y'' (except at $t = 0$). One additional differentiation to get y''' guarantees full column rank equal to three for the coefficient matrix of vector x. For the example, this rank condition is equivalent to the 1-full condition on the Jacobian $\mathcal{O}_{j,k}$ of (11) for the time-varying linearization. (We may take $j = 1$ and $k = 3$.) Thus we have smooth observability on the interval $[0, T]$ for the time-varying linearized system. Then, provided A6) holds (perhaps after additional differentiation, see Theorem 3), Theorem 2 applies and guarantees smooth observability of the nonlinear system on $[0, T]$ in a neighborhood of $x = 0$. □

Based on Theorem 2, many special results may now be deduced, involving conditions on a linearization (8), (9), or on submatrices of the full Jacobian (7), that guarantee smooth observability in $\mathcal{W}$ about a trajectory, in particular for semi-explicit systems where (7) simplifies somewhat. For example, the analysis in [13] shows that the hypotheses in Theorem 2 will hold for Hessenberg DAE systems supplied with an effective output function, that is, an output for which $\mathcal{O}_{j,k}$ in (11) is 1-full with respect to x on $\mathcal{I}$.

Finally, the special u-dependence dealt with in (1) is not a real restriction: it is still possible to apply the condition (6) to (7), allowing Theorems 2 and 3 of Section IV to be extended to the case of a DAE with the more general u-dependence, $F(x, x', t, u) = 0$, in place of (1). The linearization is still given by (8), (9), but in this case the Jacobian (7) also depends on the variables $u^{(i)}$ for $0 \leq i \leq j$.

V. Conclusion

We have established some sufficient conditions for local observability of nonlinear DAE systems near a known trajectory. We indicated by an example the importance of these results in overcoming the inadequacy of time-invariant linearizations for determining observability of nonlinear DAEs. Our sufficient conditions for smooth observability are verifiable and are strong enough to guarantee that the full system observability Jacobian satisfies the smooth observability condition in [13]. The results of this note can provide a basis for the future development of algorithms for determining observability of nonlinear DAE systems.

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On the Convergence Rate of Ordinal Comparisons of Random Variables

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***Abstract*—The asymptotic exponential convergence rate of ordinal comparisons follows from well-known results in large deviations theory, where the critical condition is the existence of a finite moment generating function. In this note, we show that this is both a necessary and sufficient condition, and also show how one can recover the exponential convergence rate in cases where the moment generating function is not finite. In particular, by working with appropriately truncated versions of the original random variables, the exponential convergence rate can be recovered.**

***Index Terms*—Large deviations, ordinal optimization, stochastic simulation.**

I. Introduction

Estimation of the mean of a random variable by Monte Carlo simulation has convergence rate $1/\sqrt{n}$, where n is the number of samples taken. On the other hand, often one is not so interested in the actual value of the mean in the absolute sense as in its value relative to other means, e.g., if one is comparing various designs in order to select the best one. Determination of the best is carried out by using some surrogate metric for performance evaluation. In using the sample mean to decide the best design, the probability of correctly selecting the best design often exhibits an asymptotically exponential convergence rate.

Specifically, our problem setting is as follows. Among m designs, we wish to determine the one with minimum mean; doing so is called "correct selection." Without loss of generality, suppose

$$EX_1 < EX_2 < \cdots < EX_m.$$

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Define

$$I_n = \arg \min_{i\in\{1,\ldots,m\}} \left\{ \frac{\sum_{j=1}^{n} X_{i,j}}{n} \right\}$$

where $X_{i,j}$ is the jth sample of X_i. I_n is the estimated best design(s) based on sample means, so "correct selection" is defined by $I_n = \{1\}$. Clearly as $n \to \infty$, $P(I_n \neq \{1\}) \to 0$. By applying standard results in large deviations theory (e.g., [3], [7]), [1], [2], and [8] showed that when using the sample mean to select the best, this probability goes to 0 exponentially fast under appropriate conditions on the underlying random variables (or stochastic processes). The idea of exploiting this for optimization via simulation in an approach coined "ordinal optimization" was introduced by Ho *et al.* [6]; see also [5] for a recent survey of this line of research and [4] for related ideas on ranking and selection and multiple comparison procedures.

In our particular setting, [8] showed that a sufficient condition for the asymptotic exponential rate is the existence of a finite moment generating function for the underlying random variables. First, we establish that a finite moment generating function is not only sufficient but also necessary in the appropriate interpretation (Lemma 2.1 and Theorem 2.1). Unfortunately, there are many common families of distributions that do not possess a finite moment generating function, including the lognormal and certain classes of gamma distributions. Thus the second purpose of this note is to recover the exponential convergence rate in these situations. Simply truncating the random variables themselves and considering the new problem is one solution, but this is less than satisfactory, since the optimal solution to the new problem may not be optimal for the original problem of interest. In our approach, we also consider truncated random variables, but the truncation is chosen in such a way that connects conclusions in the truncated setting back to the original setting. If the truncation constant is chosen appropriately large enough, exponential convergence for correct selection in the original problem is guaranteed (Theorem 2.2). However, since determination of the appropriate value of the truncation constant can be a nontrivial problem in itself, our last result (Theorem 2.3) provides an alternative method that incorporates a means for determining whether or not the truncation constant is sufficiently large.

II. Main Results

Before giving our main results of this section, we introduce necessary and sufficient conditions for the large deviation principle.

Lemma 2.1: Consider a sequence of i.i.d. random variables $\{X_n, n \geq 1\}$ with moment generating function $M(\lambda) = E[\exp(\lambda X_1)]$. Let $S_n = \sum_{i=1}^n X_i$. If $M(\lambda)$ exists in a neighborhood $(-\varepsilon, \varepsilon)$ of $\lambda = 0$ for some $\varepsilon > 0$, then

$$P(S_n/n \geq x) \leq \exp\left(-n\Lambda_+^*(x)\right), \qquad \forall\, x \geq E[X_1] \tag{1}$$

and

$$P(S_n/n \leq x) \leq \exp\left(-n\Lambda_-^*(x)\right), \qquad \forall\, x \leq E[X_1] \tag{2}$$

where

$$\Lambda_+^*(x) = \sup_{\lambda\geq 0}\left(\lambda x - \log M(\lambda)\right)$$

and

$$\Lambda_-^*(x) = \sup_{\lambda\leq 0}\left(\lambda x - \log M(\lambda)\right).$$

Conversely, if $E[|X_1|] < \infty$ and there exist an integer $n_0 > 0$ and constants $\alpha > 0$ and $C_0 > E[X_1]$ such that

$$P(S_n/n \geq C_0) \leq \exp(-n\alpha), \qquad \forall\, n \geq n_0 \tag{3}$$

then the moment generating function $M(\lambda) = E[\exp(\lambda X_1)]$ exists in an interval $[0, \varepsilon)$ for some $\varepsilon > 0$; and if there exist an integer $n_0' > 0$ and constants $\alpha' > 0$ and $C_0' < E[X_1]$ such that

$$P(S_n/n \leq C_0') \leq \exp(-n\alpha'), \qquad \forall\, n \geq n_0'$$

then the moment generating function $M(\lambda) = E[\exp(\lambda X_1)]$ exists in an interval $(-\varepsilon, 0]$ for some $\varepsilon > 0$.

Remark: Well-known distributions that do not possess a finite moment generating function include the lognormal distribution and certain gamma distributions.

Proof: The sufficiency portion of this result can be found in [1, Th. 5.1], [2, Rem. 3.1], and [8]. We establish the reverse direction. Suppose (3) is true. Without loss of generality, assume that $E[X_1] = 0$. Then by the strong law of large numbers, there exists $m_0 > n_0$ such that

$$P\left(\frac{\sum_{i=n_0+1}^{n} X_i}{n - n_0} \geq -\frac{C_0}{2} \right) \geq \frac{1}{2}, \qquad \forall\, n \geq m_0.$$

Thus, for any integer $m \geq m_0' = \max(n_0 + 1, m_0)$

$$\begin{aligned} &\frac{1}{2} P\left(\sum_{i=1}^{n_0} X_i \geq \frac{3mC_0}{2} \right) \\ &\quad \leq P\left(\sum_{i=1}^{n_0} X_i \geq \frac{3mC_0}{2},\ \frac{\sum_{i=n_0+1}^{m} X_i}{m - n_0} \geq -\frac{C_0}{2} \right) \\ &\quad \leq P\left(\frac{\sum_{i=1}^{m} X_i}{m} \geq C_0 \right) \leq \exp(-m\alpha) \end{aligned}$$

and, therefore

$$P\left(\sum_{i=1}^{n_0} X_i \geq \frac{3mC_0}{2} \right) \leq 2\exp(-m\alpha).$$

Consequently, for any $\lambda \in [0, (2\alpha/3C_0))$

$$\begin{aligned} &\sum_{m=m_0'}^{\infty} \exp\left(\frac{3m\lambda C_0}{2}\right) P\left(\sum_{i=1}^{n_0} X_i \geq \frac{3mC_0}{2} \right) \\ &\qquad \leq 2 \sum_{m=m_0'}^{\infty} \exp\left(-m\left(\alpha - \frac{3\lambda C_0}{2}\right)\right) < \infty \end{aligned}$$

implying

$$E\left[\exp\left(\lambda \sum_{i=1}^{n_0} X_i \right) \right] = M_0 < \infty$$

for any $\lambda \in [0, (2\alpha/3C_0))$.

Now, we turn to proving the moment generating function $M(\lambda) = E[\exp(\lambda X_1)]$ exists in the interval $\lambda \in [0,\ (2\alpha/3C_0))$. Note that for any $M > 0$, $E[\exp(\lambda(X_1 \wedge M))]$ exists when $\lambda \geq 0$ (where "$\wedge$" denotes the minimum operator). Hence, for $\lambda \in [0,\ (2\alpha/3C_0))$

$$\begin{aligned}(E[\exp(\lambda(X_1 \wedge M))])^{n_0} &= E\left[\exp\left(\lambda \sum_{i=1}^{n_0} (X_i \wedge M)\right)\right] \\ &\leq E\left[\exp\left(\lambda \sum_{i=1}^{n_0} X_i\right)\right] = M_0 < \infty\end{aligned}$$

and

$$E[\exp(\lambda(X_1 \wedge M))] \leq M_0^{1/n_0} < \infty.$$

Consequently, by letting M tend to infinity and by the monotone convergence theorem

$$E[\exp(\lambda X_1)] \leq M_0^{1/n_0} < \infty$$

for $\lambda \in [0,\ (2\alpha/3C_0))$.

The second case of the necessary condition can be proved in an analogous manner. ■

Hence, if the random variables have finite moment generating functions, then the random variable with minimum expectation can be sorted out based on the sample means at an asymptotically exponential convergence rate, as presented in the next result.

Theorem 2.1: Suppose there exists an $\varepsilon > 0$ such that for all $\lambda \in (-\varepsilon,\ \varepsilon)$

$$E[\exp(\lambda X_k)] < \infty, \qquad k = 1, \ldots, m. \tag{4}$$

Then

$$P(I_n \neq \{1\}) \leq \sum_{i=2}^{m} \exp(-n\Lambda_i^*)$$

where

$$\Lambda_i^* = \sup_{\lambda \geq 0} (-\log E[\exp(\lambda(X_1 - X_i))]) > 0.$$

Proof: Suppose $I_n \neq \{1\}$. Then there exists an $i \in \{2, \ldots, m\}$ such that

$$\frac{\sum_{j=1}^{n} X_{i,j}}{n} < \frac{\sum_{j=1}^{n} X_{1,j}}{n}.$$

Thus

$$\begin{aligned}P(I_n \neq \{1\}) &\leq \sum_{i=2}^{m} P\left(\frac{\sum_{j=1}^{n} X_{i,j}}{n} < \frac{\sum_{j=1}^{n} X_{1,j}}{n}\right) \\ &= \sum_{i=2}^{m} P\left(\frac{\sum_{j=1}^{n} (X_{1,j} - X_{i,j})}{n} > 0\right).\end{aligned}$$

Note that $E[X_1 - X_i] < 0$ for any $i \neq 1$. Hence, by taking $x = 0$ in Lemma 2.1 and by (1), we arrive at the conclusion of this theorem. ■

Since the assumption of a finite moment generating function is both necessary and sufficient, it would appear that one cannot obtain exponential convergence for cases involving the lognormal distribution. Fortunately, by working with appropriately truncated versions of the random variables, the asymptotically exponential convergence rate can be recovered in an indirect manner. The idea is based on the following obvious fact that if X and Y are nonnegative and integrable random variables and $EX < EY$, then there exists a constant M large enough such that

$$E(X \wedge M) < E(Y \wedge M).$$

Define

$$I_n^M = \arg \min_{i \in \{1, \ldots, m\}} \left\{\frac{\sum_{j=1}^{n} (X_{i,j} \wedge M)}{n}\right\}$$

i.e., I_n^M is the estimated best design(s) based on the sample means of the truncated random variables. The next result says that the optimal design for the original (untruncated) problem can be sorted out at an exponential convergence rate by simulating the corresponding truncated random variable.

Theorem 2.2: Suppose $X_k \geq 0$, $k = 1, \ldots, m$, and M satisfies

$$E[X_i] - E[X_1] > E[X_i \mathbf{1}(X_i > M)], \qquad i = 2, \ldots, m. \tag{5}$$

Then

$$P\left(I_n^M \neq \{1\}\right) \leq \sum_{i=2}^{m} \exp\left(-n\Lambda_i^{M*}\right)$$

where

$$\Lambda_i^{M*} = \sup_{\lambda \geq 0} (-\log E[\exp(\lambda(X_1 \wedge M - X_i \wedge M))]) > 0.$$

Remark: For the sake of simplicity, we only consider nonnegative random variables in this result. Extension to general random variables can be handled using two-sided truncation via $(X_k \wedge M) \vee M'$, $k = 1, \ldots, m$, where $M > 0 > M'$ and "$\vee$" is the maximum operator. Theorem 2.3 can also be handled in this way.

Proof: Since $X_i \geq 0$, $E[X_1 \wedge M] \leq E[X_1]$, and $E[X_i \wedge M] \geq E[X_i] - E[X_i \mathbf{1}(X_i > M)] > 0$, then if M satisfies (5), we have for each $i = 2, \ldots, m$,

$$\begin{aligned}&E[X_i \wedge M] - E[X_1 \wedge M] \\ &\qquad \geq E[X_i] - E[X_1] - E[X_i \mathbf{1}(X_i > M)] > 0,\end{aligned}$$

implying that design 1 is still an optimal solution for the truncated process in terms of expectation.

Hence, by Theorem 2.1

$$P\left(I_n^M \neq \{1\}\right) \leq \sum_{i=2}^{m} \exp\left(-n\Lambda_i^{M*}\right)$$

since $E[\exp(\lambda(X_k \wedge M))] < \infty$, $k = 1, \ldots, m$, completing the proof. ■

The above result still suffers from a drawback, i.e., it may be difficult to determine a constant M satisfying (5). The last result in this note provides the basis for an alternative method that incorporates determination of an appropriate truncation constant. To this end, we introduce some notation. Given $i \in \{1, 2, \ldots, m\}$, let

$$I_{n,i}^M = \arg\min \left\{\min_{l \in \{1, \ldots, m\},\, l \neq i} \left\{\frac{\sum_{j=1}^{n} X_{l,j}}{n}\right\},\ \frac{\sum_{j=1}^{n} (X_{i,j} \wedge M)}{n}\right\}$$

i.e., $I^M_{n,i}$ is the set containing the estimated best design(s) based on sample means from the truncated random variable $X_i \wedge M$ and from the remaining untruncated random variables $X_l, l \neq i$.

Theorem 2.3: Suppose $X_k \geq 0, k = 1, \ldots, m$.

i) If M satisfies

$$E[X_i \wedge M] > E[X_1], \qquad i = 2, \ldots, m$$

i.e., after truncation, the design with minimal mean is the same one as for the untruncated case, then

$$P\left(I^M_{n,1} \neq \{1\}\right) \leq \sum_{j=2}^{m} \exp\left(-n\tilde{\Lambda}_j^{M*}\right) \tag{6}$$

where

$$\tilde{\Lambda}_j^{M*} = \sup_{\lambda \geq 0} \left(-\log E\left[\exp\left(\lambda(X_1 \wedge M - X_j)\right)\right]\right) > 0$$

and for $i \neq 1$

$$\lim_{n\to\infty} P\left(I^M_{n,i} \neq \{i\}\right) = 1. \tag{7}$$

ii) If there exists a subset $\mathcal{I} \subseteq \{2, \ldots, m\}$ such that M satisfies

$$E[X_i \wedge M] < E[X_1], \quad i \in \mathcal{I}$$

i.e., after truncation, the minimum mean design is different, then for $i \in \mathcal{I} \cup \{1\}$

$$P\left(I^M_{n,i} \neq \{i\}\right) \leq \sum_{j=1,\, j\neq i}^{m} \exp\left(-n\tilde{\Lambda}_{j,i}^{M*}\right) \tag{8}$$

where

$$\tilde{\Lambda}_{j,i}^{M*} = \sup_{\lambda \geq 0} \left(-\log E\left[\exp\left(\lambda(X_i \wedge M - X_j)\right)\right]\right) > 0$$

and for $i \notin \mathcal{I} \cup \{1\}$

$$\lim_{n\to\infty} P\left(I^M_{n,i} \neq \{i\}\right) = 1. \tag{9}$$

iii) Suppose $E[X_1^2] < \infty$. If M satisfies

$$E[X_{i_0} \wedge M] = E[X_1], \qquad i_0 \neq 1,$$

i.e., after truncation, some other design has a mean equal to the untruncated minimum mean design, then

$$P\left(I^M_{n,1} \neq \{1\}\right) \leq \sum_{j=2}^{m} \exp\left(-n\tilde{\Lambda}_j^{M*}\right); \tag{10}$$

$$\lim_{n\to\infty} P\left(I^M_{n,i} \neq \{i\}\right) = 1, \qquad i \in \{2, \ldots, m\} - \{i_0\} \tag{11}$$

$$P\left(I^M_{n,i_0} \not\subseteq \{1, i_0\}\right) \leq \sum_{j=1,\, j\neq\{1, i_0\}}^{m} \exp\left(-n\tilde{\Lambda}_{j,i_0}^{M*}\right), \tag{12}$$

and, moreover

$$\lim_{n\to\infty} P\left(i_0 \notin I^M_{n,i_0}\right) \neq 1. \tag{13}$$

Proof:

i) Since

$$E[X_1 \wedge M] \leq E[X_1] < E[X_2] < \cdots < E[X_m]$$

(6) can be proved as in Theorem 2.2.

For $i \neq 1$, we have

$$P\left(I^M_{n,i} \neq \{i\}\right) \geq P\left(\frac{\sum_{j=1}^{n}(X_{i,j} \wedge M)}{n} > \frac{\sum_{j=1}^{n} X_{1,j}}{n}\right)$$

and, therefore, by the law of large numbers

$$\lim_{n\to\infty} P\left(\frac{\sum_{j=1}^{n}(X_{i,j} \wedge M)}{n} > \frac{\sum_{j=1}^{n} X_{1,j}}{n}\right) = 1$$

since $E[X_i \wedge M] > E[X_1]$, implying (7).

ii) The proofs for (8) and (9) are analogous to those for (6) and (7), respectively.

iii) Again, the proofs for (10) and (11) are analogous to those for (6) and (7), respectively, so it remains to show (12) and (13). For (12), if $I^M_{n,i_0} \not\subseteq \{1, i_0\}$, then there exists $i \in \{2, \ldots, m\} - \{i_0\}$ such that

$$\frac{\sum_{j=1}^{n} X_{i,j}}{n} \leq \frac{\sum_{j=1}^{n}(X_{i_0,j} \wedge M)}{n}.$$

Thus, (12) follows from the proof of (6). Now, we turn to proving (13). Suppose

$$\lim_{n\to\infty} P\left(\{i_0\} \notin I^M_{n,i_0}\right) = 1.$$

Then, by (12)

$$\lim_{n\to\infty} P\left(I^M_{n,i_0} = \{1\}\right) = 1$$

and, therefore

$$\lim_{n\to\infty} P\left(\frac{\sum_{j=1}^{n} X_{1,j}}{n} < \frac{\sum_{j=1}^{n}(X_{i_0,j} \wedge M)}{n}\right) = 1. \tag{14}$$

However, by the Central Limit Theorem

$$\begin{aligned} &\lim_{n\to\infty} P\left(\frac{\sum_{j=1}^{n} X_{1,j}}{n} < \frac{\sum_{j=1}^{n}(X_{i_0,j} \wedge M)}{n}\right) \\ &= \lim_{n\to\infty} P\left(\frac{\sum_{j=1}^{n}\left[X_{1,j} - (X_{i_0,j} \wedge M)\right)}{\sqrt{n}\sigma\left(X_{1,j} - (X_{i_0,j} \wedge M)\right)} < 0\right) \\ &= \Phi(0) = \frac{1}{2} \end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal c.d.f. As this contradicts the previous result (14), the proof of (13) is complete. ■

Based on the above result, an algorithm could proceed as follows. Using a candidate truncation constant M, for each design ($i = 1, \ldots, m$), compare the mean of the truncated design with the untruncated means of all other designs to determine $I^M_{n,i}$. If M is appropriately chosen, then $I^M_{n,i} \to \{i\}$ for *only* the optimum ($i = 1$ in our setting), and will do so with an asymptotically exponential convergence rate. Otherwise, either $I^M_{n,i} \to \{i\}$ for more than one i and/or

$I_{n,i}^{M}$ will converge to a set that is not a singleton (the latter would be the case where the truncation constant causes a tie after truncation). In this case, one would increase M and try again. Note, however, that larger values of M result in slower exponential convergence, the limit being nonexponential convergence. Thus, there is thus a trade-off between getting better exponential convergence (smaller M) and converging to correct selection (high enough M). Too small an M may lead to convergence to the wrong design, whereas too large an M may lead to slow convergence rate. In terms of calculations, for m designs, $2m$ sample quantities must be calculated, as opposed to m for either the truncated or "purely" untruncated cases considered earlier. Note, however, that if the truncated samples are obtained by sampling from the original distribution and then truncating, no additional simulation is required in these procedures, i.e., all $2m$ quantities can be generated from m raw (untruncated) samples.

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Stabilization of Discrete Time Linear Systems by Static Output Feedback

Germain Garcia, Bernard Pradin, and Fanyou Zeng

***Abstract*—This note proposes a systematic approach for the static output feedback control design for discrete time linear systems. It is shown that if the open loop system satisfies some particular structural conditions, a static output feedback gain can be calculated easily, using a formula only involving the original system matrices. Among the conditions the system has to satisfy, the stronger relies on a minimum phase argument. Square and nonsquare systems are considered.**

***Index Terms*—Discrete systems, linear quadratic optimal control, Riccati equation, static output feedback.**

I. INTRODUCTION

The static output feedback (SOF) problem has been investigated by many people and the literature concerning this topic is immense. In practice, it is not always possible to have full access to the state vector and only a partial information through a measured output is available. This explains why this problem has challenged many researchers in control theory.

Give a complete historical bibliography is not easy because there exist various unconnected approaches. However, among the proposed results, we can distinguish solvability conditions expressed from structural properties of the open loop system [1], [2], constructive approaches based on the resolution of Riccati equations (inverse linear quadratic problems) [3]–[6], or optimization techniques (usually convex and conservative) [7]–[13] , pole or eigenstructure assignment techniques [14]–[20]. For more details, the reader is referred to some good surveys giving the state of the art like [21]–[23].

Recently, an important step was done concerning the complexity of the static output feedback problem which was found to be NP complete [22]. This means that if a general algorithm for solving the SOF is derived, it is an exponential time algorithm. Despite this important limitation for practice and due to the genericity of some SOF solvability conditions, it is pointed out in [23] that restricting the class of systems under consideration can be advantageous to derive algorithms or methods with a reduced computational cost.

This note proposes a systematic approach for the SOF control design for discrete time linear systems. It is shown that if the open loop system satisfies some particular structural conditions, a static output feedback gain can be calculated easily, using a formula only involving the original system matrices. Among the conditions the system has to satisfy, the stronger relies on a minimum phase argument. Square and non square systems are considered. The main results are derived invoking the limiting behavior of the optimal linear quadratic regulator for discrete time systems and then the proposed static output control is optimal for a particular quadratic criterion. Some other implications are presented. This note is organized as follows: the next section presents some fundamental properties of optimal linear quadratic regulator for discrete time systems used extensively in this note. Section III gives the

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