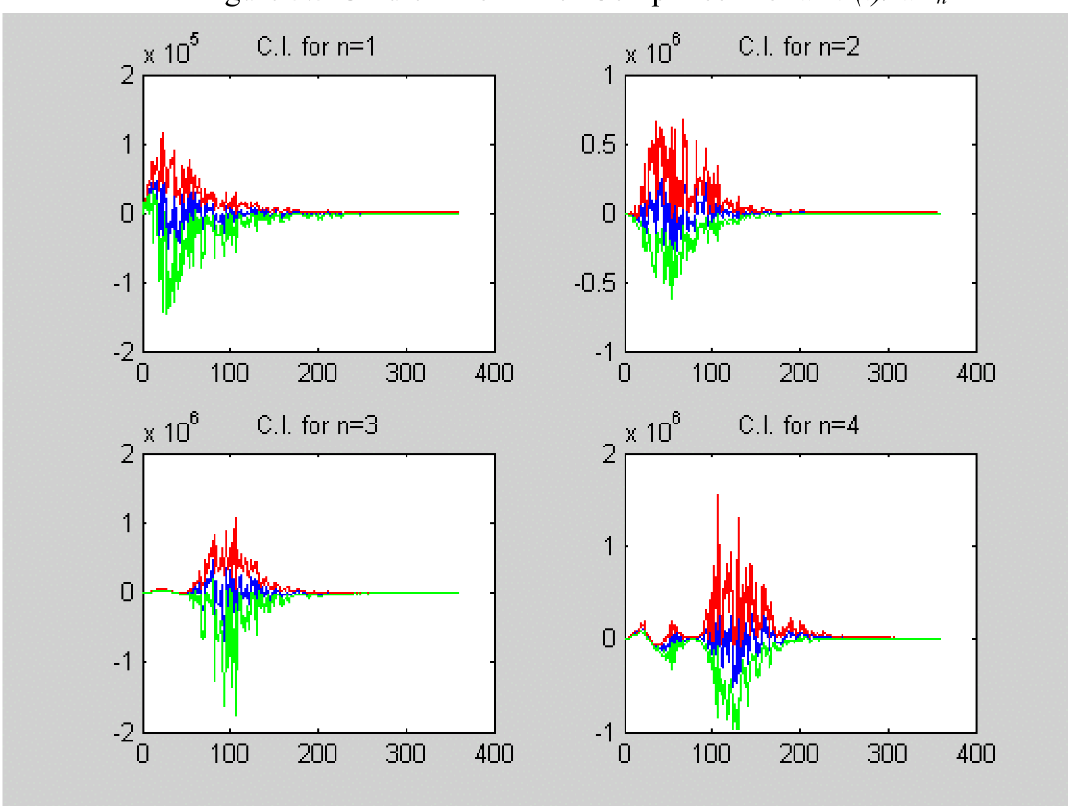


Figure 5.4 95% Confidence Interval for $dPV(t)/d\Delta_n$

Comparison of Vega Estimators for FRM

In this section, we also compare the FD and PA estimators for the gradient *w.r.t.* interest volatility: Vega. Figure 5.5 shows the FD estimator, PA estimator, their difference, standard deviation of their difference for $\frac{\partial d(t)}{\partial \sigma}$. Also there exists a pattern in the difference of these two estimators. This will also be explained later in the error analysis section. Figure 5.6 shows the gradient estimators for cash flow *c(t)*: they are pretty close, and the difference behaves as random noise. Figure 5.7 shows us the $\frac{dPV(t)}{d\sigma}$, and figure 5.8 shows the 95% confidence interval for $\frac{dPV(t)}{d\sigma}$, and we can see that 0 is always contained in the 95% confidence interval.

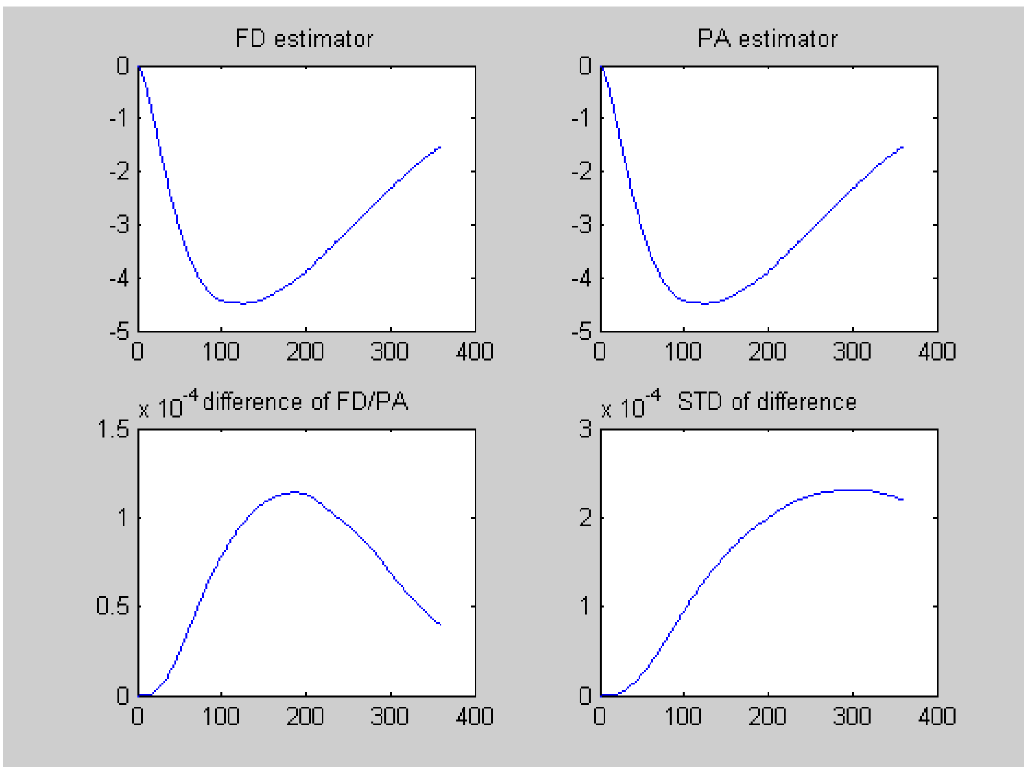


Figure 5.5 Gradient Estimator Comparison for $\partial d(t)/\partial\sigma$

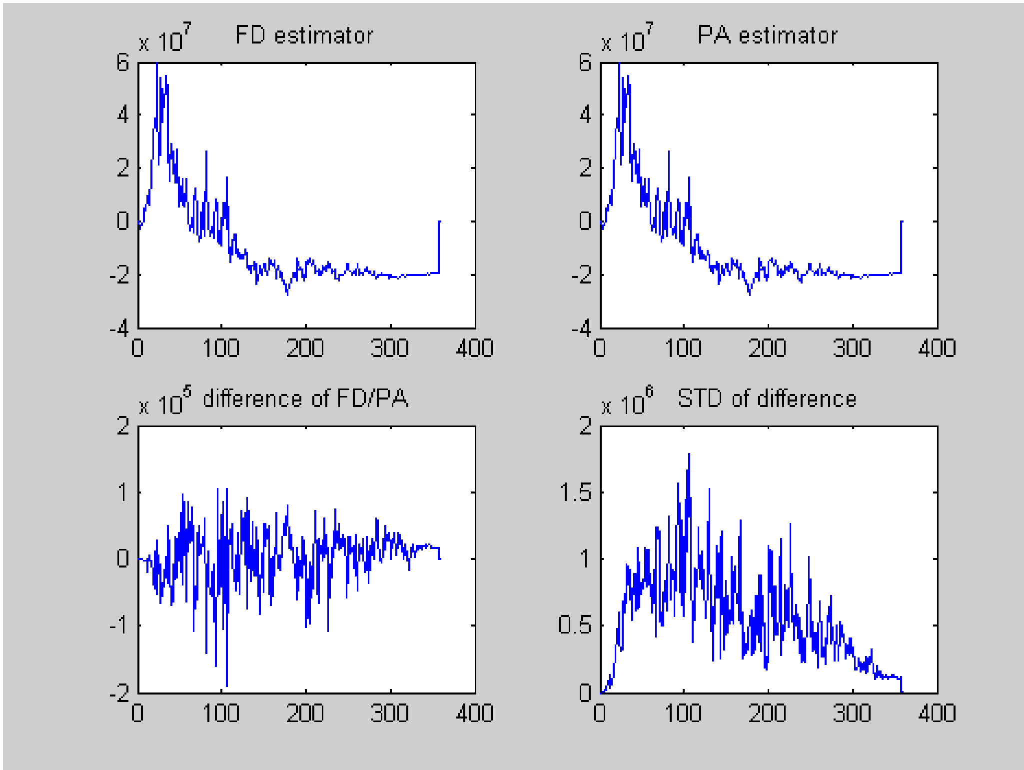


Figure 5.6 Gradient Estimator Comparison for $\partial c(t)/\partial \sigma$

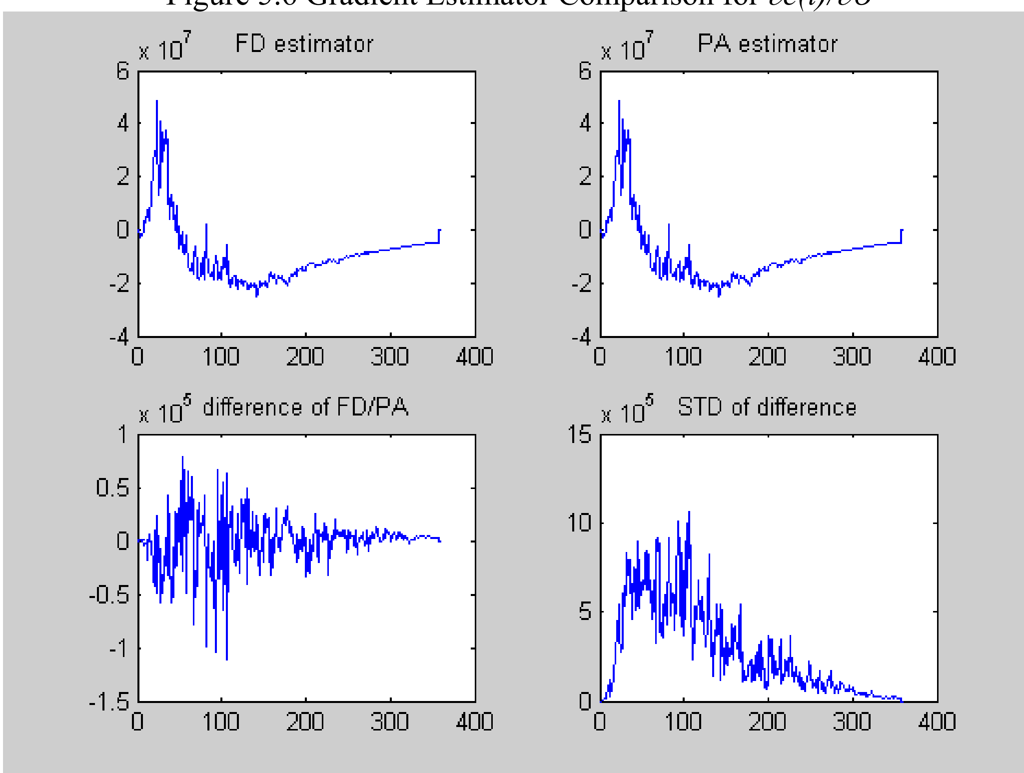


Figure 5.7 Gradient Estimator Comparison for $dPV(t)/d\sigma$

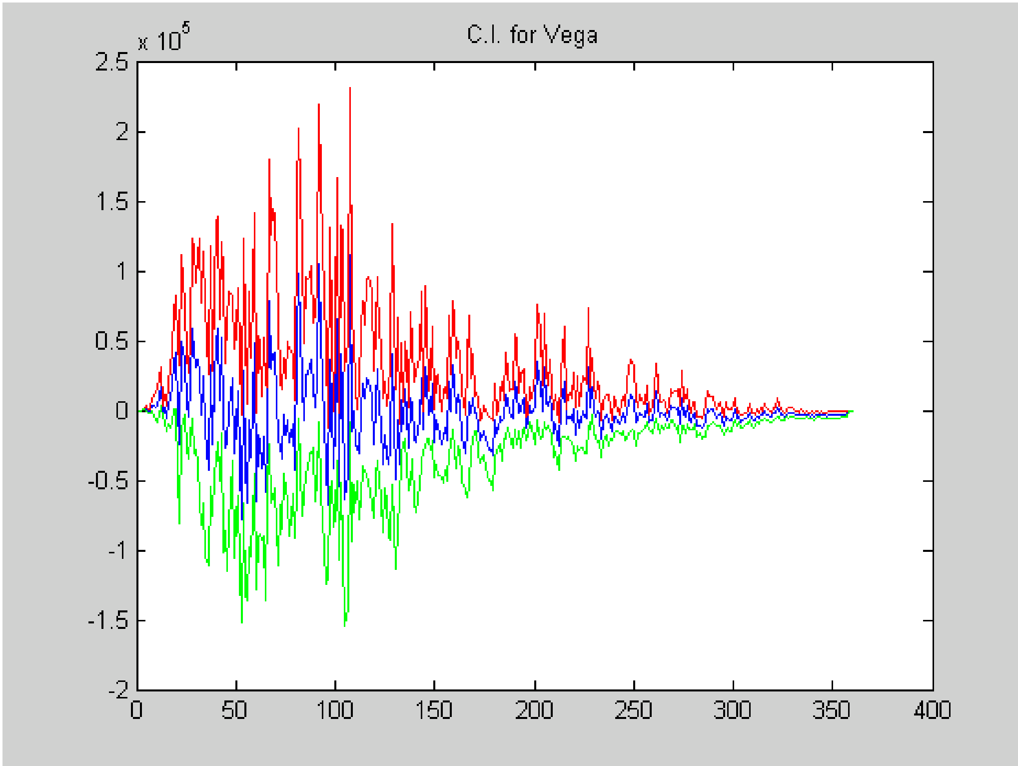


Figure 5.8 95% Confidence Interval for $dPV(t)/d\sigma$

Comparison of Gamma Estimators for FRM

For gamma estimation, $\theta = [\Delta_1\ \Delta_2\ \Delta_3\ \Delta_4]'$. So $\frac{\partial^2 d(t)}{\partial \theta^2}$, $\frac{\partial^2 c(t)}{\partial \theta^2}$, *or* $\frac{\partial^2 PV(t)}{\partial \theta^2}$ is a 4x4 matrix. If we want to estimate this matrix by the FD method, we would need 144 points to estimate 48 first order derivatives and to estimate 16 second order derivatives.

The following figures show the FD, PA estimators, the difference and STD of difference for diagonal gamma elements.

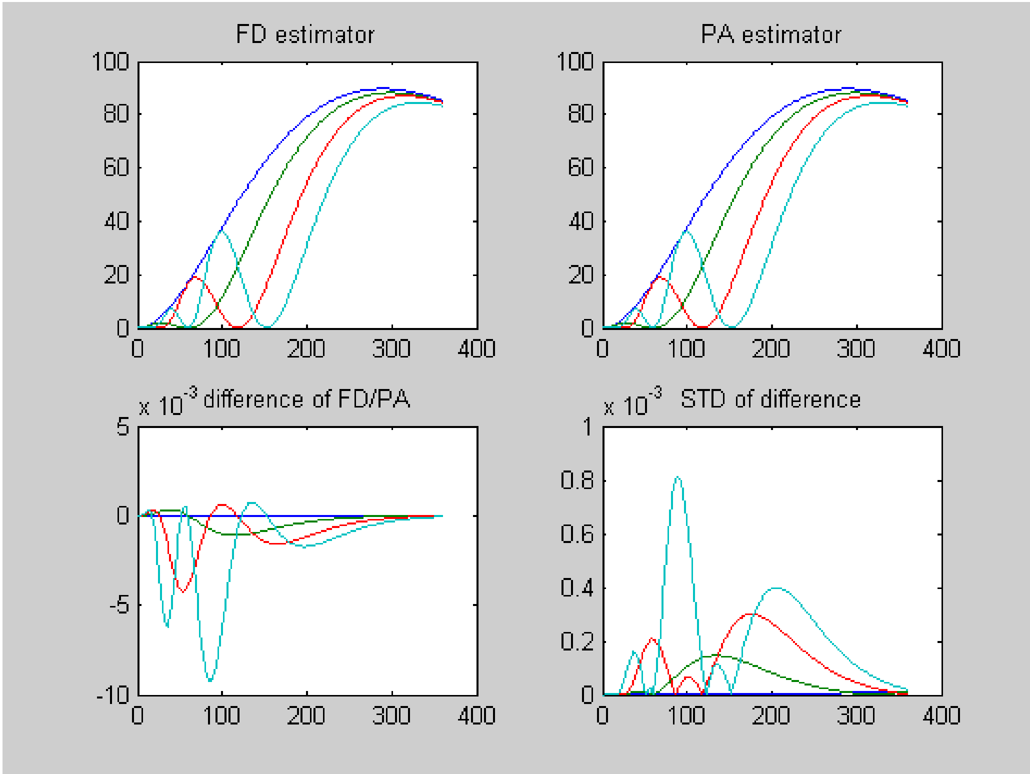


Figure 5.9 gamma estimators for $\frac{\partial^2 d(t)}{\partial \Delta_i^{\ 2}}$, *i=1, 2, 3, 4;*

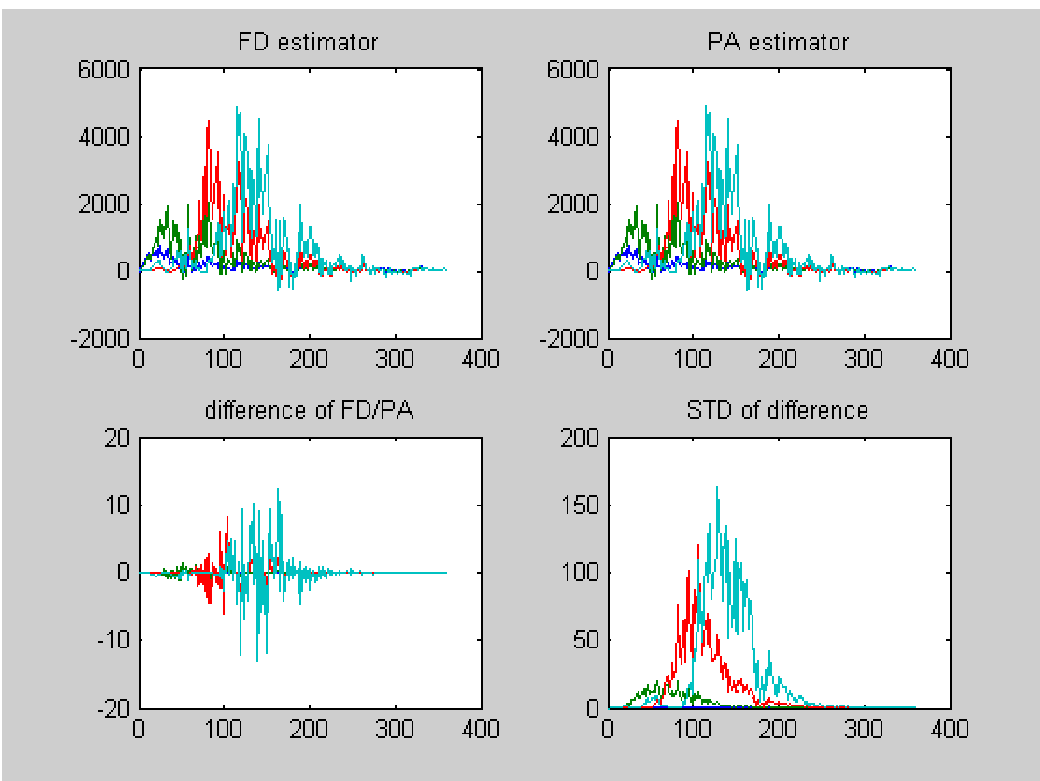


Figure 5.10 gamma estimators for $\frac{\partial^2 CPR(t)}{\partial \Delta_i^{\ 2}}$, *i=1, 2, 3, 4*

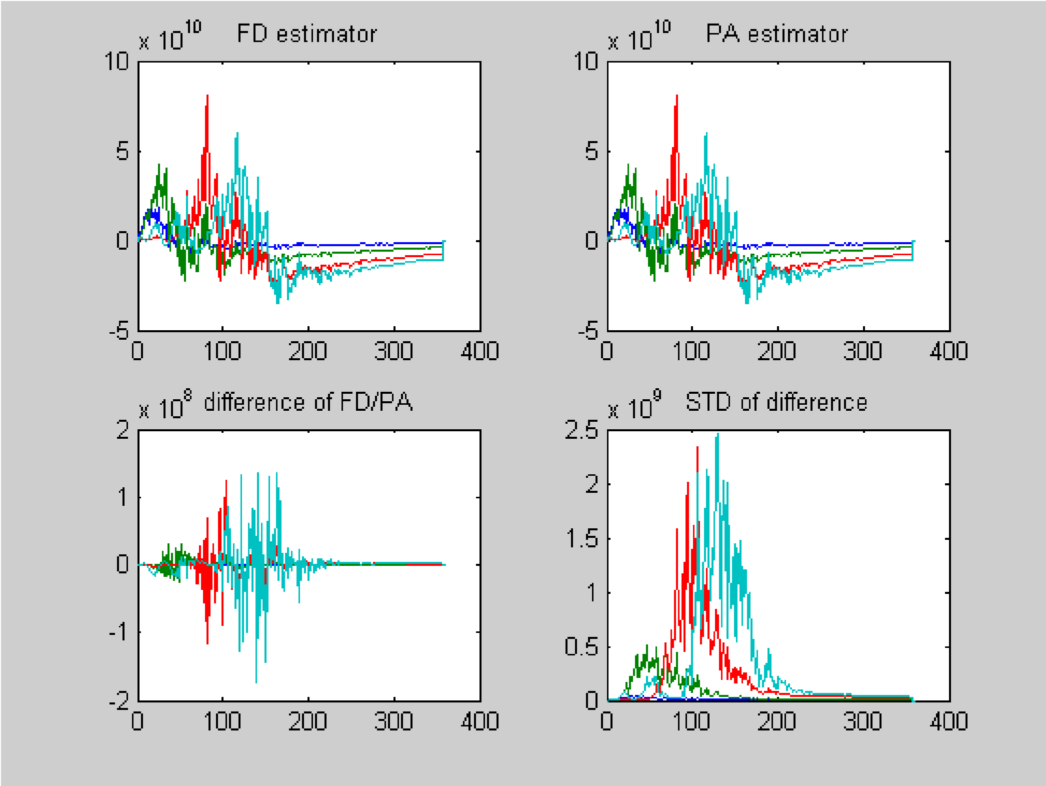


Figure 5.11 gamma estimators for $\frac{\partial^2 CF(t)}{\partial \Delta_i^{\ 2}}$, *i=1, 2, 3, 4*

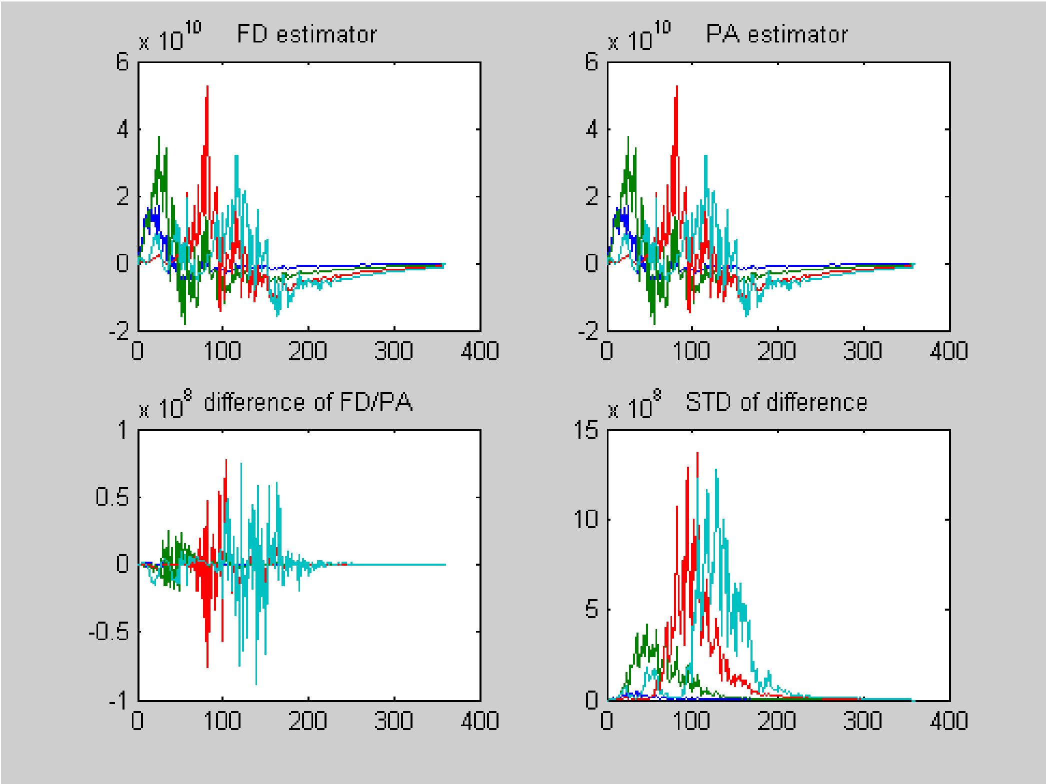


Figure 5.12 gamma estimators for $\frac{\partial^2 PV(t)}{\partial \Delta_i^{\ 2}}$, *i=1, 2, 3, 4*

Comparison of ARM gradient estimators

For ARM products, we basically have the same set of PA gradient estimators to compare with FD gradient estimators, with one additional set of estimators for $\frac{\partial WAC(t)}{\partial \Delta_i}$ (figure 5.13). To illustrate the accuracy of our simulation in a brief way, we only show the FD and PA gradient estimator comparison for one ARM product, 1-Year ARM with index of 1-Year Treasury rate, adjusted annually.

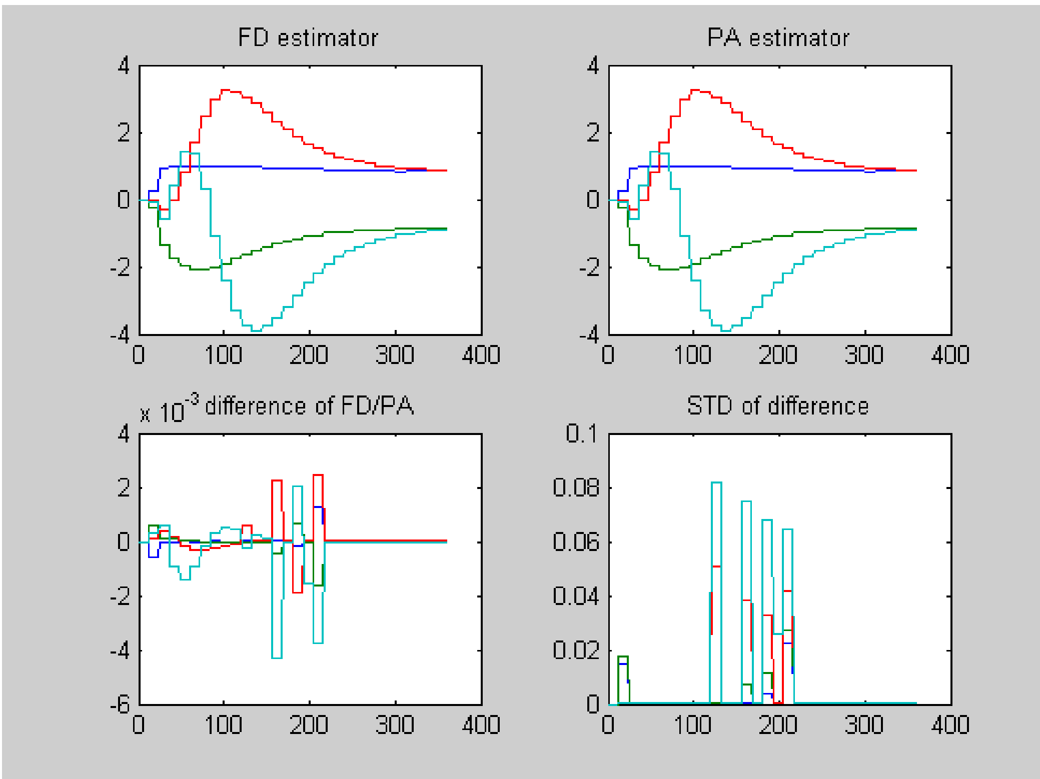


Figure 5.13 Gradient Estimator Comparison for $\frac{\partial WAC(t)}{\partial \Delta_i}$, *i=1, 2, 3, 4*

Figures 5.14 and 5.15 show the FD/PA gradient estimator comparison for $\frac{\partial PV(t)}{\partial \Delta_i}$ and $\frac{\partial PV(t)}{\partial \sigma}$ for this ARM product, respectively.

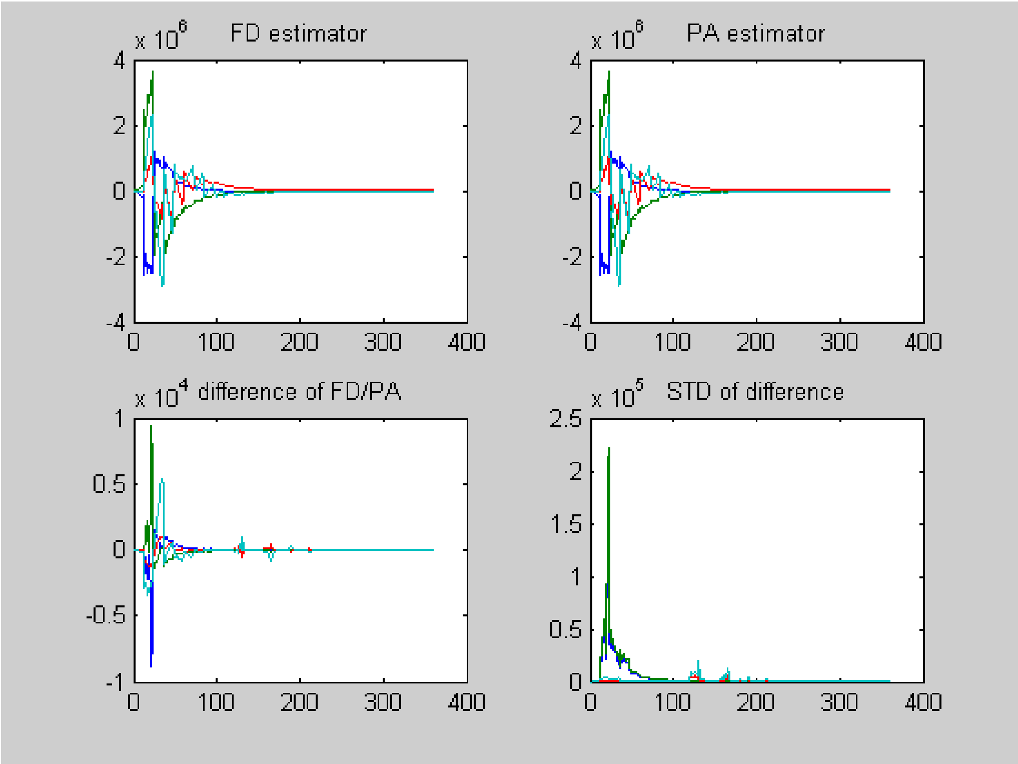


Figure 5.14 Gradient Estimator Comparison for $\frac{\partial PV(t)}{\partial \Delta_i}$, *i=1, 2, 3, 4*

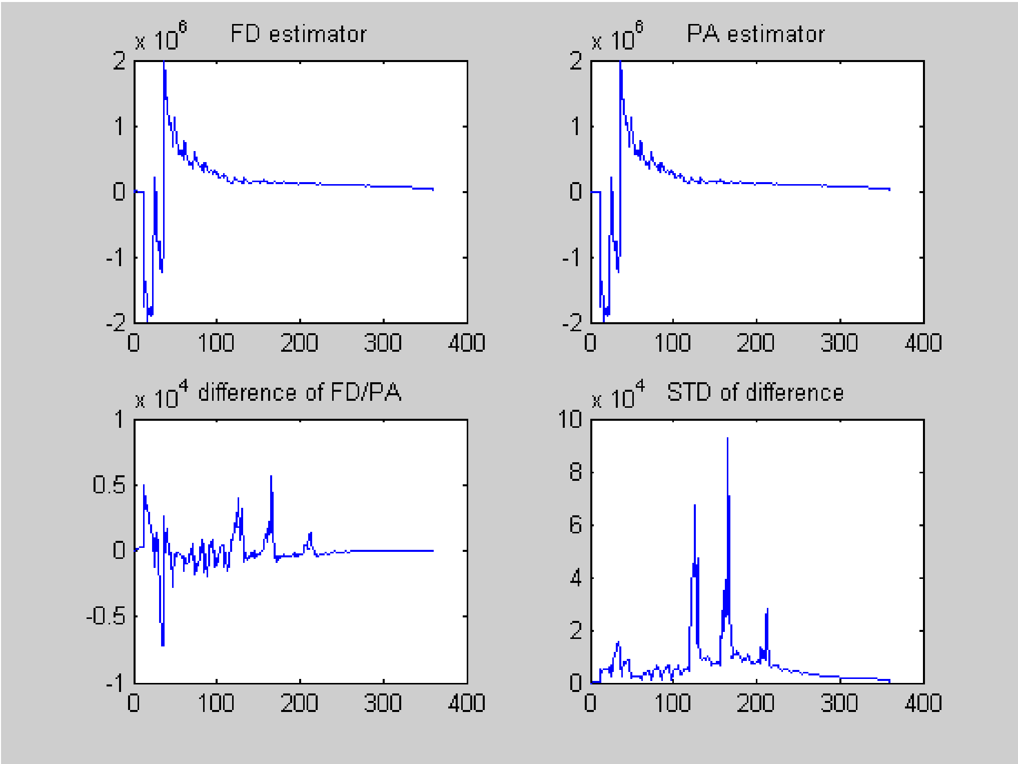


Figure 5.15 Gradient Estimator Comparison for $\frac{\partial PV(t)}{\partial \sigma}$

5.3 Result Analysis

Efficiency Analysis

In financial practice, people are more interested in duration, which is the percentage change for a security, once there is a minor shift in one parameter, which mathematically is expressed as

$$duration = \frac{dNPV(\theta)}{d\theta}\frac{1}{NPV}. \tag{5.2}$$

Actually, there should be a minus sign before the expression, since the original duration of fixed income securities measures the percentage price drop resulting from an increase in the interest rate. Yet for our analytical purpose, we do not need the duration always to be positive, since from the following numbers, we see that durations can also be negative. Table 5.1 shows the FD and PA durations for FRM30, their 95% confidence interval, and the error range of the mean.

Harmonic Order	0	1	2	3	Vega
PA estimator	-6.4816±0.1017	3.1012±0.1860	-0.5705±0.1817	0.6269±0.1189	-6.7567±0.6712
FD estimator	-6.4814±0.1017	3.1001±0.1860	-0.5695±0.1816	0.6259±0.1188	-6.7565±0.6712
Absolute Error	-0.0002	0.0011	-0.001	0.001	-0.0002
Relative Error	0.0031%	0.0355%	0.1753%	0.1595%	0.0030%

Table 5.1 Comparison of PA/FD Duration

We can see that the error size is very small, and the 95% confidence intervals are almost the same. Thus from the accuracy point of view, we can use PA estimator to replace FD estimator without causing too much problem. And the improvement in computation efficiency is enormous. The FD duration estimator works in the following way:

$$\frac{dNPV(t,\theta)}{d\theta}*\frac{1}{NPV} = \frac{NPV(\theta+\Delta\theta)-NPV(\theta-\Delta\theta)}{2\theta}\frac{1}{NPV(\theta)}. \tag{5.3}$$

Thus for each parameter, we need two additional simulations. In our case, we need 2x5+1=11 simulations to estimate the FD duration. However, by PA estimator, we only need one simulation. Ignoring the costs of middle steps, and middle variables, we can reduce the computational time by 10/11, or 90.9%. When we consider the second order derivative, gamma, the computational efficiency improves even more.

The following table shows the comparison of convexity estimators for FRM. Convexity is calculated as following:

$$convexity = \frac{d^2NPV(\theta)}{d\theta^2}\frac{1}{NPV}. \tag{5.3}$$

As we have mentioned earlier, we only estimated part of the FD gamma estimators, via using the PA delta estimators. Because to fully estimate one set of 25 gamma estimators, we would need to simulate 225 times to get all of them. And each element is a 360 by 300 (time length by simulation path) matrix.

Convexity =	Gamma/Mortgage Value				
Mortgage Value =	4.22E+08				
FD estimator	0	1	2	3	Vol
0	-246.5944896	N/A	N/A	N/A	N/A
1	N/A	-1871.407927	N/A	N/A	N/A
2	N/A	N/A	-1854.492905	N/A	N/A
3	N/A	N/A	N/A	-2000.544882	N/A
Vol	N/A	N/A	N/A	N/A	-4751.605032
PA estimator					
0	-246.6418706	951.9319609	-646.2296558	161.8535453	919.0969179
1	951.9319609	-1871.360546	1251.356282	-233.9200682	-1435.431523
2	-646.2296558	1251.356282	-1854.208619	1106.51252	1223.425174
3	161.8535453	-233.9200682	1106.51252	-2000.92393	-715.5006989
Vol	919.1916799	-1435.407832	1223.377793	-715.4533179	-4755.158608
Harmonic Order	0	1	2	3	Vol
PA estimator	-246.5944896	-1871.407927	-1854.492905	-2000.544882	-4751.605032
FD estimator	-246.6418706	-1871.360546	-1854.208619	-2000.92393	-4755.158608
Absolute Error	0.047381014	-0.047381014	-0.284286087	0.379048115	3.553576082
Relative Error	0.0192%	0.0025%	0.0153%	0.0189%	0.0748%

Table 5.2 Comparison of Convexity Estimators

So from the above analysis, we can see that by the conventional FD method, to estimate one full set of duration and convexity estimators with 5 free variables, would require 11 plus 225 simulations. Since we achieve almost the same accuracy by a single simulation in PA analysis, the simulation cost is reduced roughly by more than 99.5%. However, we also need to contemplate the introduced costs of intermediate variables as a tradeoff of the PA method.

We did all the simulations on a Pentium III 800 MHz processor, with 512 MB memory, in Matlab Release 12.0 under Windows 2000. Here is the simulation comparison.

Method	FD	PA
Memory Required	17 MB	54 MB
Simulation Time for 300 paths	115.5	765.8
Number of Duration Measures	5	5
Simulation required for estimating Duration	11	1
Number of Convexity Measures	25	25
Simulation required for estimating Convexity	225	1
Total Simulation	236	1
Total Expected Simulation Time	27257.7	765.8
Efficiency Improvement		97.2%

Table 5.3 Comparison of Compute Costs

Accuracy Analysis

In order to validate the predictive power of our PA estimator, we setup a test case to compare the predicted percentage change in the MBS price with the real percentage change.

The test case is set up as following:

$$Perturbed\_R(0,t) = R(0,t) + \sum_{n=0}^{3} \Delta_n \cos(n\pi(1 - e^{-t/T_0}));$$
$$\Delta_n = 5e-5, n = 0,1,2,3; \tag{5.4}$$
$$\Delta\sigma = 5e-5.$$
$$\frac{\Delta NPV}{NPV} = \frac{Perturbed\_NPV - NPV}{NPV}$$
$$= \text{-5.0474e-004} \ .$$

While the predicted change in NPV is calculated as following:

$$\frac{\Delta NPV}{NPV} \approx \frac{(\frac{\partial NPV}{\partial \theta})' \times \Delta\theta}{NPV} + \frac{1}{2}\frac{\Delta\theta' \times (\frac{\partial^2 NPV}{\partial \theta^2}) \times \Delta\theta}{NPV}$$
$$= duration' \times \Delta\theta + \frac{1}{2}\Delta\theta' \times convexity \times \Delta\theta \tag{5.5}$$
$$= \text{-5.0414e-004}$$

where $\Delta\theta = [\Delta_1\ \Delta_2\ \Delta_3\ \Delta_4\ \Delta\sigma]$.

We can see that the relative error by using both duration and convexity measures is only 0.0056%, while using duration measures only would produce a relative error of 0.1403%. So this test validates the predictive power of our PA gradient estimators. In the next section, we are going to show that PA estimator not only is more efficient than FD estimator, but also is a more accurate estimator.

Error Analysis

Figure 5.1 and 5.5 show that there exists a pattern in the difference of gradient estimator of discounting factor *d(t)*. Actually this has two reasons: the calculation of forward rates *f(0,t)* and the finite difference estimator of *d(t)*. This could be verified by figure 5.5, which shows the difference of FD and PA $\frac{\partial f(0,t)}{\partial \Delta_n}$ estimators.

We know that in the Hull-White model, *f(0,t)* is determined by (4.4). However, generally we do not have an explicit function form for *R(0,t)*. Instead, we only have discrete points for term structure, so *R(0,t)* is estimated by interpolation. And *f(0,t)* is further estimated by calculating the difference between adjacent points on *R(0,t)* as $\frac{\partial R(0,t)}{\partial t}$, which is not so accurate. The detailed calculation is given below.

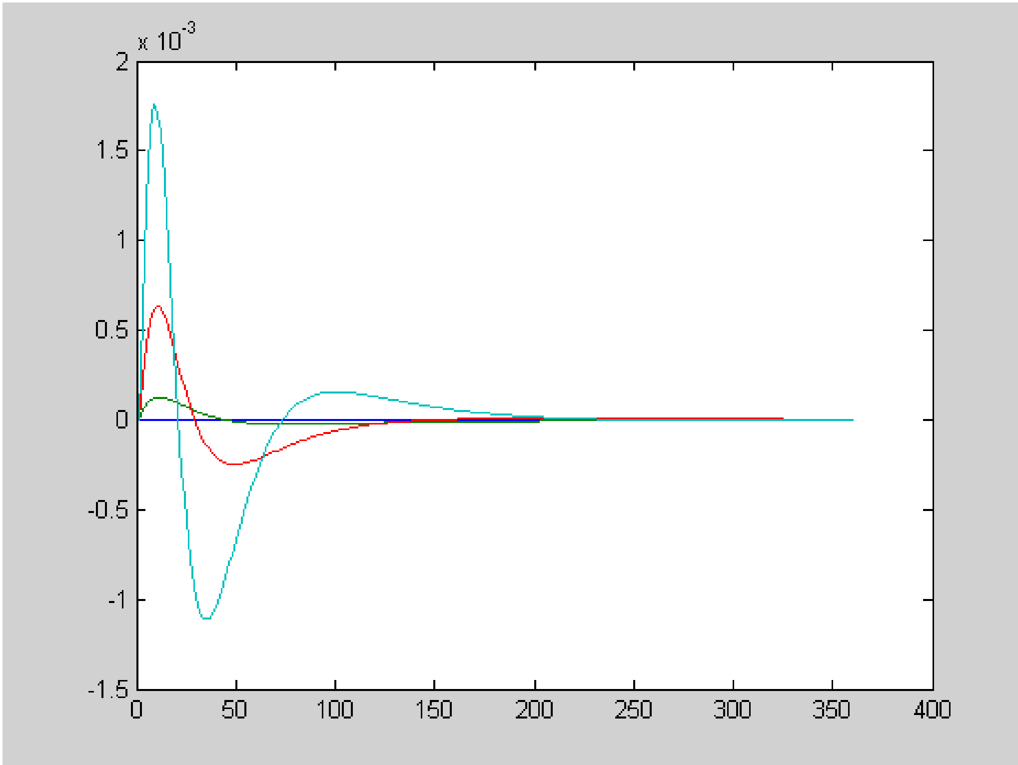


Figure 5.9 Difference of FD/PA ∂f(0, t)/ $\partial\Delta_n$ estimators

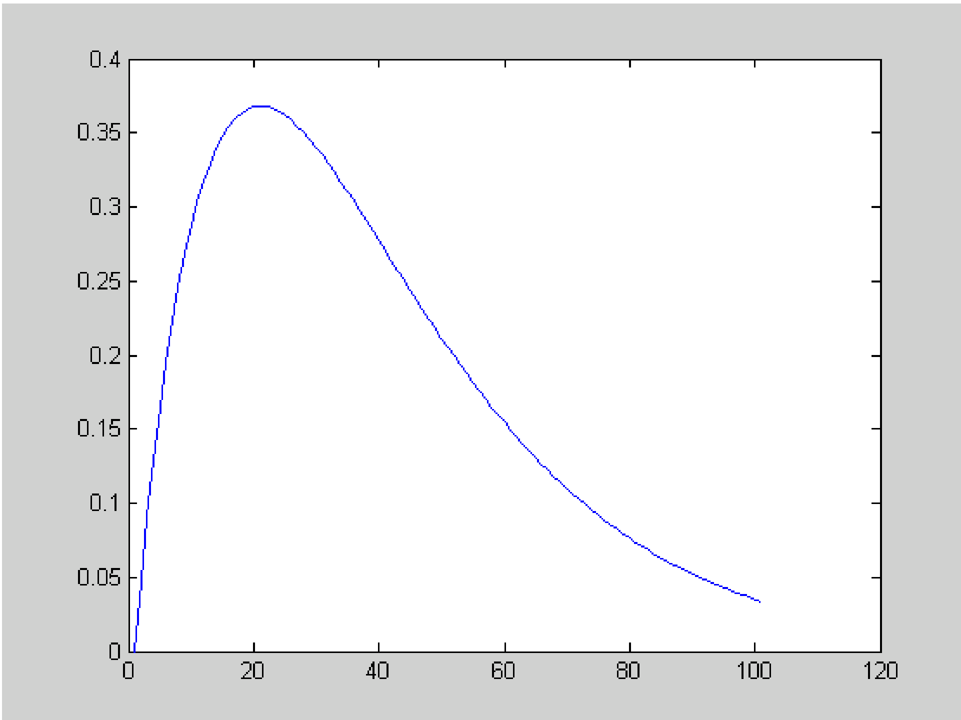
$$\frac{\partial R(0,t)}{\partial t} = \begin{cases} \frac{R(0,\Delta t) - R(0,0)}{\Delta t}, t = 0 \\ \frac{R(0,t+\Delta t) - R(0,t-\Delta t)}{2\Delta t}, 0 < t < T \\ \frac{R(0,T) - R(0,T-\Delta t)}{\Delta t}, t = T, \text{T is the maximum term} \end{cases} \tag{5.6}$$

So using FD method to calculate the f(0,t) will result inaccuracy in FD estimator of $\frac{\partial r(t)}{\partial \Delta_n}$, and this will result inaccuracy in *d(t)*. Also we know that *d(t)* takes the following form:

$$d(t) = \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\}\text{, and}$$

$$\frac{\partial d(t)}{\partial \theta} = \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\}\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial \theta})\Delta t = d(t)\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial \theta})\Delta t\,. \tag{5.7}$$

However, when we use FD method to estimate the first order derivative of e^{-x}, the FD estimator is always greater in the absolute value, because e^{-x} is a convex function. So FD estimator of $\frac{\partial d(t)}{\partial \theta}$ is always biased, the bias decreases as the FD step width reduces. The bias increases linearly, while *d(t)* decreases exponentially. As a result, the bias takes the form of xe^{-x}. Compare the difference of FD/PA $\frac{\partial d(t)}{\partial \sigma}$ gradient estimators and the figure of xe^{-x} as follows:



For the PA method, $\frac{\partial f(0,t)}{\partial \Delta_n}$ is estimated by the following formula,

$$\frac{\partial f(0,t)}{\partial \Delta_n} = t[-\sin(\frac{n\pi t}{t+T_0})\frac{n\pi T_0}{(t+T_0)^2}] + \cos(\frac{n\pi t}{t+T_0}), \tag{5.8}$$

which does not involve the FD estimation of $\frac{\partial R(0,t)}{\partial t}$. And $\frac{\partial d(t)}{\partial \theta}$ is directly estimated using its analytical form of first order derivative. So the PA estimator is more accurate than the FD estimator.

6 Interpretation of the Results

In this section, we briefly present the durations for various mortgage products, which show different trends for harmonic duration of different order. And we try to interpret how the harmonic shocks of different order would affect the discounting factors and the cash flows, and then the present value (PV) of the mortgage. Then we analyze the relationship of mortgage prepayment option and mortgage duration. Based on these analysis, we propose a potential new ARM product, which could reduce the duration over any of the existing mortgages, while having a less volatile index than most existing mortgages. This product would benefit both the investors who want to reduce the interest risk, and the mortgage borrowers who want to have a fairly stable coupon rate.

6.1 Overview of the Results

The following table shows the durations for various ARM and FRM products we specified and priced in section 5:

Harmonic Order	0	1	2	3	Vega
ARM TSY 1	-1.7761	-4.313	6.5674	-2.3822	-3.4618
FP 3/1 ARM	-2.8441	-2.9642	7.1814	-3.4784	-4.1601
FP 5/1 ARM	-3.8514	-1.1609	5.9506	-5.3355	-5.2456
FP 7/1 ARM	-4.3054	-0.3064	4.9272	-5.4472	-5.6651
FP 10/1 ARM	-5.4256	1.6401	1.7592	-2.6933	-6.6163
FRM30	-6.4816	3.1012	-0.5705	0.6269	-6.7567

Table 6.1 Durations of Different Products

The relation can be better illustrated with the figure 6.1. The zeroth order harmonic duration (with respect to Δ_0) is the same as Option Adjusted Duration (OAD), which measures the price percentage change to a parallel interest term structure shift. Other harm durations are the same measure, with respect to other interest term structure changes. Vega measures the price percentage change to an interest volatility change. As we can see, for OAD and Vega, the most important hedge measures, FRM30 has the highest numbers, and 1-Year ARM has the lowest. For everything between pure FRM and pure ARM, there exists a monotonic relationship with the product's approximation to an FRM30. For example, the Fixed Period 10/1 ARM is more like an FRM30 than a Fixed Period 7/1 ARM, so it has higher OAD, and higher Vega.

This means that ARM products have a lower interest risk than FRM products, since an ARM borrower takes more interest risk than an FRM borrower. This result is consistent with Kau et al.[1990,1992,1993] and Chiang [1997].

However, an interesting phenomenon is that the first order harmonic duration (with respect to Δ_1) actually decreases, and changes sign as volatility of the coupon rate decreases. This indicates that an opposite move of the long-term and short-term rates would not only affect ARMs with a different magnitude, but also has a reverse effect from FRMs. Here is the explanation for this. The first order harmonic duration models the following changes in term structure:

- Short-term rate increases;
- Intermediate term rate (e.g. 10 year rate) doesn't change, or moves only a little bit;
- Long-term rate decreases.

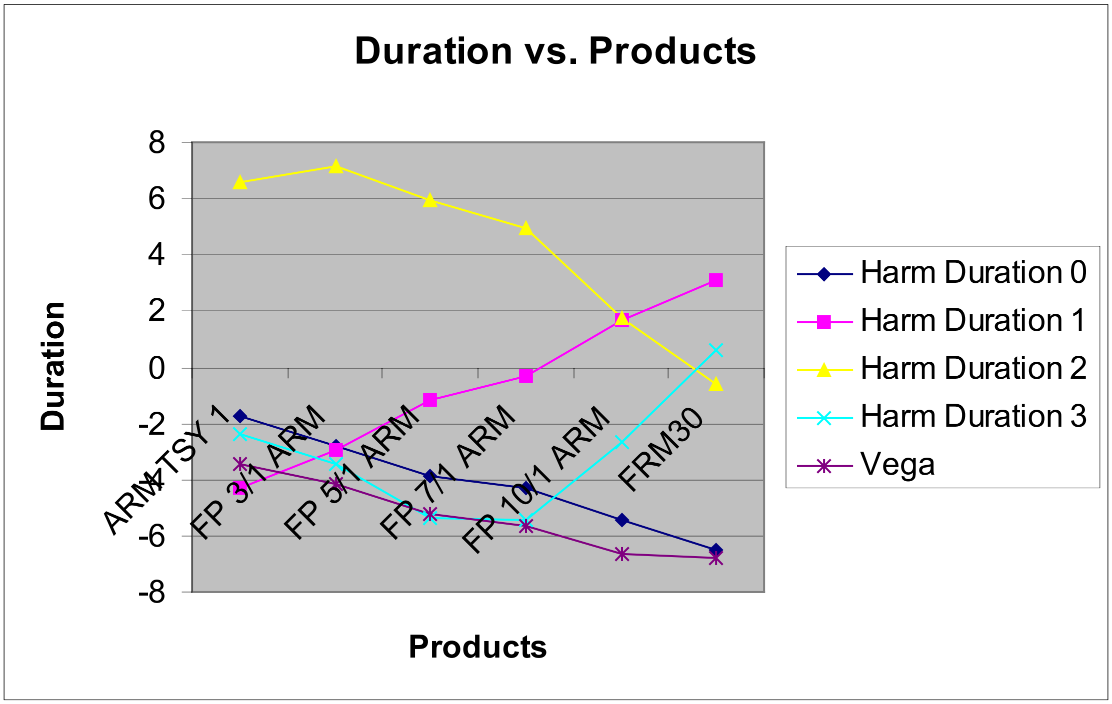


Figure 6.1 Duration vs. Products

In this scenario, people with a short-term ARM, e.g. 1 year ARM are burnt the hardest, so they are going to refinance anyway, even if the prevailing mortgage rate does not change a lot. This will create huge prepayment, and reduce the NPV of the ARM mortgage. People with FRM, on the other hand, have no incentive to refinance, since the refinance mortgage rate (highly correlated with 10-Year Treasury rate) does not change a lot. This will make the future cash flow more stable and valuable, since they are discounted at a lower long-term interest rate, and increase the NPV of the FRM mortgage.

The above analysis is based mainly on intuition, and does not show how will this term structure shock affect the discounting factors, cash flows, and NPV of MBS. In the following section, we will see what effect each one of the harmonic functions has on these components of MBS for various mortgage products.

6.2 Harmonic Shock Impact

The following 8 charts will show different harmonic shocks on term structure *R(0,t)*, and their impact on *d(t)*, *CF(t)*, and *PV(t)*.

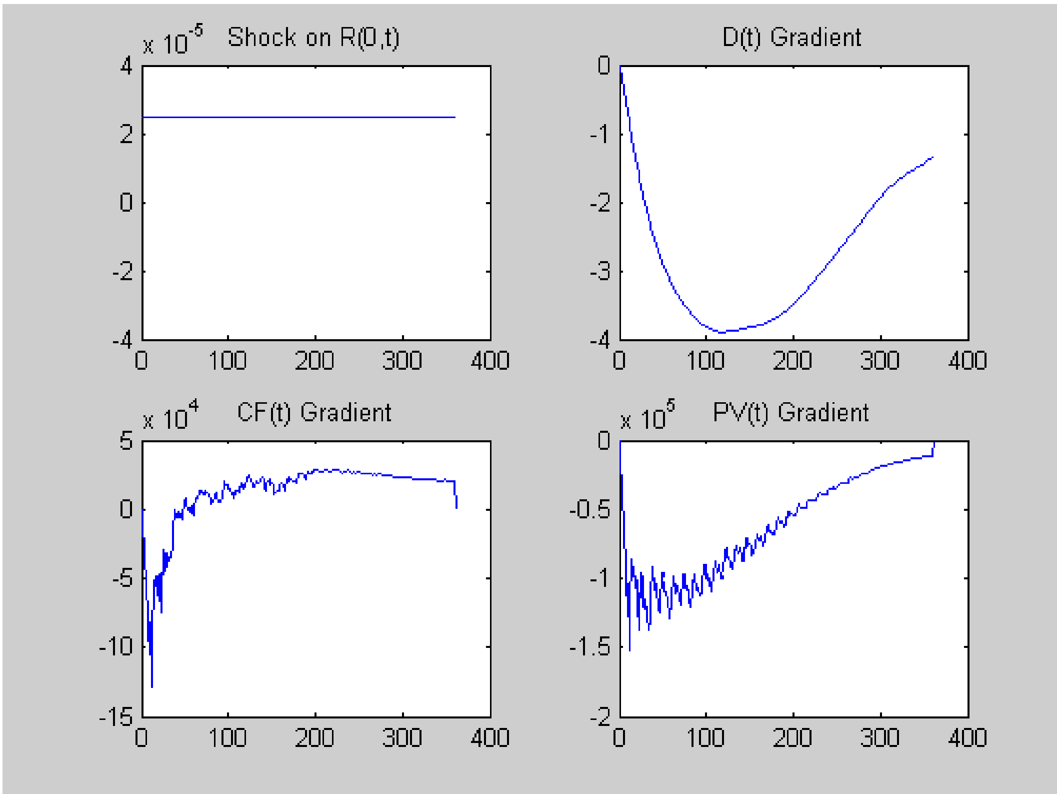


Figure 6.2 The Impact of Harmonic Function Order 0 on FRM30

Explanation: A parallel shift in the upward slope term structure will have a negative impact on the discounting factor. Also people are less likely to prepay in the near future, which reduces the cash flow in the short term, and increase the cash flow in the long term a little bit. However, the overall effect of such a shift on present value is negative, and thus reduces the NPV of this MBS.

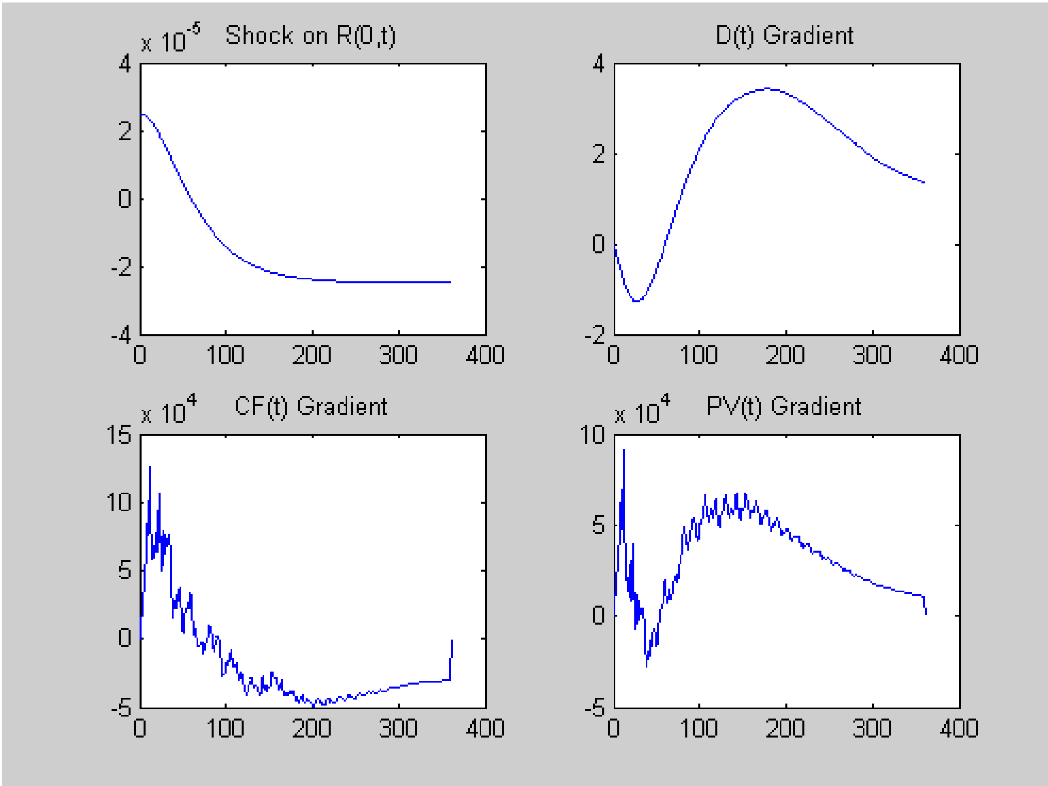


Figure 6.3 The Impact of Harmonic Order 1 on FRM30

Explanation: A shift of this shape in the upward slope term structure will have a mixed impact on the discounting factor: decrease it in the short term, but increase it in the long term. Also people are more likely to prepay in the near future, which increases the cash flow in the short term, and reduces the cash flow in the long term a little bit. However, the overall effect of such a shift on present value is positive, and thus increases the NPV of this MBS.

The Impact of Harmonic Order 2 on FRM30 (next page)

Explanation: A shift of this shape in the upward slope term structure will have a mixed impact on the discounting factor: increase it in the middle term, but decrease it in the long term. There is little incentive for people to prepay in the near future, and they will also cling to their current coupon rate in the middle term, because at that time the refinance rate will increase. However, the overall effect of such a shift on present value is cancelled out, and has little impact on the NPV of this MBS.

The Impact of Harmonic Order 3 on FRM30 (next page)

Explanation: same as Harmonic Order 2

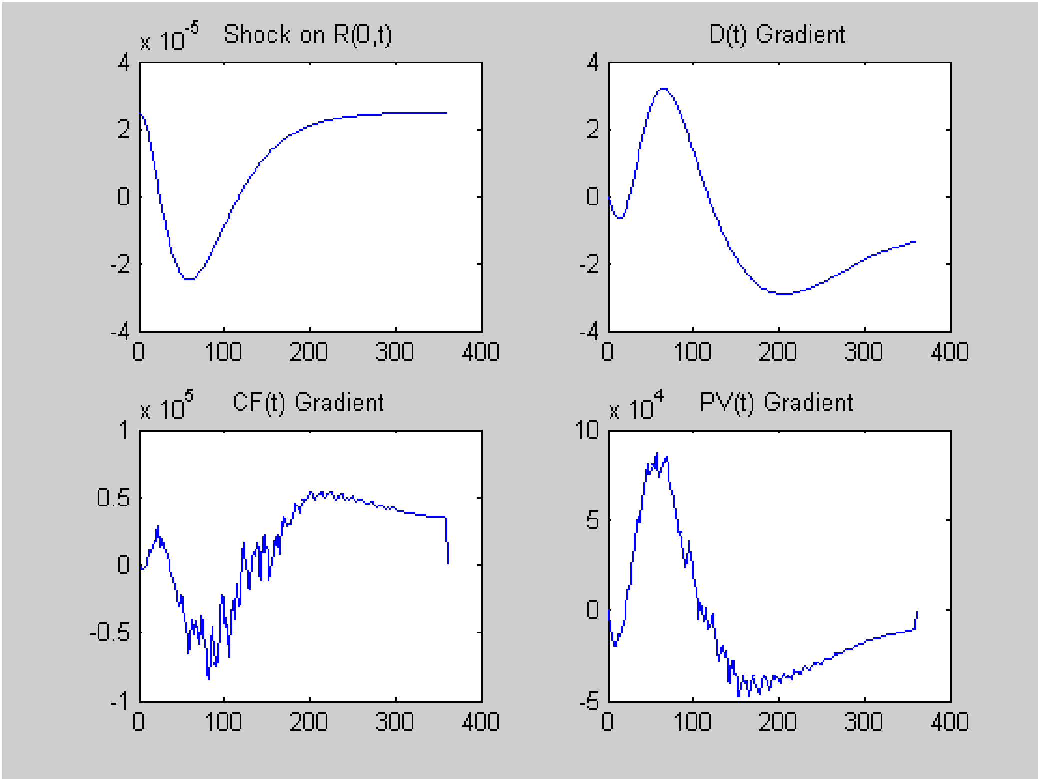


Figure 6.4 The Impact of Harmonic Order 2 on FRM30

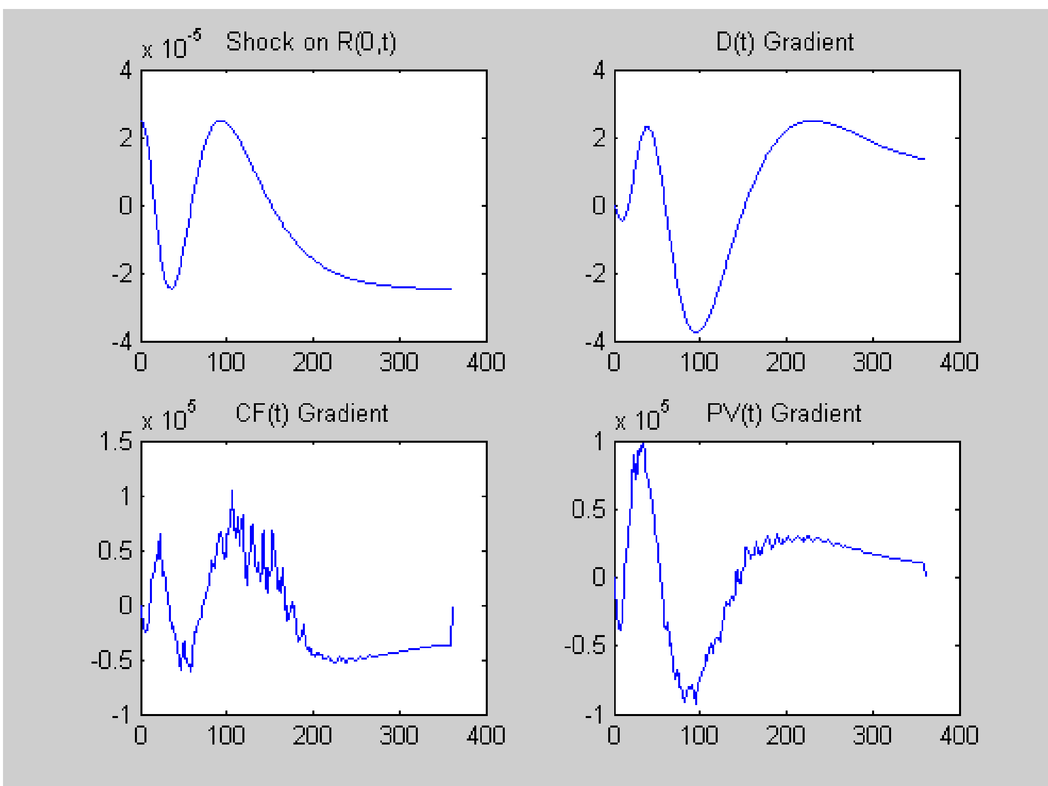


Figure 6.5 The Impact of Harmonic Order 3 on FRM30

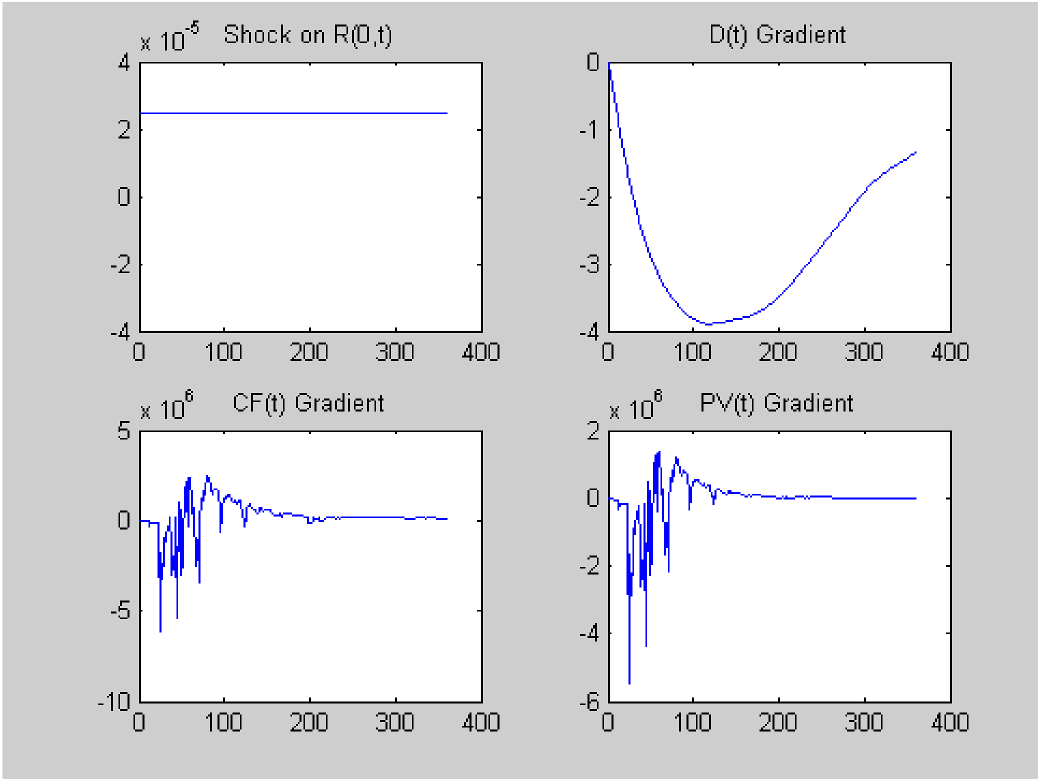


Figure 6.6 The Impact of Harmonic Order 0 on ARM TSY 1

Explanation: A parallel shift in the upward slope term structure will have a negative impact on the discounting factor. Also people with ARM are less likely to prepay in the near future, because they have a lower ARM rate than the refinance. Yet they will start prepay in the middle term, because short term rate at that time will increase, due to the upward slop term structure. This behavior will reduce the cash flow in the short term, and increase the cash flow in the middle term. However, the overall effect of such a shift on present value is negative, and thus reduces the NPV of this MBS. Yet the impact will be much smaller than that for FRM.

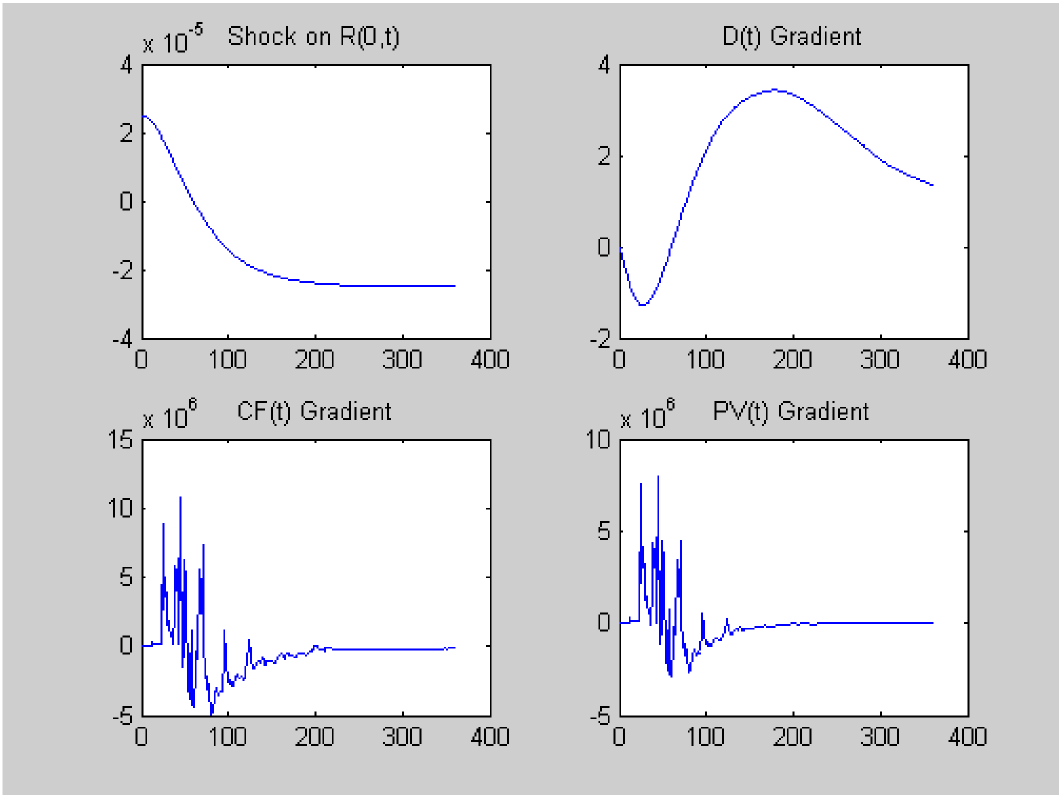


Figure 6.7 The Impact of Harmonic Order 1 on ARM TSY 1

Explanation: A shift of this shape in the upward slope term structure will have a mixed impact on the discounting factor: decrease it in the short term, but increase it in the long term. Also people are more likely to prepay in the near future, which increase the cash flow in the short term, and reduce the cash flow in the long term a little bit. The overall effect of such a shift on present value is negative, and thus decreases the NPV of this MBS.

The Impact of Harmonic Order 2 on ARM TSY 1 (next page)

Explanation: A shift of this shape in the upward slope term structure will have a mixed impact on the discounting factor: increase it in the middle term, but decrease it in the long term. People will cling to their low ARM rate for the first few years, but then start to prepay in the middle term, since short term rate will increase at that time. The overall effect of such a shift on present value is positive, due to the increase cash flow and discounting factor in the middle term.

The Impact of Harmonic Order 3 on ARM TSY 1 (next page)

Explanation: much like Harmonic Order 3, yet because the reverse effect of discounting factor, the overall effect will be negative.

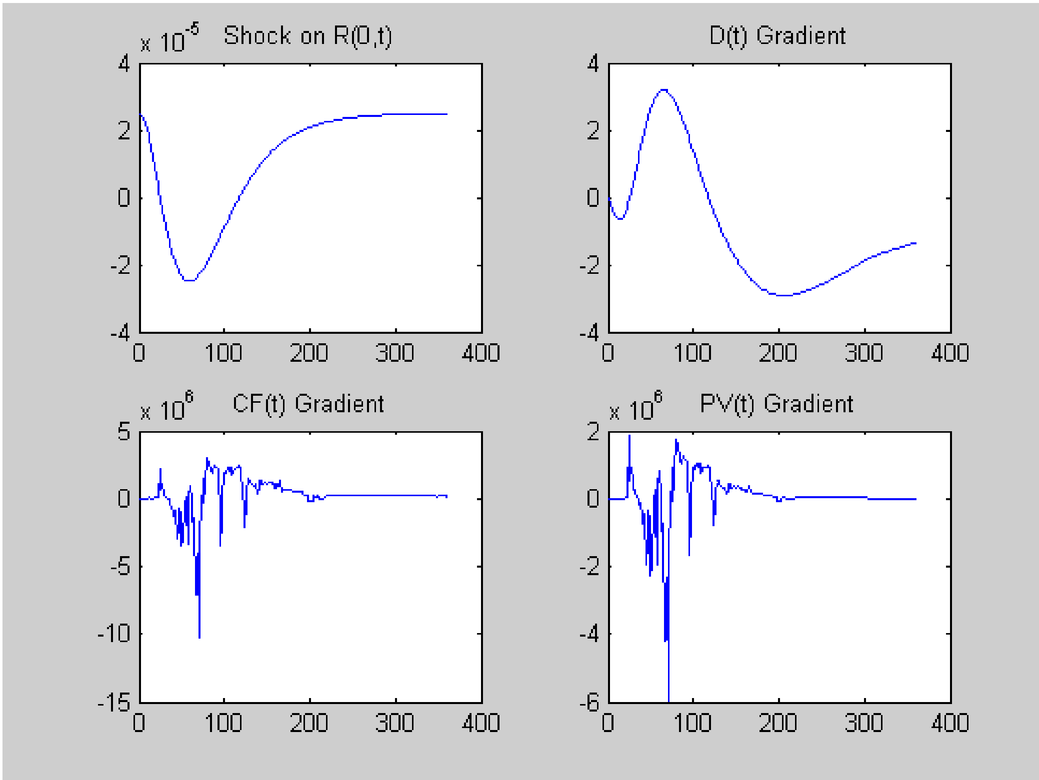


Figure 6.8 The Impact of Harmonic Order 2 on ARM TSY 1

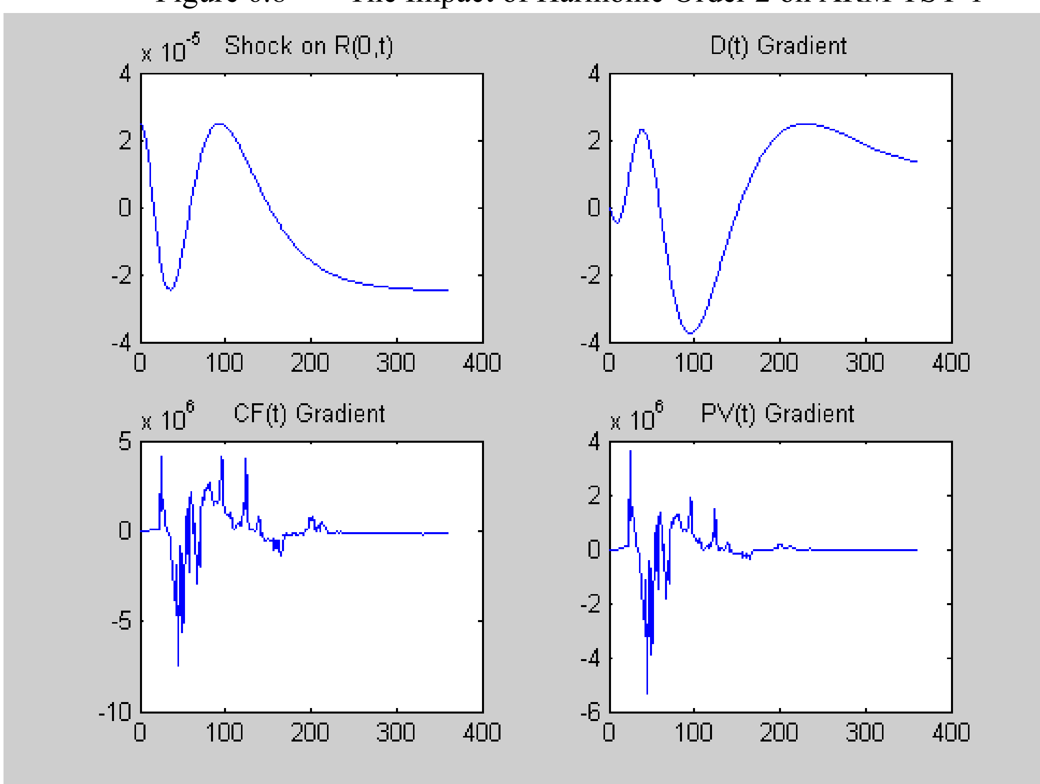


Figure 6.9 The Impact of Harmonic Order 3 on ARM TSY 1

6.3 Potential New ARM Product

Duration is used to measure the interest risk of a fixed income security. The higher the duration is, the more interest risk that security bears. From the investor's perspective, she will benefit if interest rates fall, and suffer if interest rates climb, if the security is non-callable (no prepayment option). From the mortgage borrower's point of view, he will exercise his prepayment option if interest rates drop, and thus reduce the benefit for the investor. He will be able to lock in the low mortgage rate (for FRM), in case interest rates climb, and thus hurt the investor more. However, for the ARM borrower, he benefits from the rate drop, so he does not prepay like the FRM; thus the MBS investor will also benefit. And he also pays the high coupon rate when interest rates increase, and the ARM MBS investor will not suffer like the FRM MBS investors. From this perspective, the ARM should have a lower duration compared to FRM.

ARM borrower's coupon rate fluctuates with the current interest rate, which is correlated with the prevailing mortgage rate. Because of this, she will have less incentive to prepay when interest rate drops. So the prepayment option value for a FRM borrower will be larger than that of an ARM borrower. This is compatible with the market, where FRM mortgages are sold with the highest rate (borrower pays for the valuable prepayment option), and ARM, that adjust most frequently are offered with the lowest rate.

In option theory, we know that option value generally increases as the volatility of underlying asset increases. However, from the above analysis, we also know that the option value for a FRM is generally greater than for an ARM, while an ARM bears a more volatile coupon rate than a FRM. This looks like a contradiction to the option-volatility relationship. In fact, it's not, because the underlying asset of a prepayment option is not its coupon rate, but the difference between the coupon rate and the prevailing mortgage rate. In most cases, the more volatile the coupon rate is, the less the difference will be, and the less valuable the option will be. However, the borrower does not like the volatility, which put her at risk when interest rate jumps. The investor, on the other hand, does not like the prepayment, which reduce her investment value. It seems that no product can both reduce the coupon rate volatility and the prepayment option at the same time. Is this true? We will see that we can achieve both goals in a potential new ARM product.

We have mentioned that the underlying asset for the prepayment function is the spread between the coupon rate and the prevailing mortgage rate. So an ARM bearing a volatile index does not necessarily indicate a less volatile spread. From historical data, we know that 10-Year Treasury rate is highly correlated with conventional (FRM30) mortgage rate. Figure 6.10 shows the two rates for last 30 years. And the correlation calculated is 0.978876. Figure 6.10 also shows the 10-Year Treasury Rate and the 1-Year Treasury rate, which is the most commonly used index in ARM. As we can see, the 1-Year Treasury rate is more volatile than the 10-Year Treasury Rate. The calculated volatility is 2.7890 for the 1-Year Treasury rate, and 2.6309 for the 10-Year Treasury rate. Obviously 10-Year Treasury rate has a lower volatility and also a lower volatile spread. The spread between conventional mortgage rate (FRM30) and 10-Year Treasury rate and the spread between conventional mortgage rate (FRM30) and 1-Year Treasury rate are also shown, which indicates that ARM with index of 1-Year Treasury rate has a more volatile spread.

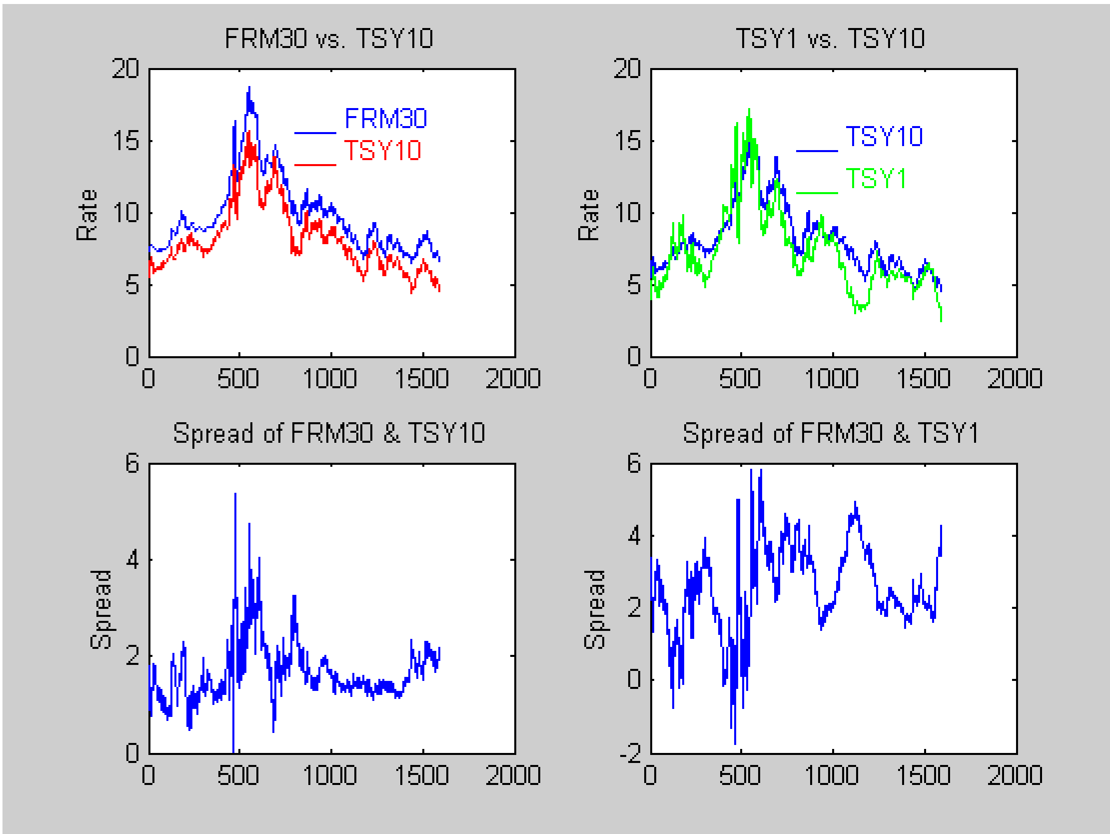


Figure 6.10 10-Year T Rate, 1-Year T Rate, and mortgage rate

Thus if we construct an ARM with index of 10-Year Treasury rate, and reset more frequently, we could expect a less duration. So we construct such an ARM with the adjustment period of 12 months. This ARM does not exist at present; it is for illustration purpose only. We then got the duration as following:

Harmonic Order	0	1	2	3	Vega
ARM TSY 10	-1.2741	-4.0635	3.1819	-0.4894	-1.8855

Figure 6.11 shows the new ARM product's duration against duration of other mortgage products we calculated earlier. We compare this set of durations with table 6.1, and we can see that this product has the least durations for harmonic function order 0 and 3, as well as for vega. The durations for harmonic function order 1 and 2 are not very high. And we know that generally when there is a shock on the term structure, the biggest magnitude would be that of the first order harmonic function, and volatility is also a big impact. So this product would actually have the least percentage change during a common term structure shift, which satisfies the needs of investors.

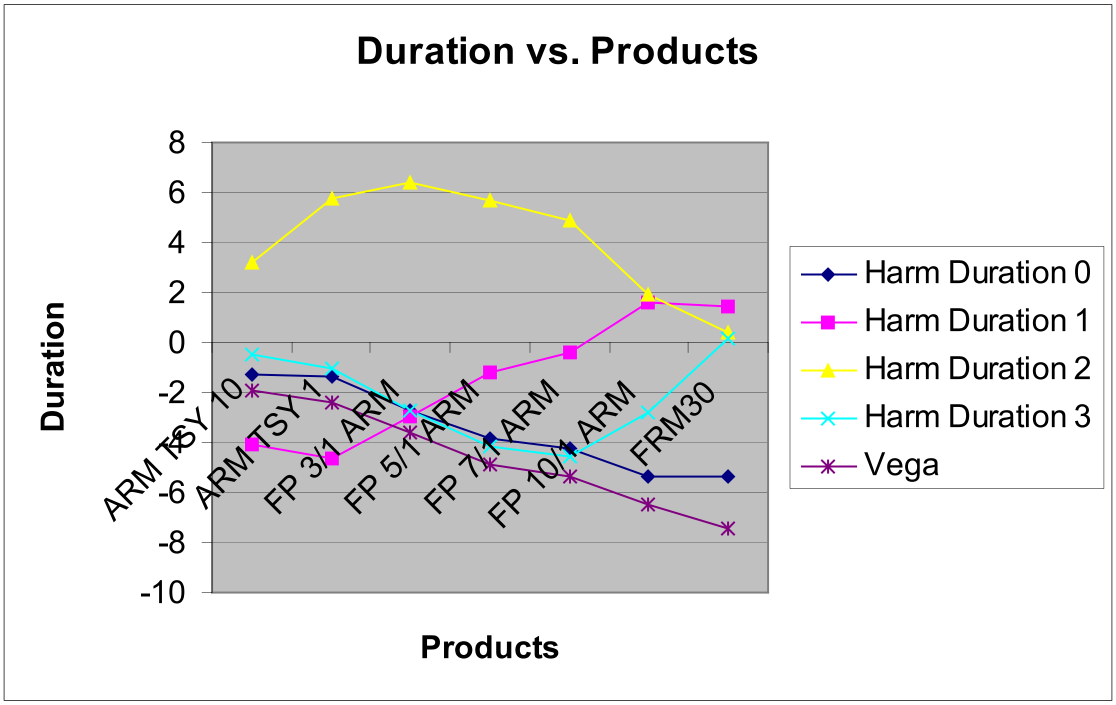


Figure 6.11 New ARM TSY 10 Durations

So we could predict that if there exist such a mortgage, it would have the least refinancing incentive, which would be a better product to suit investors' needs, and it will also have a less volatile index, which suits borrowers' needs.

7 Conclusion

This paper applies perturbation analysis (PA) method to estimate MBS sensitivities. The sensitivity estimators include most interest risk measures like duration (equivalent to delta), convexity (equivalent to gamma), and vega. MBS products covered includes fixed rate mortgages (FRMs) and adjustable rate mortgages (ARMs).

We first derive a general framework to derive the PA estimators of MBS, without restriction to MBS type, interest rate model, or prepayment model. Then we apply the PA estimator to both FRM and ARM products, in the setup of a one-factor Hull-White model and a commonly used prepayment model. We compare the PA estimators with finite difference (FD) estimators, and find that PA method can achieve at least the same accuracy as FD method, with a much lower computational cost. In the case we presented, the computational time is reduced by 95.7%, while the memory requirement increases only by a factor of 3, which can be handled by current computer technology with ease. Then we analyze the results of PA estimated sensitivity measures for various MBS products. We justify why and how different term structure shock would affect FRM and ARM differently. Based these analysis, we propose a potential new ARM product which could benefit both the MBS investor and the mortgage borrower.

Future research includes applying this method to other MBS-like securities, since the PA method proposed in section 3 is a very general framework. These include other asset-backed securities, e.g. securities backed by student loans, car loans, credit card receivables. It is pretty straightforward to expand this framework to those securities, since all that is required is to apply a specific interest rate model and prepayment model.

Another area for further research is to incorporate more complicated prepayment and/or default models into the MBS pricing scheme. For MBS investors, the major concerns are price sensitivities to interest changes, which we have covered in detail. However, the MBS guarantor/insurer and issuer might have other concerns, e.g., how will the interest rate change affect the default behavior of the mortgage borrowers? Our framework would be able to serve this purpose as well. By applying the default model that same way as we apply a prepayment model, the default cash flow will take the place of payment cash flow, so the default cost sensitivities could be easily estimated.

References

1. Åkesson, F., and Lehoczky, J. P., Path Generation for Quasi-Monte Carlo Simulation of Mortgage-Backed Securities, *Management Science*, Vol. 46, No. 9, September 2000.

2. Caflisch, R.E., Morokoff, W. and Owen, A.B., Valuation of Mortgage Backed Securities using Brownian Bridges to Reduce Effective Dimension*, Journal of Computational Finance*, Vol. 1, No. 1, 1997.

3. Chiang, R., Gosnell, T. F., Heuson, A. J., Evaluating the Interest-Rate Risk of Adjustable-Rate Mortgage Loans, *Journal of Real Estate Research*, Vol. 13, November 1997.

4. Fabozzi, F. J., Fixed Income Mathematics, Irwin, 1993.

5. Fu, M. C., Hu, J. Q., Sensitivity Analysis for Monte Carlo Simulation of Option Pricing, *Probability in the Engineering and Information Sciences*, Vol.9, No.3, 1995.

6. Fu, M. C., Madan, D.P., and Wang, T., Pricing Continuous Asian Options: A Comparison of Monte Carlo and Laplace Transform Inversion Methods, *Journal of Computational Finance*, Vol.2, No.2, 1999.

7. Fu, M. C., Laprise, S.B., Madan, D.B., Su, Y., and Wu, R., Pricing American Options: A Comparison of Monte Carlo Simulation Approaches, *Journal of Computational Finance*, Vol.4, No.3, Spring 2001.

8. Golub, B. W., Toward a New Approach to Measuring Mortgage Duration, Chapter 25 in Frank J. Fabozzi, *The Handbook of Mortgage-Backed Securities*, McGraw-Hill, 2001.

9. Broadie, M., and Glasserman, P., Estimating Security Price Derivatives using Simulation, *Management Science*, Vol. 42, 269-285, 1996.

10. Glasserman, P., Heidelberger, P., and Shahabuddin, P., Importance Sampling in the Heath-Jarrow-Morton Framework, *the Journal of Derivatives*, Fall 1999.

11. Glasserman, P., and Zhao, X., Fast Greeks in Forward LIBOR Model, working paper, Graduate School of Business, Columbia University, January 1999.

12. Heath, D., Jarrow, R. and Morton, A., Bond Princing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation, *Econometrica* Vol. 60, No.1, 77–105, 1992

13. Hull, J., and White, A., Pricing interest rate derivative securities, *Rev. Financial Studies,* Vol. 3, 573–592, 1990.

14. Jamshidian, F., Libor and swap market models and measures, *Finance and Stochastics*, Vol.1, 293–330, 1997.

15. Kau, J. B., Keenan, D. C., The Theory of Housing and Interest Rates (in Competitive Paper Winners), *Journal of Financial and Quantitative Analysis*, Vol. 15, No. 4, Proceedings of 15th Annual Conference of the Western Finance Association, June 19-21, 1980, San Diego, California. November 1980.

16. Kau, J. B., Keenan, D. C., Muller, W. J. III, Epperson, J. F., The Valuation of Adjustable Rate Mortgage, *Management Science*, Vol. 36, No. 12, December, 1990.

17. Kau, J. B., Keenan, D. C., Muller, W. J. III, Epperson, J. F., A Generalized Valuation Model for Fixed-Rate Residential Mortgages, *Journal of Money, Credit and Banking*, Vol. 24, No. 3, August, 1992.

18. Kau, J. B., Keenan, D. C., Muller, W. J. III, Epperson, J. F., Option Theory and Floating-Rate Securities with a Comparison of Adjustable- and Fixed-Rate Mortgages, *Journal of Business*, Vol. 66, No. 4, October 1993.

19. Kpprasch, R., Understanding Duration and Convexity, Chapter 5 in Frank J. Fabozzi and Irving Pollack, *The Handbook of Fixed Income Securities*, Dow Jones-Irwin, 1987.

20. Robert A. Ott, Jr., The Duration of an Adjustable-Rate Mortgage and the Impact of the Index, *Journal of Finance*, Vol. 41, No. 4. (Sep., 1986), pp. 923-933.

21. Stanton, R., Wallace, N., Anatomy of an ARM: The Interest Rate Risk of Adjustable Rate Mortgage, *Journal of Real Estate Finance and Economics,* Vol.19, 1999

22. Suri, R., Perturbation Analysis: The State of the Art and Research Issues Explained via the GI/G/1 Queue, *Proceedings of the IEEE*, Vol. 77, No.1, January 1989.

23. Wu, R., Fu, M. C., Optimal Exercise Policies and Simulation-based Valuation for American-Asian Options," working paper, Robert H. Smith School of Business, University of Maryland, 2001.