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Pricing Continuous Asian Options: A Comparison of Monte Carlo and Laplace Transform Inversion Methods

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Abstract

In this paper, we investigate two numerical methods for pricing Asian options: Laplace transform inversion and Monte Carlo simulation. In attempting to numerically invert the Laplace transform of the Asian call option that has been derived previously in the literature, we point out some of the potential difficulties inherent in this approach. We investigate the effectiveness of two easy-to-implement algorithms, which not only provide a cross-check for accuracy, but also demonstrate superior precision to two alternatives proposed in the literature for the Asian pricing problem. We then extend the theory of Laplace transforms for this problem by deriving the double Laplace transform of the continuous arithmetic Asian option in both its strike and maturity.

We contrast the numerical inversion approach with Monte Carlo simulation, one of the most widely used techniques, especially by practitioners, for the valuation of derivative securities. For the Asian option pricing problem, we show that this approach will be effective for cases when numerical inversion is likely to be problematic. We then investigate ways to improve the precision of the simulation estimates through the judicious use of control variates. In particular, in the problem of correcting the discretization bias inherent in simulation when pricing continuous-time contracts, we find that the use of suitably biased control variates can be beneficial. This approach is also compared with the use of Richardson extrapolation.

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The goal of this paper is to compare and contrast the effectiveness of Monte Carlo simulation with numerical inversion methodologies in the valuation of Asian options. Asian options pay the excess over strike of the average of the asset price over an interval of time. If the average is computed using a finite sample of asset price observations taken at a set of regularly spaced time points we have a discrete Asian option, as opposed to a continuous Asian option that is obtained by computing the average via the integral of the price path over an interval of time. In addition, the average itself can be either arithmetic (summed) or geometric (product). Since the geometric Asian is analytically tractable in closed form (cf. Kemna and Vorst 1990 and Vorst 1996), the primary focus of our work here is valuation of the continuous arithmetic Asian option, whose value at time $t \leq T$ is defined by

$$c_{t,T}(K) = e^{-r(T-t)} E \left[\frac{1}{T} \int_0^T S(u) du - K \right]^+ , \tag{1}$$

where $S(t)$ denotes the stock price at time t, T is the expiration date, and K is the strike price.

Although arithmetic Asian options cannot be priced in closed form, Geman and Yor (1993) derived an analytical expression for the Laplace transform in maturity for the continuous call option case when the asset price follows geometric Brownian motion, and numerical inversion of this transform was considered briefly in Geman and Eydeland (1995). For the discrete time case, no similar analytical results exist, although various approximations have been proposed, such as the one by Turnbull and Wakeman (1991) based on moment approximations; see also Curran (1992). In the absence of closed-form results, researchers in the past have employed Monte Carlo simulation methods (see Broadie and Glasserman 1996, Kemna and Vorst 1990, and Boyle 1977). Due to their dependence on the entire path of the underlying asset, Asian options appear to be particularly suited for Monte Carlo simulation.

Since both numerical inversion and Monte Carlo methods can be applied, valuation of continuous Asian options provides a good example on which to investigate, compare, and contrast the various issues that arise in their practical implementation. Monte Carlo methods can be computationally expensive without the enhancement of variance reduction techniques, and in the case of pricing the continuous Asian, one must also account for the inherent discretization bias resulting from the approximation of continuous time processes through discrete sampling (see also Broadie et al. 1999). On the other hand, numerical inversion is subject to errors introduced by approximation techniques embedded in numerical integration routines employed, and can encounter instabilities for certain parameter values.

Numerically inverting the Geman and Yor (1993) Laplace transform is a non-trivial problem, as it involves the transform parameter in the index of a Bessel function, or as an argument of a Gamma function. As we shall see, naïve implementation may even lead to reasonable looking results that are grossly inaccurate. We employ inversion algorithms proposed by Abate and Whitt (1995) based on the Euler and Post-Widder methods and compare our results with those of Geman and Eydeland (1995) and Shaw (1998). We investigate the sources of problems for numerical inversion of the transform. As a possible alternative possibility for numerical inversion, we also derive a double Laplace transform of the Asian call option price in maturity and strike and the relationship in general of this transform to the transform of the density of the underlying asset price process.

In applying simulation to the Asian option valuation problem, the choice and use of control variates plays a large role in obtaining accurate estimates, and past research has concluded that the geometric Asian call option price serves as a very effective control variate (Kemna and Vorst 1990). The other concern for the continuous Asian problem is the bias introduced by using a sum to approximate an integral. To address both of these issues simultaneously, we consider using the continuous geometric Asian option as a control variate. Since the simulation actually estimates the price of a discrete geometric Asian option, subtracting the value of a continuous geometric Asian option price leads to a biased control variate. Computational experiments indicate that the two discretization biases roughly cancel out. For comparison, we also apply Richardson extrapolation to the continuous Asian option pricing problem when the unbiased control variate is used. We then investigate the idea of using biased control variates further, by considering estimation of the geometric Asian call option itself, where closed-form analytical results can be used to evaluate the effectiveness more accurately.

In sum, our computational experiments lead to the following general findings. Inversion of the Laplace transform via numerical integration can lead to numerical problems for low volatilities and short maturities. These problems appear to be independent of the technique used for the inversion, being a result of the slowly decaying oscillatory nature of the integrand for such parameter values, although the point at which each integration routine beings to degenerate may differ. Among the numerical inversion techniques considered, the Euler method provides a high degree of accuracy for a reasonable amount of computation time. For parameter values approaching the "danger zone" alluded to above, we recommend using the Post-Widder approach as a check. If speed is of the essence, then the approximate fast Fourier inversion technique used by Geman and Eydeland (1995) is recommended. On the other hand, Monte Carlo simulation performs particularly well in the domains of low volatility and short maturity, since the stochastic variation inherent in sampling is lower in these ranges. Moreover, the Monte Carlo method inherits its primary advantage of generality, in that it can easily incorporate important factors such as stochastic volatility and stochastic interest rates without any significant increase in effort or computation time. We note that other numerical approaches not considered here include numerical solutions of the associated partial differential equation (e.g., Wilmott et al. 1993), quasi-Monte Carlo methods, and binomial methods.

The rest of the paper is organized as follows. Section 1 presents two Laplace transform inversion techniques from Abate and Whitt (1995) and applies them to the Geman and Yor (1993) Laplace transform. Numerical experiments are conducted to investigate the sensitivity and convergence of the algorithms with respect to various user-specified parameters, and to compare the two inversion algorithms with inversion results in Geman and Eydeland (1995) and Shaw (1998). Experiments investigating the accuracy of Monte Carlo simulation estimates are provided in Section 2, the focus being on the discretization bias and the use of control variates to increase the efficiency of the technique. The idea of using a biased control variate is investigated further numerically in Section 3, using the analytically tractable geometric Asian option. Section 4 derives the double Laplace transform of the Asian call option price, and Section 5 concludes.

1 Inverting the Geman and Yor (1993) Call Option Price Laplace Transform in Maturity

Geman and Yor (1993) derived the Laplace transform in maturity of the continuous Asian call option price, i.e., they showed that (1) may be expressed as

$$c_{t,T}(K) = \frac{e^{-r(T-t)}}{T} \frac{4S(t)}{\sigma^2} C^{(\nu)}(h,q), \tag{2}$$

where $\nu = 2r/\sigma^2 - 1,\ h = (\sigma^2/4)(T-t),\ q = \frac{\sigma^2}{4S(t)}(KT - ta(t))$, $a(t) = \int_0^t S(u)du/t$ is the average to time t, and the Laplace transform of $C^{(\nu)}(\cdot, q)$ in the first parameter is given by

$$\hat{C}(\lambda, q) = \int_0^\infty e^{-\lambda h} C^{(\nu)}(h, q) dh = \frac{\int_0^{1/(2q)} e^{-x} x^{(\mu-\nu)/2-2} (1-2qx)^{(\mu+\nu)/2+1} dx}{\lambda(\lambda - 2 - 2\nu)\Gamma((\mu-\nu)/2-1)}, \quad (3)$$

where $\mu = \sqrt{2\lambda + \nu^2}$.

However, naïve attempts at inverting this transform using commercially available software will lead to erroneous results such as those we obtained in applying the Laplace transform inversion routine found in Mathematica 2.0 and also in using an IMSL subroutine. Using Mathematica 3.0, however, Shaw (1998) has recently shown that the transform can be successfully inverted for some cases, so we will compare the results with those obtained by the techniques discussed here. The transform is difficult to invert due to the transform parameter in the index of a Bessel function, or as an argument of a Gamma function. We employ the methods of Euler and Post-Widder, as proposed by Abate and Whitt (1995). The Euler method applies the trapezoidal rule over intervals of an appropriate length that reduce the cosine function to alternating signs. Euler summation is employed to speed the convergence of the alternating series generated by this method. Specifically let, $\hat{f}(\lambda)$ be the transform of $f(y)$. Then defining

$$a_k(y) = \mathcal{R}e\left\{\hat{f}\left(\tfrac{A+2k\pi i}{2y}\right)\right\},$$

we define the partial sums,

$$s_n(y) = \frac{e^{A/2}}{2y}\mathcal{R}e\left\{\hat{f}\left(\tfrac{A}{2y}\right)\right\} + \frac{e^{A/2}}{y}\sum_{k=1}^{n}(-1)^k a_k(y)$$

and approximate $f(y)$ by the sum,

$$f(y) \cong \sum_{k=0}^{m}\binom{m}{k}2^{-m}s_{n+k}(y).$$

Implementation of the Euler algorithm involves the selection of three parameters: m, n, A, where increasing the values of the former two increases the accuracy of the algorithm. Shortly, we will investigate empirically the effect of these parameters on the convergence to the correct option value, but in general, the values recommended by Abate and Whitt (1995) of $m = 11$, $n = 15$, and $A = 18.4$ consistently provide accuracy to at least three significant digits.

The Post-Widder (PW) method is based on the Post-Widder theorem. The function $f(y)$ is approximated in the algorithm by:

$$
\begin{aligned}
f_n(y) &= \frac{n+1}{2ynR^n}\sum_{k=1}^{2n}(-1)^k \mathcal{R}e\left\{\hat{f}(\frac{n+1}{y}(1-Re^{\pi ki/n}))\right\} \\
&= \frac{n+1}{2ynR^n}[\hat{f}((n+1)(1-R)/y)+(-1)^n\hat{f}(n+1)(1+R)/y) \\
&\quad +2\sum_{k=1}^{n-1}(-1)^k \mathcal{R}e\left\{\hat{f}(\frac{n+1}{y}(1-Re^{\pi ki/n}))\right\}], \qquad (4)
\end{aligned}
$$

which involves the two parameters n and R. However, the convergence is quite slow, so in order to enhance the accuracy and expedite the convergence, the following linear combination of the terms f_n is adopted in Abate and Whitt (1995):

$$
f_{j,m}(y) = \sum_{k=1}^{m} w(k,m) f_{j\cdot k}(y), \qquad (5)
$$

where $w(k,m) = (-1)^{m-k}\frac{k^m}{k!(m-k)!}$. The algorithm consists of using $f_{j,m}$, which as defined in (5) is a weighted sum of $f_{j\cdot k}(y)$'s defined by (4) with $R = 10^{-E/2n}$, to approximate f. Implementation of the PW algorithm also involves the selection of three parameters: j, m, E, where again increasing the values of the former two increases the accuracy of the algorithm. As we shall see, the recommended settings in Abate and Whitt (1995) of $j = 10$, $m = 6$, and $E = 8$ do not always give the desired level of accuracy.

To investigate the accuracy and implementation of the two inversion algorithms for the Geman and Yor (1993) Laplace transform given by (2) and (3), we conducted a series of numerical experiments. Table 1 gives the results of the first set of experiments, using the recommended parameter settings of Abate and Whitt (1995). Based on an annual interest rate of $r = 9\%$, the continuous Asian call option values are computed at the starting date $t = 0$ for a contract with initial stock price $S(0) = 100$, for six maturities $T = 0.4, 0.5, 1.0, 2.0, 3.0, 5.0$, and five strike prices of $K = 90, 95, 100, 105, 110$. In all cases, we have taken $a(0) = 0$, i.e., we calculate the prices at the beginning of the averaging period. The values are increasing in maturity and decreasing in the strike price as expected. Slowly converging numerical integration led to difficulties in the inversion for maturities below 0.25. We will return to analyze this difficulty in more depth shortly. The computational times for the inversions were between 19 and 53 CPU seconds for the Euler algorithm and between 440 to 640 CPU seconds for the

PW algorithm, using Mathematica 2.0 on a Sun Sparc 10 workstation. These times are meant to be used primarily as a relative guide to the computational burden of the two algorithms, where it is clear that the PW algorithm requires at least an order of magnitude more computational time.

Except for the lengthiest maturity case, it is clear that the results differ in the second decimal place. Further investigation will in fact reveal that it is the PW algorithm that requires further refinement, i.e., an adjustment of the algorithm parameters.

We next conducted a series of numerical experiments in which the user-specified parameters were varied: A, m, n in the Euler algorithm, and E, j, m in the PW algorithm. This would serve two purposes: (i) to get an idea of the convergence rate of the algorithms as a function of the parameters, and (ii) to determine which algorithm was in fact giving the more accurate result at the recommended settings. Tables 2 and 3 present the results, from which it appears that the recommended settings of Abate and Whitt (1995) generally provide at least three decimal places of accuracy for the Euler algorithm with rapid convergence. Given that it is also the much more computationally efficient of the two algorithms, it is clearly the recommended one, with the usefulness of the PW algorithm as a cross-check, especially on the more numerically difficult cases, as we shall soon demonstrate.

The next experiments were used to compare the two inversion algorithms with other numerical algorithms in the literature, using the cases from Geman and Eydeland (1995). Table 4 reports on a comparison of the Euler and PW inversions using the recommended parameter values with the Geman and Eydeland (1995) inversion approximations, the Shaw (1998) Mathematica 3.0 implementation, the Turnbull and Wakeman (1991) approximation, and Monte Carlo results based on 10 daily readings and 100 daily readings (with standard errors based on 10,000 replications given in parentheses). We observe from these calculations that for these 6 cases, the Euler and PW inversions – using the recommended parameter values – are in agreement to 3 decimal places in all cases except the fourth case. This case has the lowest values of $\sigma^2 T$, which give some indication of the difficulty of inversion, as the integral diverges as this quantity goes to zero. In fact, it leads to huge errors (an order of magnitude off) for the Shaw (1998) implementation, and a nearly 4% error for the Geman and Eydeland approximation. The second most difficult case (the fifth one) also leads to a similar degree of error for the latter case. Investigating further the fourth case, we note that to five decimal places, the values obtained using the recommended parameter settings for the Euler and PW algorithms were 0.05599 and 0.06237, respectively. Increasing the Euler parameters a bit and the PW parameters substantially leads to agreement to six decimal places at 0.055989, indicating that the Euler algorithm value at the recommended settings was accurate to at least five decimal places for even this difficult case.

To understand further the numerical instabilities arising from lower volatilities, we provide in Figure 1 various graphs for the behavior of the Laplace transform as a function of the transform parameter at three different volatilities, $\sigma = 0.1, 0.5, 1.0$ (from top to bottom, respectively), while the other parameters are held constant at $S(0) = 2.0, K = 2.0, r = 0.02, T = 1.0$ years. Specifically, what is plotted is 10^5 times the real part of $\hat{C}(19.1 + si, \sigma^2/4)$ versus s, where $i = \sqrt{-1}$ here. The graphs in the left column are plotted on the same scale — to provide a quick eyeball comparison of the decaying tail, whereas those on the right column are not — to magnify the actual magnitude of the tail decay. The graph for $\sigma = 0.1$ is still oscillatory even after $s = 500$, i.e., it will cross 0 again into positive territory for larger s. Since the inversion requires essentially an integration of this function after multiplication by a sinusoidal-like function, it can be seen that the slowly decaying tail for low volatilities will lead to numerical problems, since convergence will require an increasingly large domain of integration. In particular, the magnitude of the plotted function for $\sigma = 1.0$ has decayed to well under 0.1 by $s = 100$, whereas for $\sigma = 0.1$, the magnitude is still greater than 0.1 at $s = 500$.

For the Shaw (1998) implementation, the numerical difficulties in the fourth case are also due to a slowly decaying tail, causing problems with the numerical integration. The original implementation used a range of integration up to 500. For the fourth case, the graphs clearly show that this is not sufficient. Even ranges up to 2000 gave very poor results (0.142). Not until the integration range was increase to 20000 was a result of 0.05598 obtained, and this required over 16200 CPU seconds on the Sun Sparc 10 workstation (compared to under 60 seconds for the Euler algorithm). In fact, for all six cases, the Shaw implementation encounters the following slow convergence message from Mathematica 3.0 [1]:

```
NIntegrate::"slwcon":
```

[1] The prescribed solution from the error message did not alleviate the problem.

```
"Numerical integration converging too slowly; suspect one of the
following: singularity, value of the integration being 0, oscillatory
integrand, or insufficient WorkingPrecision. If your integrand is
oscillatory, try using the option Method->Oscillatory in NIntegrate."
```

We note that Geman and Eydeland (1995) do not recommend their algorithm if $\sigma^2(T-t)/4 < 0.3$, which is true for all of the cases here. It is likely that the right-hand side should have been on the order of .003 instead of 0.3, but in any case, it indicates the likeliness of numerical problems when the values for volatility and time to maturity are too low. In those cases, linear interpolation is suggested. Our numerical experiments with low volatilities and short maturities yield the same conclusion, although the Euler method seems to be able to handle these cases slightly better than the others. In contrast, the Turnbull and Wakeman (1991) approximation works the best in this regime, as larger biases are reported for higher volatility. Monte Carlo methods are also more efficient in these regimes, as indicated by the lower standard errors. In summary, for numerical inversion we recommend from the perspectives of speed and accuracy the Euler inversion method, which is rapidly convergent to four significant digits of accuracy at the recommended parameter settings. The PW algorithm is recommended as a check for those cases where it is suspected that numerical difficulties due to a slowly decaying integrand may arise.

2 A Comparison of Simulated and Analytical Continuous Asian Call Option Valuations

Simulation is an effective and simple tool for valuation that is readily applicable to the problem of valuing some of the most complicated contracts in some of the most complex contexts. In contrast to analytical approaches, simulation methods can easily accommodate complex formulations of volatility dependence on the current state or even formulations that make volatility specifications path dependent. As a result, it is a method that is heavily relied upon by researchers and practitioners seeking option value solutions for models that are econometrically better specified. On the other hand, simulation can be computationally intensive compared to methods such as numerical inversion, so it is important to investigate mechanisms for decreasing the costs of and increasing the accuracy of simulation.

For simple European option contracts, Boyle (1977) confirmed that control variate methods with 5000 trials provide sufficient pricing quality. The situation is considerably more complex when dealing with continuous-time path-dependent options like the continuous Asian call option. Continuous Asian call options have been valued using simulation earlier by Kemna and Vorst (1990), where they noted that the use of geometric Asian call prices served as a high quality control variate. They did not however have an analytical value to compare the results against. The results of Geman and Yor (1993) enable such a comparison. We shall show that the use of control variates introduces some subtleties, from which we can draw some important lessons on simulation strategies.

For the purpose of simulating the continuous-time stock price motion under the geometric Brownian motion model, the time interval was partitioned into days, and the days were partitioned further uniformly into $1, 10, 100, 1000$ or 10,000 readings per day. The number of replications of sample paths was $1000, 10000, 100000$ or a million replications. For the generation of the path we used a straight update of the stock price given by

$$S(t+h) = S(t)e^{rh+\sigma\sqrt{h}\epsilon_{t+h}-\sigma^2 h/2},$$

where the random variables ϵ_{t+h} were drawn from a standard normal distribution.

We now present some results, shown in Table 5, where the *continuous* geometric Asian option is used as a control variate for estimating the continuous arithmetic Asian option. From top to bottom, the number of daily readings increases by multiples of 10 from 1 to 1000, but the number of replications is kept fixed at 10,000, and so the width of the confidence band remains approximately the same. As may be observed from this table, the pricing quality of the continuous control variate estimate, as reported in the column labeled "ContCV Value" and measured by the "% Error" column, is excellent, with the case of one daily reading delivering a price that is within .02% of the numerical inversion value, reported in the "NumInv Value" column and obtained via the Euler and Post-Widder inversions matching to four decimal places indicated. The continuous geometric Asian option control variate is therefore highly effective for the pricing of continuous Asian options. Kemna and Vorst (1990) observed this effectiveness as viewed from the perspective of reduced standard errors but not in terms of a comparison with an analytical value. The comparison here is therefore particularly interesting, as the case of one daily reading with the control variate matches the analytical value to two decimal places in all cases.

A subtle point in the use of the continuous geometric Asian call option price as a control variate is that the simulated geometric Asian call price is actually a discrete-time Asian option, so the continuous geometric option price is technically a biased control variate. Even so, it performed extremely well. Unexpected, however, are the results reported in the column labeled "DiscCV Value" of Table 5, where the use of the correct unbiased control variate led to a deterioration in the pricing quality! In particular, with one daily reading, the theoretical value falls outside the 95% confidence interval (approximately 2 standard errors in half-width) for all five cases. This indicates that if we are interested in the continuous Asian option, then the bias in the continuous geometric Asian control variate is comparable to the inherent discretization bias introduced by simulation. Thus, in general when simulation is being used to attack continuous problems, there might be substantial benefit in the use of appropriately biased control variates to offset the discretization bias. To investigate this possibility further, we considered a situation where we have analytical values for the continuous and discrete case. This is the case of the geometric Asian option taken up in the next section.

First, however, we investigate the alternative of using Richardson extrapolation (introduced to option pricing by Geske and Johnson 1984) combined with control variates. We consider only the unbiased control variate estimator (that uses the discrete geometric Asian option), since the extrapolation is intended to correct for the discretization bias. To carry out the extrapolation, the maturity T is partitioned into T/m subintervals (m a multiple of 4) with a stock price generated for each epoch. Three estimates, say p_1, p_2, and p_3, for the Asian option are obtained using $m/4$, $m/2$ and m stock prices in the average, respectively. In Table 6, the entries "Biased CV Value" and "Unbiased CV Value" again represent the estimates with the continuous (biased) and discrete (unbiased) control variates, respectively. A three-point Richardson extrapolation expression for the option price p is given by the following:

$$p = \frac{8}{3}p_3 - 2p_2 + \frac{1}{3}p_1.$$

Similarly, if we denote the option price using $m/2$ stock prices in the average p_1 and the one using m prices p_2, then the two-point Richardson extrapolation is given by the following:

$$p = 2p_2 - p_1.$$

The two columns "2-pt Extrap" and "3-pt Extrap" in Table 6 present the corresponding extrapolated results for the continuous Asian option prices. In all cases, the extrapolated values improve upon the original (Unbiased CV Value) estimate and the three-point extrapolation is always better than the two-point version. For this limited set of cases, the biased control variate estimator outperforms the three-point extrapolation estimate most of the times (19 out of 25), but a more comprehensive set of cases would need to be tested to be more conclusive.

3 A Comparison of Biased and Unbiased Control Variates for the Geometric Asian Option

The geometric Asian option — both the discrete and continuous versions — can be accurately priced analytically and by simulation. The option is also path dependent, and in the continuous case the dependence is on the entire continuous, nowhere differentiable, path of Brownian motion. It therefore provides us with an interesting case on which to fine tune our simulation techniques.

In the previous section, we have seen that biased control variates appear to have a role in constructing valuations of continuous contracts. In this section we address the question as to whether the use of an unbiased control variate, here the discrete geometric Asian option value, results in simulated estimates matching the analytical discrete geometric Asian option price, whereas the use of the biased continuous geometric Asian option price leads to values from the same simulation to now match the continuous geometric Asian option price. Hence we wish to observe whether or not in the geometric Asian option context, where we have exact analytical values for both the discrete and continuous cases, we can get a biased control variate to outperform the unbiased control variate from the perspective of valuing the continuous contract.

To address the issue of the possible performance enhancement associated with the use of a biased control variate, we simulated geometric Asian option prices for the five strike levels with $r = .09$, $\sigma = .2$, $S = 100$ and $T = .4$ using 10,000 replications with the number of partitions of the maturity, i.e., total number of stock price readings in the discrete average, set at $5, 10, 15, 20$ and 25, and employing both biased and unbiased control variates. The number of partitions is intentionally kept on the low side, as our goal is not to approximate the continuous case, but rather to assess how well the biased control variate performs in correcting a relatively discrete situation.

The control variate was as usual the price of a geometric Asian option with the same maturity and number of partitions. However, we cannot have a control that exactly matches the contract being priced. We note that in the case of the arithmetic Asian contract, the control is biased by being the price of a geometric contract. The magnitude of this bias is by Jensen's inequality of the order of $e^{\sigma^2/2}$, which for our parameter settings is 1.008. We therefore take for a control variate the value of the same contract evaluated at a slightly different volatility. The control volatilities we tried were $.19, .195, .205$ and $.21$. We report here just the control volatilities of .19 and .21, since the other results are similar.

Table 7 presents the results, where the columns labeled "Cont. Value" and "Disc. Value" give the analytical values of the respective continuous and discrete geometric Asian call options; columns 5 and 7 present the simulated option values using the unbiased and biased control variates, respectively, for a control volatility of .19, with corresponding % relative errors provided in Columns 6 and 8; and columns 9 through 12 present the same corresponding results for the control volatility of .21.

We observe from the rows for 25 stock price readings that using the biased control variate, for both values of the control volatilities, consistently and accurately estimates the continuous geometric Asian option price, while the use of an unbiased control variate provides a consistent and accurate estimate of the discrete geometric Asian option price. The same observation can be made in the case of only 5 readings of stock prices prior to the expiration date with a slight reduction in accuracy. As noted in the headings of Table 7, the standard errors of the reported estimates with these control variates were all below .0035. The simulations took 4 CPU seconds for 5 readings and 11 seconds for 25 readings using 10000 replications.

These results confirm our conjecture that control variates can have a dual role in simulation. The first is variance reduction, but secondly, as observed here in two cases, if the object of interest is the continuous limit, then the discretization bias inherent in simulation may be corrected by the judicious choice of similarly biased control variates. Here of course we have the benefit of knowing the analytical answer and can easily assess whether a proposed biased control is correctly and fruitfully biased. In more general and complex applied contexts, the nature and usefulness of the bias could be assessed by a fine simulation of sample paths with a large number of readings per day, but then once this has been done, in operation one could cut the cost of simulation by taking a cruder path simulation and employing the biased control variate.

4 A Double Laplace Transform for the Continuous Asian Option

For the continuous Asian option, we derive in this section the double Laplace transform of the call option price in both its strike and its maturity. This is done by relating in general the Laplace transform of the call price in the strike to the Laplace transform of the density of the average price on which the call is written. We next obtain the transform in maturity of this transform of the density. The two results are put together to obtain the double Laplace transform.

Consider first the transform in strike of a call option written on a general real valued random variable A. In particular, suppose we have a call option maturing at time T, on a real valued random variable A, that pays at T the excess of A over a pre-specified strike of K. For a standard European call option on a stock, A is the price of a stock at the maturity T. In the case of the continuous Asian option A is the integral of the stock price from time 0 to time T, divided by T. Further suppose that risk free investment at the constant continuously compounded interest rate of r per unit time is available and let the risk neutral measure Q be the one associated with discounting by the money market accumulation factor e^{rt}.

Further we suppose that the risk neutral density, or the density of A under the measure Q at time t is well defined and is $f_{t,T}(a)$, with the Laplace transform given by

$$\psi_{t,T}(\lambda) = \int_0^\infty e^{-\lambda a} f_{t,T}(a) da. \tag{6}$$

Let $c_{t,T}(K)$ be the price at t of a call option on A maturing at T, with strike K. It follows from martingale theory pricing principles, that

$$c_{t,T}(K) = e^{-r(T-t)} \int_K^\infty (a-K) f_{t,T}(a) da. \tag{7}$$

Define the Laplace transform in K of the call option price by

$$\phi_{t,T}(\lambda) = \int_0^\infty e^{-\lambda K} c_{t,T}(K) dK. \tag{8}$$

The double Laplace transform of the continuous Asian call option price is based on first generally relating $\psi_{t,T}(\lambda)$ of equation (6) to $\phi_{t,T}(\lambda)$ of equation (8). We then develop an expression for the transform in T of $\psi_{t,T}(\lambda)$ and hence obtain the double Laplace transform.

Theorem 1 <u>*Relationship between the Density and Call Price Transforms*</u>

The Laplace transform of the call option price in the strike price $\phi_{t,T}(\lambda)$ is related to the Laplace transform of the risk neutral density $\psi_{t,T}(\lambda)$ by

$$\phi_{t,T}(\lambda) = e^{-r(T-t)} \frac{E_{t,T}[A]\lambda + \psi_{t,T}(\lambda) - 1}{\lambda^2}, \tag{9}$$

where $E_{t,T}[A]$ is the mean of the density $f_{t,T}(a)$.

Proof. From equation (7) we may write

$$c_{t,T}(K) = e^{-r(T-t)} \int_K^\infty a f_{t,T}(a) da - K e^{-r(T-t)}(1 - F_{t,T}(K)), \tag{10}$$

where $F_{t,T}(a)$ is the distribution function associated with the density $f_{t,T}(a)$. Evaluating the first integral in (10) by parts we have that

$$\int_K^\infty a f_{t,T}(a) da = -a(1 - F_{t,T}(a)) \mid_K^\infty + \int_K^\infty (1 - F_{t,T}(a)) da,$$

which yields on evaluation of the first part that

$$\int_K^\infty a f_{t,T}(a) da = K(1 - F_{t,T}(K)) + \int_K^\infty (1 - F_{t,T}(a)) da. \tag{11}$$

Premultiplying (11) by $e^{-r(T-t)}$ and rearranging we observe from (10) and (11) that

$$c_{t,T}(K) = e^{-r(T-t)} \int_K^\infty (1 - F_{t,T}(a)) da.$$

From the definition of $\phi_{t,T}(\lambda)$ in (8) it follows that,

$$\phi_{t,T}(\lambda) = e^{-r(T-t)} \int_0^\infty e^{-\lambda K} \int_K^\infty (1 - F_{t,T}(a)) da dK. \tag{12}$$

Integrating (12) by parts we have that,

$$\phi_{t,T}(\lambda) = e^{-r(T-t)} \left[-\frac{e^{-\lambda K}}{\lambda} \int_K^\infty (1 - F_{t,T}(a)) da \mid_0^\infty - \int_0^\infty \frac{e^{-\lambda K}}{\lambda} (1 - F_{t,T}(K)) dK \right].$$

On evaluation of the first term in the square bracket we obtain that

$$\phi_{t,T}(\lambda) = e^{-r(T-t)} \frac{E_{t,T}[A]}{\lambda} - \frac{e^{-r(T-t)}}{\lambda} \int_0^\infty e^{-\lambda K} (1 - F_{t,T}(K)) dK. \tag{13}$$

Performing the integration in (13) by parts we obtain

$$\int_0^\infty e^{-\lambda K}(1 - F_{t,T}(K))dK = -\frac{e^{-\lambda K}}{\lambda}(1 - F_{t,T}(K)) \mid_0^\infty - \int_0^\infty \frac{e^{-\lambda K}}{\lambda} f_{t,T}(K)dK. \quad (14)$$

Evaluating (14) using the definition of $\psi_{t,T}(\lambda)$ and substituting back into (14) we get that

$$\phi_{t,T}(\lambda) = e^{-r(T-t)}\frac{E_{t,T}[A]\lambda + \psi_{t,T}(\lambda) - 1}{\lambda^2}. \qquad \blacksquare$$

Equation (9) provides us with a useful relationship between call option prices and the density of the underlying risk. For many stochastic processes with independent and identically distributed increments, the density is not available in analytical form, but the Laplace transform is known. Equation (9) can be used to obtain option prices in cases where the stock price motion is given by these processes. Our application here uses (9) by first developing an expression for the Laplace transform of the density of the integral of a geometric Brownian motion.

Suppose now that the stock price process under the risk neutral measure is given by a geometric Brownian motion. Let $S(t)$ be the stock price at time t and $b(t)$ be standard Brownian motion. We suppose that $S(t)$ satisfies the stochastic differential equation

$$dS(t) = rS(t)dt + \sigma S(t)db(t), \quad (15)$$

with solution given by

$$S(t) = S(0)e^{rt+\sigma b(t)-\sigma^2 t/2}.$$

Consider now a continuous Asian option written on this underlying stock at time 0, that matures at time T, with a strike price of K. Let the price process of the Asian option be represented by $w(t)$. The payoff of the option at maturity, or equivalently $w(T)$, is given by

$$w(T) = (a(T) - K)^+,$$

where $a(t) = A(t)/t = \int_0^t S(u)du/t$. The Laplace transform of $a(T)$ at t is given by

$$\psi_{t,T}(\lambda) = E_t^Q\left[e^{-\lambda a(T)}\right]. \quad (16)$$

To obtain the double Laplace transform of the continuous Asian option call price we obtain the transform in T of $\psi_{t,T}(\lambda)$ and then use (9) to transform $\phi_{t,T}(\lambda)$ in T as well.

We first reduce (16) in terms of the remaining uncertainty at time t. Note that,

$$\begin{aligned}
\psi_{t,T}(\lambda) &= E_t^Q\left[e^{-\lambda a(T)}\right] = E_t^Q\left[e^{-(\lambda/T)A(T)}\right] \\
&= E_t^Q\left[e^{-(\lambda/T)A(t)-(\lambda/T)(A(T)-A(t))}\right] \\
&= e^{-(\lambda t/T)a(t)}E_t^Q\left[e^{-(\lambda/T)\int_t^T S(u)du}\right]. \qquad (17)
\end{aligned}$$

Define the Laplace transform of the remaining uncertainty by

$$\Phi(t,\lambda,T) = E_t^Q\left[e^{-\lambda\int_t^T S(u)du}\right], \qquad (18)$$

then by (17) we may write that

$$\psi_{t,T}(\lambda) = e^{-(\lambda t/T)a(t)}\Phi(t,\lambda/T,T).$$

In developing an expression for the Laplace transform for the remaining uncertainty we note the similarity of (18) to the pricing of pure discount bonds. Indeed the expression for $\Phi(t,\lambda,T)$ is precisely the price of a pure discount bond if the interest rate process is taken to be $\lambda S(t)$. We therefore solve for Φ following the methods of Cox, Ingersoll and Ross (1985), noting that $S(t)$ in the exponent of (18) is a homogeneous Markov process. It follows from Cox, Ingersoll and Ross (1985) and the stochastic differential equation (15) that,

$$\Phi(t,\lambda,T) = \Psi(t,S,T,\lambda), \qquad (19)$$

where $S(t) = S$, and Ψ satisfies the partial differential equation

$$\Psi_t + rS\Psi_S + \frac{1}{2}\sigma^2S^2\Psi_{SS} = \lambda S\Psi \qquad (20)$$

subject to the boundary condition

$$\Psi(t,S,t,\lambda) = 1. \qquad (21)$$

Theorem 2 *The Maturity Transform of the Remaining Uncertainty*

The solution to the partial differential equation (20) subject to the boundary condition (21) is given by

$$\Psi(t,S,T,\lambda) = U(S,T-t,\lambda), \qquad (22)$$

where the Laplace transform of $U(x,\tau,\lambda)$ in the maturity τ is given by $W(S,\lambda,v)$ defined as

$$W(S,\lambda,v) = \int_0^\infty e^{-v\tau}U(S,\tau,\lambda)d\tau$$

and

$$W(S,\lambda,v)=\frac{1}{v}(1-\frac{\sigma^2}{2(r-v)})(1+\frac{\sigma^2}{2(r-v)}\frac{2\lambda}{\sigma^2}S)+\frac{1}{v}\frac{\sigma^2}{2(r-v)}{}_1F_2(1;1-\alpha_1,1-\alpha_2;\frac{2\lambda}{\sigma^2}S),\tag{23}$$

for α_1 and α_2 satisfying, $\alpha_1+\alpha_2=(1-2r/\sigma^2)$, $\alpha_1\alpha_2=-2v/\sigma^2$. Here, ${}_1F_2$ is the generalized hypergeometric function (see Erdélyi, et. al. 1953).

Proof. As the underlying process is a homogeneous Markov process, the function Ψ depends only on the time difference $(T-t)$. Consider a solution for Ψ in the form of (22). The partial differential equation (20) in Ψ may be written in terms of U as

$$U_\tau-rSU_S-\frac{\sigma^2}{2}S^2U_{SS}+\lambda SU=0\tag{24}$$

with boundary condition $U(S,0,\lambda)=1$. Premultiplying the partial differential equation in U, (24) by $e^{-v\tau}$ and integrating we obtain, on employing the boundary condition for U, the ordinary differential equation in W, the Laplace transform of U, as

$$S^2W_{SS}+\frac{2r}{\sigma^2}SW_S-\frac{2}{\sigma^2}(v+\lambda S)W=-\frac{2}{\sigma^2}.\tag{25}$$

For boundary conditions, note that when $S=0$, by construction $U(0,\tau,\lambda)=1$ and hence $W(0,v,\lambda)=1/v$. Furthermore, also by construction

$$U_S(0,\tau,\lambda)=E\left[-\lambda\int_0^\tau e^{ru+\sigma b(u)-\sigma^2u/2}du\right]=-\frac{\lambda}{r}(e^{r\tau}-1)$$

and hence that

$$W_S(0,v,\lambda)=-\int_0^\infty e^{-v\tau}\frac{\lambda}{r}(e^{r\tau}-1)d\tau=\frac{\lambda}{v(r-v)}.$$

Consider a solution for (25) of the form

$$W(S,v,\lambda)=L(\frac{2\lambda}{\sigma^2}S,v).\tag{26}$$

The differential equation (25) in W can be expressed in terms of L as

$$z^2L_{zz}+\frac{2r}{\sigma^2}zL_z-(\frac{2v}{\sigma^2}+z)L=-\frac{2}{\sigma^2}.\tag{27}$$

The boundary conditions for L are obtained from those for W and are $L(0,v)=1/v$ and $L_z(0,v)=\frac{1}{v}\frac{\sigma^2}{2(r-v)}$. Equation (27) is solved for the function L by the method of analytic coefficients in the Appendix, and has the solution

$$L(z,v)=\frac{1}{v}(1-\frac{\sigma^2}{2(r-v)})(1+\frac{\sigma^2}{2(r-v)}z)+\frac{1}{v}\frac{\sigma^2}{2(r-v)}{}_1F_2(1;1-\alpha_1,1-\alpha_2;z),\tag{28}$$

where α_1 and α_2 are the roots of the quadratic $x^2+(2r/\sigma^2-1)x-2v/\sigma^2$. The result for W follows on substituting $(2\lambda/\sigma^2)S$ for z in (28), as is required by (26). ∎

Equation (23) identifies the Laplace transform in maturity of the integral of geometric Brownian motion, given by (18) and (19) as a particular generalized hypergeometric function that has a convergent series representation valid over the whole complex plane. For the purposes of inversion it is useful to employ an integral representation of the generalized hypergeometric function given in Erdélyi (1937). This representation is

$$ {}_1F_2(a; b_1, b_2; x) = \int_0^1 \Gamma(b_2)(tx)^{-(b_2-1)/2} I_{(b_2-1)}(2\sqrt{tx}) \frac{\Gamma(b_1)}{\Gamma(a)\Gamma(b_1-a)} t^{a-1}(1-t)^{b_1-a-1} dt, $$

where $I_\nu(w)$ is the Bessel function of imaginary argument and $\Gamma(x)$ is the gamma function. In this representation one must ensure that $\mathrm{Re}(a) < \mathrm{Re}(b_1)$. For our particular case of $a = 1, b_1 = 1-\alpha_1$ and $b_2 = 1-\alpha_2$ we have

$$ {}_1F_2(1; 1-\alpha_1, 1-\alpha_2; z) = \int_0^1 \Gamma(1-\alpha_2)(tz)^{\alpha_2/2} I_{-\alpha_2}(2\sqrt{tz})(-\alpha_1)(1-t)^{-(1+\alpha_1)} dt. $$

The specific values of α_1 and α_2 are

$$ \alpha_1 = 1/2 - r/\sigma^2 - \sqrt{(1/2 - r/\sigma^2)^2 + 2v/\sigma^2}, \quad \alpha_2 = 1/2 - r/\sigma^2 + \sqrt{(1/2 - r/\sigma^2)^2 + 2v/\sigma^2}, $$

and the condition that $\mathrm{Re}(a) < \mathrm{Re}(b_1)$ is met by always choosing α_1 to be the root with the negative real part.

Theorems 1 and 2 may be combined to derive the double Laplace transform of the Asian call option in both the strike and maturity. First we note that the uncertainty in the value of the call at any time t arises from the integral of the stock price over the interval $[t, T]$. In fact, a continuous Asian call option originating at time 0 with maturity T and strike K is the equivalent of $1/T$ units of a call option on the integral over the interval $[0, T]$ with a strike of KT, and this in turn is the equivalent of a call option on the integral of the stock price over the remaining time interval $[t, T]$ with a strike of $(KT - \int_t^T S(u)du)$. Hence, without loss of generality, we may restrict attention to the case $t = 0$, and consider options on the integral of the stock price over the interval $[0, T]$.

Let $C(K, T, S, r, \sigma)$ be the price of a call option on the integral

$$ R = \int_0^T S(u)du, \tag{29} $$

with a strike of K and a maturity of T, for an initial stock price of S when the interest rate is r and the volatility on the stock is σ. Define the double Laplace transform of C by

$$\Lambda(\lambda, v, S, r, \sigma) = \int_0^\infty \int_0^\infty e^{-vT} e^{-\lambda K} C(K, T, S, r, \sigma) dK dT.$$

Theorem 3 *The Continuous Asian Call Double Laplace Transform*

The double Laplace transform of the continuous Asian call option, $\Lambda(\lambda, v)$ is given by

$$\begin{aligned}\Lambda(\lambda, v, S, r, \sigma) &= \frac{S}{r\lambda}\left[\frac{1}{v} - \frac{1}{r+v}\right] - \frac{1}{(r+v)\lambda^2} + \frac{1}{(r+v)\lambda^2} \times \\ &\quad \left[(1+\frac{\sigma^2}{2v})(1-\frac{\lambda S}{v}) - \frac{\sigma^2}{2v}{}_1F_2(1; 1-\alpha_1, 1-\alpha_2; \frac{2\lambda}{\sigma^2}S)\right], \end{aligned} \tag{30}$$

where α_1 and α_2 are

$$\alpha_1 = 1/2 - r/\sigma^2 - \sqrt{(1/2 - r/\sigma^2)^2 + 2(r+v)/\sigma^2}$$

and

$$\alpha_2 = 1/2 - r/\sigma^2 + \sqrt{(1/2 - r/\sigma^2)^2 + 2(r+v)/\sigma^2}.$$

Proof. Let $\psi_{0,T}(\lambda)$ be the Laplace transform of the integral of the stock price over the interval $[0, T]$ so that

$$\psi_{0,T}(\lambda) = E[e^{-\lambda R}], \tag{31}$$

then by Theorem 1, the Laplace transform of the call option price $C(K, T, S, r, \sigma)$

$$\phi_{0,T}(\lambda) = E\left[\int_0^\infty e^{-\lambda K} C(K, T, S, r, \sigma) dK\right]$$

is given by

$$\phi_{0,T}(\lambda) = e^{-rT} \frac{E[R]\lambda + \psi_{0,T}(\lambda) - 1}{\lambda^2}. \tag{32}$$

By direct evaluation we note that

$$E[R] = S\frac{e^{rT} - 1}{r}. \tag{33}$$

Substitution of (33) into (32) yields

$$\phi_{0,T}(\lambda) = \frac{S}{r\lambda}(1 - e^{-rT}) + e^{-rT}(\psi_{0,T}(\lambda) - 1)/\lambda^2. \tag{34}$$

The double Laplace transform is obtained on transforming (34) in T to get

$$\Lambda(\lambda, v) = \frac{S}{r\lambda}\left[\frac{1}{v} - \frac{1}{r+v}\right] - \frac{1}{(r+v)\lambda^2} + \frac{\int_0^\infty e^{-(r+v)T}\psi_{0,T}(\lambda)dT}{\lambda^2}. \qquad (35)$$

Since $\psi_{0,T}(\lambda)$ is by equations (31), (29) and (18) equal to $\Phi(0, \lambda, T)$ which by equation (22) equals $U(S, T, \lambda)$, the transform in T required in (35) is given by Theorem 2 and equation (23). Evaluating the transform (23) at v equal to $r + v$ as required by (35) and substitution of the result into (35) yields the result (30). ∎

5 Conclusion

We investigated the problem of pricing continuous Asian options using numerical inversion and Monte Carlo methods. In numerically inverting the Geman-Yor Laplace transform in maturity, we find that numerical inversion techniques encounter numerical instabilities for low volatilities and short maturities, with the quantity $\sigma^2(T-t)$ providing a good relative yardstick for difficulty. In general, the Euler algorithm developed by Abate and Whitt (1995) is computationally efficient and extremely accurate, with the Post-Widder algorithm providing a nice cross-check if there is doubt about convergence at low volatilities and/or short maturities. Difficulties with the Euler algorithm seem to begin when $\sigma^2(T-t) < 0.01$. In light of the numerical difficulties associated with the Geman-Yor transform, we derived a double Laplace transform of the arithmetic continuous Asian option in both its strike and maturity, which might be more amenable to numerical inversion techniques, although this is left for future research.

In terms of Monte Carlo simulation, in addition to finding that the pricing performance of simulation is considerably enhanced by the judicious choice of control variates, we observe that if the continuous Asian option price is sought, then there is an advantage to using suitably biased control variates that correct for the discretization bias inherent in simulation. An example of such a biased control is the simulated daily geometric Asian option price corrected by not its theoretical daily geometric theoretical option price but the price of a continuous geometric option price. Hence we observe that control variates can play a dual role of variance reduction and when suitably constructed, of bias correction. Our findings raise questions for future research into the design of appropriately biased control variates for particular classes of problems.

Table 1: Inverting the Geman-Yor Continuous Asian Call Option Laplace transform: Euler vs Post-Widder at Abate-Whitt recommended settings; $r = 0.09,\ \sigma = 0.2, S(0) = \$100$, CPU times in secs

T(yrs)	K	Euler	CPU	PW	CPU
0.4	90	11.5293	53	11.5176	640
	95	7.2131	44	7.1981	631
	100	3.8087	42	3.8196	628
	105	1.6465	45	1.6623	623
	110	0.5761	38	0.5728	625
0.5	90	11.9247	34	11.9241	613
	95	7.7249	33	7.7185	591
	100	4.3696	33	4.3759	601
	105	2.1175	32	2.1290	585
	110	0.8734	32	0.8753	595
1.0	90	13.8372	26	13.8439	518
	95	9.9998	26	10.0029	513
	100	6.7801	26	6.7823	514
	105	4.2982	25	4.3010	503
	110	2.5473	25	2.5450	503
2.0	90	17.1212	24	17.1297	475
	95	13.6763	23	13.6830	473
	100	10.6319	23	10.6370	473
	105	8.0436	23	8.0474	469
	110	5.9267	22	5.9295	474
3.0	90	19.8398	21	19.8495	468
	95	16.6740	21	16.6822	467
	100	13.7974	21	13.8042	467
	105	11.2447	20	11.2502	462
	110	9.0316	20	9.0360	465
5.0	90	24.0861	19	24.0861	453
	95	21.3774	22	21.3774	442
	100	18.8399	19	18.8399	449
	105	16.4917	21	16.4917	441
	110	14.3442	21	14.3442	453

Table 2: Numerical Inversion Results Using Euler Algorithm:
$r = 0.09,\ \sigma = 0.2, S(0) = \$100$

A	m	K	n=10	n=15	n=20	n=30	n=50
18.4	11	90	11.52632	11.52925	11.52909	11.52909	11.52909
		95	7.21418	7.21312	7.21319	7.21318	7.21318
		100	3.80888	3.80865	3.80863	3.80863	3.80863
		105	1.64615	1.64648	1.64648	1.64648	1.64648
		110	0.57616	0.57610	0.57611	0.57611	0.57611
18.4	20	90	11.52913	11.52909	11.52909	11.52909	11.52909
		95	7.21310	7.21318	7.21318	7.21318	7.21318
		100	3.80868	3.80863	3.80863	3.80863	3.80863
		105	1.64647	1.64648	1.64648	1.64648	1.64648
		110	0.57610	0.57611	0.57611	0.57611	0.57611
18.4	40	90	11.52909	11.52909	11.52909	11.52909	11.52909
		95	7.21318	7.21318	7.21318	7.21318	7.21318
		100	3.80863	3.80863	3.80863	3.80863	3.80863
		105	1.64648	1.64648	1.64648	1.64648	1.64648
		110	0.57611	0.57611	0.57611	0.57611	0.57611
19.1	11	90	11.52610	11.52927	11.52908	11.52909	11.52909
		95	7.21416	7.21311	7.21318	7.21318	7.21318
		100	3.80894	3.80865	3.80863	3.80863	3.80863
		105	1.64614	1.64648	1.64648	1.64648	1.64648
		110	0.57615	0.57610	0.57611	0.57611	0.57611
19.1	20	90	11.52914	11.52909	11.52909	11.52909	11.52909
		95	7.21309	7.21318	7.21318	7.21318	7.21318
		100	3.80868	3.80863	3.80863	3.80863	3.80863
		105	1.64647	1.64648	1.64648	1.64648	1.64648
		110	0.57610	0.57610	0.57611	0.57611	0.57611
19.1	40	90	11.52909	11.52909	11.52909	11.52909	11.52909
		95	7.21318	7.21318	7.21318	7.21318	7.21318
		100	3.80863	3.80863	3.80863	3.80863	3.80863
		105	1.64648	1.64648	1.64648	1.64648	1.64648
		110	0.57611	0.57611	0.57611	0.57611	0.57611
23	11	90	11.52483	11.52935	11.52909	11.52909	11.52909
		95	7.21383	7.21311	7.21319	7.21318	7.21318
		100	3.80932	3.80864	3.80863	3.80863	3.80863
		105	1.64610	1.64649	1.64648	1.64648	1.64648
		110	0.57608	0.57610	0.57611	0.57611	0.57611
23	20	90	11.52928	11.52908	11.52909	11.52909	11.52909
		95	7.21303	7.21319	7.21318	7.21318	7.21318
		100	3.80869	3.80863	3.80863	3.80863	3.80863
		105	1.64648	1.64648	1.64648	1.64648	1.64648
		110	0.57610	0.57610	0.57611	0.57611	0.57611
23	40	90	11.52909	11.52909	11.52908	11.52909	11.52909
		95	7.21318	7.21318	7.21318	7.21318	7.21318
		100	3.80863	3.80863	3.80863	3.80863	3.80863
		105	1.64648	1.64648	1.64648	1.64648	1.64648
		110	0.57610	0.57610	0.57610	0.57610	0.57610

Table 3: Numerical Inversion Results Using Post-Widder Algorithm:
$r = 0.09,\ \sigma = 0.2, S(0) = \$100$

E	m	K	$j = 10$	$j = 12$	$j = 18$	$j = 20$
6	6	90	11.3790	11.4107	11.4727	11.4851
		95	7.2180	7.2012	7.1913	7.1924
		100	3.9908	3.9373	3.8616	3.8495
		105	1.8278	1.7771	1.7043	1.6923
		110	0.6103	0.5930	0.5763	0.5749
6	12	90	11.5294	11.5302	11.5297	11.5296
		95	7.2078	7.2105	7.2130	7.2132
		100	3.8106	3.8094	3.8088	3.8088
		105	1.6512	1.6490	1.6471	1.6469
		110	0.5750	0.5756	0.5761	0.5762
6	18	90	11.5294	11.5293	11.6104	11.4569
		95	7.2134	7.2134	7.2134	7.2133
		100	3.8089	3.8088	3.8088	3.8088
		105	1.6468	1.6467	1.6467	1.6466
		110	0.5763	0.5763	0.5763	0.5763
8	6	90	11.3788	11.4105	11.4725	11.4848
		95	7.2177	7.2010	7.1911	7.1921
		100	3.9906	3.9371	3.8614	3.8492
		105	1.8276	1.7769	1.7041	1.6921
		110	0.6101	0.5928	0.5761	0.5747
8	12	90	11.5292	11.5300	11.5295	11.5294
		95	7.2076	7.2102	7.2128	7.2123
		100	3.8104	3.8092	3.8086	3.8086
		105	1.6510	1.6488	1.6469	1.6467
		110	0.5748	0.5754	0.5760	0.5760
8	18	90	11.5291	11.5291	11.5292	11.5290
		95	7.2131	7.2132	7.2133	7.2131
		100	3.9463	3.8337	3.8087	3.8086
		105	1.6465	1.6465	1.6465	1.6465
		110	0.5761	0.5761	0.5761	0.5761
10	6	90	11.3788	11.4105	11.4725	11.4848
		95	7.2177	7.2010	7.1911	7.1921
		100	3.9906	3.9371	3.8614	3.8492
		105	1.8276	1.7769	1.7041	1.6921
		110	0.6101	0.5928	0.5761	0.5747
10	12	90	11.5292	11.5300	11.5295	11.5294
		95	7.2075	7.2103	7.2128	7.2130
		100	3.8104	3.8092	3.8085	3.8086
		105	1.6510	1.6488	1.6469	1.6467
		110	0.5748	0.5754	0.5760	0.5760
10	18	90	11.5292	11.5291	11.5293	11.5292
		95	7.2132	7.2131	7.2133	7.2132
		100	3.8085	3.8086	3.8086	3.8087
		105	1.6465	1.6465	1.6466	1.6466
		110	0.5761	0.5761	0.5761	0.5762

Table 4: Comparison of Numerical Techniques for Pricing Continuous Asian Options: Geman-Eydeland (GE), Shaw, Euler, Post-Widder (PW), Monte Carlo (MC), and the Turnbull-Wakeman (TW) approximation; $K = 2.0$ and $t = 0$ throughout, T in years

r	σ	T	$S(0)$	GE	Shaw	Euler	PW	TW	MC10	MC100	std err
0.05	0.5	1	1.9	0.195	0.193	0.194	0.194	0.195	0.192	0.196	(0.004)
0.05	0.5	1	2.0	0.248	0.246	0.247	0.247	0.250	0.245	0.249	(0.004)
0.05	0.5	1	2.1	0.308	0.306	0.307	0.307	0.311	0.305	0.309	(0.005)
0.02	0.1	1	2.0	0.058	**0.520**	.0560	**.0624**	.0568	.0559	.0565	(.0008)
0.18	0.3	1	2.0	0.227	0.217	0.219	0.219	0.220	0.219	0.220	(0.003)
.0125	0.25	2	2.0	0.172	0.172	0.172	0.172	0.173	0.173	0.172	(0.003)
0.05	0.5	2	2.0	0.351	0.350	0.352	0.352	0.359	0.351	0.348	(0.007)

Table 5: Simulation Results Using Geometric Asian Options as Control Variate $r = 0.09,\ \sigma = 0.2,\ S_0 = \$100,\ T = 0.4$yrs, #replications: 10,000

K	Daily Reads	NumInv Value	ContCV Value	% Error	Standard Error	DiscCV Value	% Error	CPU Secs
90	1	11.529	11.527	0.02	0.0015	11.525	0.04	73
90	10	11.529	11.524	0.05	0.0014	11.524	0.05	560
90	100	11.529	11.524	0.04	0.0015	11.524	0.04	5405
90	1000	11.529	11.525	0.03	0.0015	11.525	0.03	53791
95	1	7.213	7.213	.003	0.0014	7.209	0.05	73
95	10	7.213	7.209	0.05	0.0013	7.209	0.06	560
95	100	7.213	7.210	0.04	0.0013	7.210	0.04	5405
95	1000	7.213	7.212	0.02	0.0013	7.212	0.02	53791
100	1	3.809	3.809	.005	0.0012	3.804	0.14	73
100	10	3.809	3.806	0.08	0.0011	3.805	0.09	560
100	100	3.809	3.807	0.06	0.0011	3.806	0.06	5405
100	1000	3.809	3.808	.008	0.0011	3.808	.008	53791
105	1	1.647	1.647	.006	0.0010	1.642	0.29	73
105	10	1.647	1.645	0.08	0.0009	1.645	0.11	560
105	100	1.647	1.645	0.11	0.0009	1.645	0.11	5405
105	1000	1.647	1.647	0.01	0.0009	1.647	0.01	53791
110	1	0.576	0.576	0.03	0.0008	0.573	0.50	73
110	10	0.576	0.576	0.05	0.0008	0.576	0.10	560
110	100	0.576	0.576	0.02	0.0008	0.576	0.02	5405
110	1000	0.576	0.576	0.05	0.0008	0.576	0.05	53791

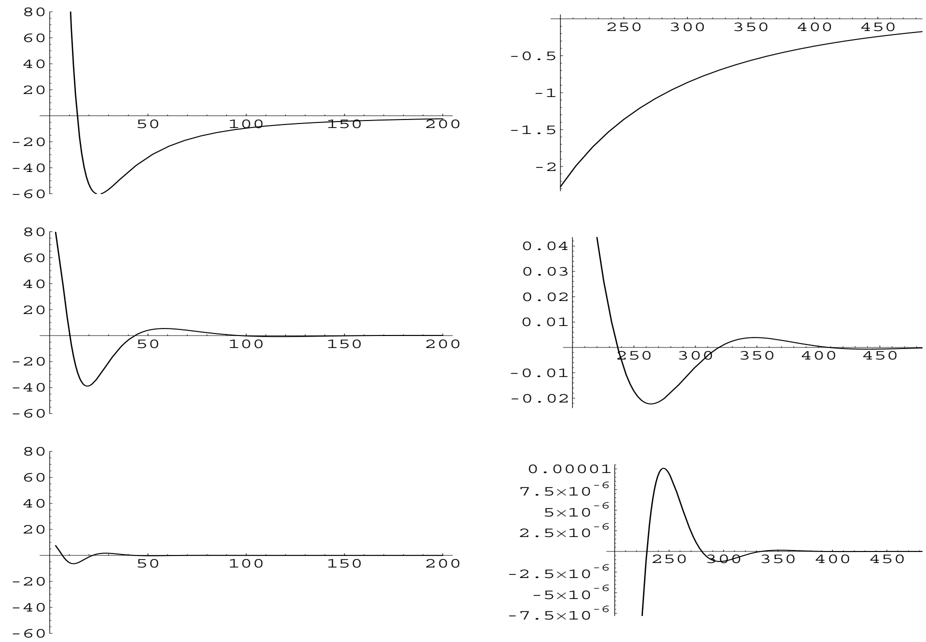

Figure 1: Comparison of Laplace transforms for different volatilities: $10^5\mathrm{Re}\{\hat{C}(19.1+si,\sigma^2/4)\}, 0<s\leq 500,\ S(0)=2.0, K=2.0, r=0.02, T=1.0$, for $\sigma=0.1,0.5,1.0$, respectively

Table 6: Simulation Results using CVs and Extrapolation
$r = 0.09,\ \sigma = 0.2,\ S_0 = \$100,\ T = 0.4\text{yrs}, \#\text{replications: } 10,000,$
standard errors between .0013 and .0016

K	Total Reads	Num Inv Value	Biased CV Value	% Err	Unbiased CV Value	% Err	2-pt Extrap	% Err	3-pt Extrap	% Err	CPU secs
90	20	11.529	11.531	.014	11.517	.106	11.512	.151	11.533	.028	22
90	40	11.529	11.526	.027	11.519	.088	11.516	.115	11.528	.013	38
90	80	11.529	11.527	.017	11.524	.048	11.522	.061	11.528	.009	70
90	160	11.529	11.527	.017	11.526	.033	11.525	.039	11.528	.014	133
90	320	11.529	11.525	.039	11.524	.047	11.524	.050	11.525	.036	260
90	640	11.529	11.525	.035	11.525	.039	11.525	.041	11.525	.034	513
95	20	7.213	7.215	.032	7.189	.336	7.184	.400	7.202	.151	22
95	40	7.213	7.212	.019	7.198	.207	7.195	.243	7.206	.103	38
95	80	7.213	7.212	.015	7.205	.110	7.204	.128	7.209	.058	70
95	160	7.213	7.213	.001	7.209	.049	7.209	.058	7.211	.023	133
95	320	7.213	7.210	.038	7.209	.062	7.208	.067	7.209	.049	260
95	640	7.213	7.210	.038	7.209	.050	7.209	.052	7.210	.042	513
100	20	3.809	3.811	.060	3.774	.906	3.770	1.03	3.787	.559	22
100	40	3.809	3.808	.011	3.789	.505	3.787	.566	3.796	.325	38
100	80	3.809	3.809	.004	3.799	.246	3.798	.279	3.803	.146	70
100	160	3.809	3.809	.001	3.804	.125	3.803	.140	3.806	.079	133
100	320	3.809	3.806	.062	3.804	.125	3.804	.133	3.805	.100	260
100	640	3.809	3.808	.028	3.806	.059	3.806	.063	3.807	.046	513
105	20	1.647	1.648	.099	1.614	1.94	1.610	2.19	1.627	1.20	22
105	40	1.647	1.647	.045	1.630	.999	1.628	1.13	1.637	.602	38
105	80	1.647	1.648	.065	1.639	.463	1.638	.527	1.642	.273	70
105	160	1.647	1.648	.077	1.643	.189	1.643	.223	1.645	.086	133
105	320	1.647	1.646	.035	1.644	.168	1.643	.186	1.645	.116	260
105	640	1.647	1.645	.071	1.644	.138	1.644	.147	1.645	.112	513
110	20	0.576	0.578	.259	0.557	3.33	0.554	3.84	0.565	1.86	22
110	40	0.576	0.577	.088	0.566	1.75	0.564	2.01	0.570	1.00	38
110	80	0.576	0.577	.192	0.572	.740	0.571	.860	0.574	.376	70
110	160	0.576	0.577	.182	0.574	.287	0.574	.355	0.576	.085	133
110	320	0.576	0.575	.239	0.573	.474	0.573	.506	0.574	.379	260
110	640	0.576	0.575	.123	0.575	.241	0.575	.255	0.575	.198	513

Table 7: Geometric Asian Options Pricing Using Biased & Unbiased Control Variates
$r = 0.09,\ \sigma = 0.2,\ S_0 = \$100,\ T = 0.4$yrs, #replications: 10,000

K	Cont. Value	Total Reads	Disc. Value	Unbiased CV σ=.19	% Err	Biased CV σ=.19	% Err	Unbiased CV σ=.21	% Err	Biased CV σ=.21	% Err
90	11.3998	5	11.3522	11.3673	.28	11.4079	.071	11.3362	.56	11.391	.075
90	11.3998	10	11.3736	11.3740	.23	11.3963	.031	11.3722	.24	11.402	.024
90	11.3998	15	11.3817	11.3828	.15	11.3982	.014	11.3811	.16	11.402	.019
90	11.3998	20	11.3860	11.3878	.10	11.3996	.002	11.3842	.14	11.400	.003
90	11.3998	25	11.3886	11.3890	.09	11.3985	.011	11.3886	.10	11.401	.015
95	7.1014	5	7.0082	7.0277	1.04	7.1120	.15	6.9891	1.58	7.0910	.15
95	7.1014	10	7.0507	7.0535	.67	7.0994	.028	7.0475	.76	7.1030	.023
95	7.1014	15	7.0665	7.0640	.53	7.0956	.082	7.0689	.46	7.1070	.079
95	7.1014	20	7.0748	7.0761	.36	7.1002	.017	7.0735	.39	7.1026	.017
95	7.1014	25	7.0800	7.0792	.31	7.0986	.039	7.0806	.29	7.1040	.037
100	3.7175	5	3.5869	3.6045	3.04	3.7276	.27	3.5694	3.98	3.7074	.272
100	3.7175	10	3.6469	3.6516	1.77	3.7182	.019	3.6420	2.03	3.7166	.024
100	3.7175	15	3.6691	3.6688	1.31	3.7145	.081	3.6693	1.30	3.7205	.081
100	3.7175	20	3.6807	3.6817	.96	3.7164	.030	3.6797	1.02	3.7186	.030
100	3.7175	25	3.6878	3.6879	.80	3.7159	.043	3.6877	.80	3.7191	.043
105	1.5790	5	1.4600	1.4833	6.06	1.5948	1.00	1.4372	8.98	1.5637	.97
105	1.5790	10	1.5145	1.5216	3.63	1.5820	.190	1.5076	4.52	1.5762	.18
105	1.5790	15	1.5348	1.5366	2.68	1.5780	.063	1.5331	2.91	1.5801	.07
105	1.5790	20	1.5454	1.5475	1.99	1.5791	.006	1.5433	2.26	1.5791	.006
105	1.5790	25	1.5518	1.5529	1.65	1.5784	.038	1.5509	1.78	1.5798	.051
110	0.5348	5	0.4628	0.4905	8.28	0.5545	3.68	0.4352	18.6	0.5153	3.65
110	0.5348	10	0.4954	0.5070	5.20	0.5421	1.36	0.4842	9.46	0.5280	1.27
110	0.5348	15	0.5077	0.5107	4.51	0.5349	.019	0.5045	5.67	0.5346	.037
110	0.5348	20	0.5141	0.5159	3.53	0.5343	.093	0.5125	4.17	0.5355	.13
110	0.5348	25	0.5181	0.5178	3.18	0.5326	.41	0.5184	3.07	0.5370	.41

Appendix: Solution of equation (27) by the method of analytic coefficients.

The equation in L we wish to solve is of the form

$$z^2 L_{zz} + \alpha z L_z - (\beta + z)L = -\gamma \tag{36}$$

for $\alpha = 2r/\sigma^2, \beta = 2v/\sigma^2$ and $\gamma = 2/\sigma^2$, for the boundary conditions

$$L(0, v) = 1/v, \quad L_z(0, v) = \frac{1}{v}\frac{\sigma^2}{2(r-v)}.$$

Consider a solution of the form

$$L(z, v) = \sum_{m=0}^{\infty} a_m z^m. \tag{37}$$

Rewriting equation (36) as $z^2 L_{zz} + \alpha z L_z - \beta L = -\gamma + zL$, and employing (37) and equating coefficients of the powers of z, we obtain

$$-\beta a_0 = -\gamma,$$

$$(\alpha - \beta)a_1 = a_0,$$

$$(m(m-1) + \alpha m - \beta)a_m = a_{m-1}.$$

The solution of these equations yields

$$a_0 = \gamma/\beta = 1/v,$$

$$a_1 = \frac{1}{v}\frac{\sigma^2}{2(r-v)},$$

$$a_m = \frac{a_{m-1}}{m^2 + (\alpha - 1)m - \beta}.$$

Let α_1, α_2 be the roots of the equation $x^2 + (\alpha - 1)x - \beta = 0$. Then it follows that

$$a_m = \frac{a_{m-1}}{(m - \alpha_1)(m - \alpha_2)}$$

and substitution into (37) yields

$$L(z, v) = \frac{1}{v} + \frac{1}{v}\frac{\sigma^2}{2(r-v)}z + \sum_{m=2}^{\infty} \frac{1}{v}\frac{\sigma^2}{2(r-v)}\frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(m+1)}{\Gamma(m+1-\alpha_1)\Gamma(m+1-\alpha_2)}\frac{z^m}{m!}. \tag{38}$$

Note that (38) satisfies the required boundary conditions. Adding and subtracting the terms of order $m = 0$ and $m = 1$ in the infinite sum yields on simplification the result

$$L(z, v) = \frac{1}{v}(1 - \frac{\sigma^2}{2(r-v)})(1 + \frac{\sigma^2}{2(r-v)}z) + \frac{1}{v}\frac{\sigma^2}{2(r-v)}{}_1F_2(1; 1-\alpha_1, 1-\alpha_2; z).$$

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