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Convergence of a Stochastic Approximation Algorithm for the GI/G/1 Queue Using Infinitesimal Perturbation Analysis

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Abstract. Discrete-event systems to which the technique of infinitesimal perturbation analysis (IPA) is applicable are natural candidates for optimization via a Robbins-Monro type stochastic approximation algorithm. We establish a simple framework for single-run optimization of systems with regenerative structure. The main idea is to convert the original problem into one in which *unbiased* estimators can be derived from strongly consistent IPA gradient estimators. Standard stochastic approximation results can then be applied. In particular, we consider the GI/G/1 queue, for which IPA gives strongly consistent estimators for the derivative of mean system time. Convergence (w.p.1) proofs for the problem of minimizing mean system time with respect to a scalar service time parameter are presented.

Key Words. Stochastic optimization, stochastic approximation, perturbation analysis, single-server queues.

1. Introduction

In many discrete-event systems, infinitesimal perturbation analysis (IPA) provides an efficient means by which to obtain derivative estimators of performance measures with respect to controllable parameters. The motivation for obtaining such derivative estimators is for optimization of appropriate performance measures. Formally, the problem under consideration is the following:

$$\min_{\theta \in \Theta} J(\theta) \stackrel{\triangle}{=} E[L(\theta,\omega)],$$

where $\omega \in \Omega$ represents the stochastic effects, $J(\theta)$ is the performance measure, and $L(\theta,\omega)$ is a sample performance. We usually adopt the perspective in which L is a function of a sample path $X(\theta,\omega)$ of a discrete-event stochastic system defined on an underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and θ is a (possibly vector) decision parameter. We also assume $J(\theta)$ cannot be easily calculated analytically, so that we have a stochastic optimization problem.

Under suitable conditions, the minimum can be found by finding the zero of the gradient ∇J. One means of doing this is iteratively by way of a Robbins-Monro type stochastic approximation algorithm, which takes the following form:

$$\theta_{n+1} = \theta_n - a_n \tilde{\nabla} J(\theta_n, \xi_n),$$

where $\{a_n\}$ is an appropriate sequence of step sizes and $\tilde{\nabla} J(\theta,\xi)$ is an estimator of $\nabla J(\theta)$.

In this note, we apply an appropriate modification of this algorithm to the GI/G/1 queue to solve the problem of minimizing system time with repect to a scalar service time parameter using an IPA estimator. This optimization problem was initially addressed using IPA in preliminary experiments done by Suri and Zazanis (Ref.1), with more extensive experiments for the M/M/1 queue conducted by Suri and Leung (Ref.2). Suri and Zazanis (Ref.1) have proven the strong consistency of the IPA estimator for derivatives of system time for the M/G/1 queue under quite general conditions and for the GI/G/1 queue under more restrictive conditions (Ref.3). Although the main focus of this note is the GI/G/1 queue, where simulation results show quite rapid convergence, the derived optimization algorithm is in fact quite general, in that it can be applied to regenerative systems in which a strongly consistent IPA estimator is available. Glynn (Ref.4) has proven similar convergence results for regenerative systems using finite difference estimators for the estimator, and for more restrictive (Markovian) systems using a likelihood ratio estimator.

Finally, we should point out that although the proposed algorithm requires the system to have regenerative structure, the earlier investigations did not, with experimental evidence indicating that this may not be necessary for convergence in most cases (Refs. 2,5). However, by converting the problem of finding a zero of the gradient to an equivalent problem of finding the zero of another function, we can use the regenerative structure to derive *unbiased* estimators of that function from strongly consistent estimators of the gradient. Classical convergence results can then be applied in a straightforward manner, adding some theoretical support to the accumulated experimental evidence.

2. IPA for Regenerative Systems

We follow the development in Zazanis and Suri (Ref.6). Let $\{X(t) : t \geq 0\}$ be a regenerative process with state space $\Re^k$ and regenerative points $\{\beta_n : n \in \mathcal{Z}^+\}$. Define $\alpha_n = \beta_{n+1} - \beta_n =$ length of the nth regenerative period. Assume $E[\alpha_n] < \infty$ and $X(t) \Rightarrow X$ (where $\Rightarrow$ denotes convergence in distribution). Then, we have the following (see, e.g., (Ref.7)):

Theorem 2.1. Let $f : \Re^k \longrightarrow \Re$ be a measurable function, and let

$$Y_n \triangleq \int_{\beta_n}^{\beta_{n+1}} f[X(t)]dt.$$

Then, (i) $\{(Y_n, \alpha_n) : n \in Z^+\}$ is a renewal process, and

(ii) if $E|f(X)| < \infty$, then

$$E[f(X)] = \frac{E[Y_n]}{E[\alpha_n]}.$$

As a corollary, for a discrete time regenerative process $\{X_i : i = 1, 2, ...\}$ with regenerative points $\{j_n : n \in \mathcal{Z}^+\}$, and $\eta_n = j_{n+1} - j_n =$ length of the nth regenerative period, we have

$$E[f(X)] = \frac{E[Z_n]}{E[\eta_n]},$$

where

$$Z_n \triangleq \sum_{i=j_n}^{j_{n+1}-1} f(X_i).$$

For example, consider the GI/G/1 queue with $\rho < 1$ for stability, where ρ represents the traffic intensity. A continuous time regenerative process is given by the number in the system at time t, $N(t)$, with regenerative points, β_n, at the initiation of busy periods, so that the length of the busy period plus the length of the subsequent idle period would constitute the regenerative period. A discrete time regenerative process is given by the system time of the ith customer, T_i, with regenerative points at the initiation of a busy period, so that the number of customers served in the nth busy period constitutes the regenerative period. (In both cases, f is simply the identity map.)

A natural estimator for $E[f(X)] \triangleq r$ is

$$r_M = \frac{\sum_{i=1}^{M} Y_i}{\sum_{i=1}^{M} \alpha_i},$$

where M = number of regenerative periods.
By the strong law of large numbers (SLLN), we have

$$\lim_{M\to\infty} r_M = \lim_{M\to\infty} \frac{\frac{1}{M}\sum Y_i}{\frac{1}{M}\sum \alpha_i} = \frac{EY_n}{E\alpha_n} = r \text{ w.p.1},$$

i.e., r_M is a strongly consistent estimator for r. However, as is well-known, r_M is not in general *unbiased,* since

$$Er_M = E\frac{\sum Y_i}{\sum \alpha_i} \neq \frac{E\sum Y_i}{E\sum \alpha_i} = \frac{MEY_n}{ME\alpha_n} = r.$$

Now, we consider all quantities as a function of some scalar parameter θ, which we assume enters through the distribution functions. Then, assuming that EY and $E\alpha$ are differentiable with respect to θ, we have

$$\frac{dr}{d\theta} = \frac{\frac{dEY}{d\theta}E\alpha - \frac{dE\alpha}{d\theta}EY}{(E\alpha)^2}.$$

The IPA estimator is obtained by differentiating r_M:

$$\frac{dr_M}{d\theta} = \frac{\sum \frac{dY}{d\theta}\sum \alpha - \sum \frac{d\alpha}{d\theta}\sum Y}{(\sum \alpha)^2}.$$

Assuming Y and α are differentiable with respect to θ, we have by SLLN again,

$$\lim_{M\to\infty} \frac{dr_M}{d\theta} = \frac{E\frac{dY}{d\theta}E\alpha - E\frac{d\alpha}{d\theta}EY}{(E\alpha)^2}.$$

Thus, we get the necessary and sufficient condition for the IPA estimator to be strongly consistent:

$$\lim_{M\to\infty} \frac{dr_M}{d\theta} = \frac{dr}{d\theta} \text{ w.p.1} \Longleftrightarrow E\frac{dY}{d\theta}E\alpha - E\frac{d\alpha}{d\theta}EY = \frac{dEY}{d\theta}E\alpha - \frac{dE\alpha}{d\theta}EY.$$

However, with this condition, the IPA estimator defined over any finite number of regenerative periods, though strongly consistent, is again not unbiased (even if Y, $dY/d\theta$, are independent of α, $d\alpha/d\theta$).

Suppose we want $dr/d\theta = 0$. Then, since $\alpha > 0$,

$$\frac{dr}{d\theta} = 0 \text{ iff } \frac{dEY}{d\theta}E\alpha - \frac{dE\alpha}{d\theta}EY = 0$$

$$E\frac{dr_M}{d\theta} = 0 \text{ iff } E\frac{\sum \frac{dY}{d\theta}\sum \alpha - \sum \frac{d\alpha}{d\theta}\sum Y}{(\sum \alpha)^2} = 0.$$

Let

$$R \triangleq \frac{dEY}{d\theta}E\alpha - \frac{dE\alpha}{d\theta}EY$$

$$\text{and } \tilde{R}_{IPA} \triangleq \frac{dY}{d\theta}\alpha - \frac{d\alpha}{d\theta}Y.$$

Now, if $dY/d\theta$ is independent of α and Y is independent of $d\alpha/d\theta$, then

$$E\tilde{R}_{IPA} = E\frac{dY}{d\theta}E\alpha - E\frac{d\alpha}{d\theta}EY.$$

If, in addition, we have the strongly consistent NASC for $dr/d\theta$,

$$E\frac{dY}{d\theta}E\alpha - E\frac{d\alpha}{d\theta}EY = \frac{dEY}{d\theta}E\alpha - \frac{dE\alpha}{d\theta}EY,$$

then $\tilde{R}_{IPA}$ is an unbiased estimator of R, i.e., $E\tilde{R}_{IPA} = R$.

Summarizing, we have the following lemma.

Lemma 2.1.

$$\tilde{R}_{IPA} \text{ is an unbiased estimator of } R \Longleftrightarrow \frac{dr_M}{d\theta} \text{ is a strongly consistent estimator of } \frac{dr}{d\theta},$$

$$\text{i.e., } E\tilde{R}_{IPA} = R \Longleftrightarrow \frac{dr_M}{d\theta} \to \frac{dr}{d\theta} \text{ w.p.1}$$

3. Optimization

We now apply this simple lemma to the following stochastic optimization problem:

$$\min_{\theta\in\Theta} J(\theta) \triangleq E[L(\theta,\omega)] \triangleq E[f(\theta,\omega) + C(\theta)] \triangleq r(\theta) + C(\theta).$$

We think of L being decomposed into a sum of two parts – a stochastic part to be estimated by simulation and a deterministic, known cost function on the parameter of interest. As stated in the introduction, we are primarily interested in the case where the stochastic part arises from a discrete-event dynamic system, and is thus a functional of the underlying stochastic process. For simplicity in the discussion, we will restrict ourselves to scalar θ in this section.

First, we state a lemma which gives a set of simple, but very applicable, sufficient conditions for a unique minimum to exist. Conditions (3) and (4) are tailored to the optimization problem to be solved in the next section, and could be replaced with others to suit the particular problem at hand.

Lemma 3.1. Let $J(\theta)$ be defined on (a, b), where the interval can even be all of R. If

(i) J is differentiable w.r.t. θ,

(ii) J is strictly convex,

(iii) $J(\theta) \to +\infty$ as $\theta \downarrow a$,

(iv) $J(\theta) \to +\infty$ as $\theta \uparrow b$,

then $J(\theta)$ has a unique minimum $\theta_* \in (a, b)$, where $J'(\theta_*) = 0$.

Proof. The proof is quite straightforward, and so is omitted here.

With J expressed as a sum of two parts, the following corollary is usually more directly applicable to our problem:

Corollary 3.1. Let $J(\theta) = r(\theta) + C(\theta)$. Sufficient conditions on $r(\theta)$ and $C(\theta)$ for $J(\theta)$ to have a unique minimum $\theta_* \in (a, b)$ at $J'(\theta_*) = 0$ are:

(i) r and C are differentiable w.r.t. θ,

(ii) r is convex, and C is strictly convex w.r.t. θ,

(iii) r bounded at $\theta = a$, and $r(\theta) \to +\infty$ as $\theta \uparrow b$,

(iv) C bounded at $\theta = b$, and $C(\theta) \to +\infty$ as $\theta \downarrow a$.

Note that the roles of r and C in conditions (3) and (4) could be reversed.

Assuming the conditions of the lemma are satisfied, we consider

$$\frac{dJ}{d\theta} = \frac{\frac{dEY}{d\theta}E\alpha - \frac{dE\alpha}{d\theta}EY}{(E\alpha)^2} + C'(\theta).$$

Since $\alpha > 0$,

$$\frac{dJ}{d\theta} = 0 \Longleftrightarrow g(\theta) \triangleq \frac{dJ}{d\theta}(E\alpha)^2 = \frac{dEY}{d\theta}E\alpha - \frac{dE\alpha}{d\theta}EY + (E\alpha)^2 C'(\theta) = 0.$$

What we have done is convert the problem of solving for the zero of $dJ/d\theta$ to that of finding the zero of $g(\theta)$. Now, assuming we have a strongly consistent estimator of $dr/d\theta$, we can proceed as in the previous section to derive an unbiased estimator of $g(\theta)$. Strong consistency gives

$$g(\theta) = E\frac{dY}{d\theta}E\alpha - E\frac{d\alpha}{d\theta}EY + (E\alpha)^2 C'(\theta).$$

Since we have a product of expectations, in order to get an unbiased estimator for $g(\theta)$, we need appropriate pairwise independence of the quantities involved. We get such independence by exploiting the regenerative structure of the system, taking advantage of the fact that observations in different regenerative periods are independent. Using two regenerative periods, say i and j, our estimator is

$$\tilde{g} \triangleq \frac{dY_i}{d\theta}\alpha_j - \frac{d\alpha_i}{d\theta}Y_j + \alpha_i\alpha_j C'(\theta).$$

Since $E[\tilde{g}|\theta] = E[\frac{dY_i}{d\theta}\alpha_j - \frac{d\alpha_i}{d\theta}Y_j + \alpha_i\alpha_j C'(\theta)] = E\frac{dY}{d\theta}E\alpha - E\frac{d\alpha}{d\theta}EY + (E\alpha)^2 C'(\theta) = g(\theta)$, $\tilde{g}$ is an unbiased estimator of $g(\theta)$, so we have derived an unbiased estimator of $g(\theta)$ from a strongly consistent estimator of $dr/d\theta$. Under appropriate conditions, as pointed out in the introduction, a classical Robbins-Monro-like algorithm can be applied. We presently address the required conditions.

For most practical uses of stochastic approximation, a projection algorithm is useful. For the scalar projection algorithm

$$\theta_{n+1} = [\theta_n + a_n(g(\theta_n) + \xi_n + \beta_n)]_a^b,$$

$$\text{where } [x]_a^b = \begin{cases} a & \text{if } x < a \\ b & \text{if } x > a \\ x & \text{otherwise} \end{cases},$$

we have the following convergence result:

Theorem 3.1. Assume that $g(\theta)$ is continuous, $\sum a_n = \infty$, $\sum a_n^2 < \infty$, $E[\tilde{g}(\theta)] = g(\theta)$ and $E[\tilde{g}^2(\theta)] \leq \sigma^2$. Then, if $g(\theta)$ has a unique root $\theta_* \in [a, b]$ s.t. $g(\theta) < 0 \ \forall\theta < \theta_*$, then $\theta_n \to \theta_*$ w.p.1 for the projection algorithm.

Proof. It is a straightforward application of a convergence theorem in (Ref.8). See Fu (Ref.9) for details.

In the next section, we apply this to the GI/G/1 queue, for the specific problem addressed originally in (Ref.1) and later in (Refs.2,5). Specifically, simple conditions for convergence are given.

4. GI/G/1 Optimization Problem

For the GI/G/1 queue, we wish to minimize the mean steady-state system time of a customer with respect to a parameter of the service time distribution, subject to some cost on the parameter, i.e., we wish to solve

$$\min_{\theta \in \Theta} J(\theta),$$

$$\text{where } J(\theta) = ET(\theta, \omega) + C(\theta) = \bar{T}(\theta) + C(\theta),$$

$\theta =$ parameter of the service time distribution, $\Theta \stackrel{\triangle}{=} (a, b)$,
$\bar{T}(\theta) =$ mean steady-state system time of a customer,
$C(\theta) =$ cost function on the parameter.

We assume "nice" conditions on C, i.e., the conditions specified in the corollary of the previous section: C is strictly convex and continuously differentiable w.r.t.θ, C is bounded at $\theta = b$, and $C(\theta) \to +\infty$ as $\theta \downarrow a$. As for $\bar{T}(\theta)$, there exist simple sufficient conditions on the service time distribution for convexity (Ref. 10), and differentiability follows automatically once the strong consistency of the IPA estimator is assumed. We will, however, have to further assume the continuity of the derivative. Lastly, the boundedness at $\theta = a$ is meant to correspond to $\rho = 0$, with $\theta \uparrow b$ corresponding to $\rho \to 1$, where ρ denotes the traffic intensity of the system. Hence, we can apply a stochastic approximation algorithm to solve $J'(\theta) = d\bar{T}/d\theta + C'(\theta) = 0$.

As stated in Section 2, for the GI/G/1 queue with $\rho < 1$, we have

$$\bar{T} = \frac{E[Y]}{E[\eta]}, \text{ where } Y = \sum_{i=1}^{\eta} T_i,$$

T_i = system time of the ith customer in the busy period,
η= number of customers served in a busy period.

For the GI/G/1 queue, there is fortuitous simplification due to the fact that the IPA estimator of $d\eta/d\theta$ yields 0. (For more details, see (Ref.1).) Strong consistency of the IPA estimator for $d\bar{T}/d\theta$ gives

$$\frac{d\bar{T}}{d\theta} = \frac{E[\frac{dY}{d\theta}]}{E[\eta]},$$

$$\text{where } \frac{dY}{d\theta} = \sum_{i=1}^{\eta} \frac{dT_i}{d\theta} = \sum_{i=1}^{\eta}\sum_{j=1}^{\eta} \frac{\partial T_i}{\partial X_j}\frac{dX_j}{d\theta} = \sum_{i=1}^{\eta}\sum_{j=1}^{i} \frac{dX_j}{d\theta},$$

X_j= service time of the jth customer in the busy period.
However, as pointed out before, the IPA estimator over M busy periods given by

$$\frac{dT_M}{d\theta} = \frac{\sum_{m=1}^{M} \frac{dY_m}{d\theta}}{\sum_{m=1}^{M} \eta_m}$$

is *not* an unbiased estimator of $E[\frac{dY}{d\theta}]/E[\eta]$. However, using the device employed in proving Lemma 1, we see that to satisfy $J'(\theta) = 0$, we can multiply out the denominator, i.e.,

$$J'(\theta) = 0 \Longleftrightarrow g(\theta) \stackrel{\triangle}{=} J'(\theta)E[\eta(\theta)] = E[\frac{dY}{d\theta}] + E[\eta(\theta)]C'(\theta) = 0,$$

$$\text{so } \tilde{g}(\theta) = g_M(\theta) = \frac{1}{M}\sum_{m=1}^{M}\{\frac{dY_m}{d\theta} + \eta_m(\theta)C'(\theta)\}$$

is an unbiased (and strongly consistent) estimator of $g(\theta)$. In this particular problem, we don't need two regenerative periods, since $g(\theta)$ contains no products of expectations. Taking our estimator over a single busy period, i.e., $M = 1$, and

$$\tilde{g}(\theta) = \frac{dY}{d\theta} + \eta C'(\theta).$$

Note that the observation length of this estimator is random, being the length of a busy period, in contrast to the algorithms used in (Refs.1,2,5).

The conditions specified in the theorem from the last section, $E[\tilde{g}(\theta)] = g(\theta)$ and $E[\tilde{g}^2(\theta)] \leq \sigma^2$, translate into boundedness conditions on $E(\frac{dY}{d\theta})^2$ and $E\eta^2$. The following lemma gives a set of sufficient conditions:

Lemma 4.1. If, w.p.1, w.r.t.$\theta \in (a,b), C$ and X_j are continuously differentiable and $E\eta^l$, $l = 1,2,3,4$, are continuous, then $\forall [a',b'] \subset (a,b)\ \exists \sigma < \infty$ s.t. $E[\tilde{g}^2(\theta)] \leq \sigma^2 \forall \theta \in [a',b']$.

Proof. Let $[a',b'] \subset (a,b)$.
Since $E\eta^l$ is continuous on (a,b) for $l = 1,2,3,4$, and $[a',b']$ is compact, we know that $E\eta^l$ is bounded on $[a',b']$, by, say, k_l, for $l = 1,2,3,4$. Similarly, $dX_j/d\theta$ and $C'(\theta)$ are bounded, by,

say, B and K, respectively.
Then,

$$
\begin{aligned}
E(\frac{dY}{d\theta})^2 &= E(\sum_{i=1}^{\eta}\sum_{j=1}^{i}\frac{dX_j}{d\theta})^2 \\
&\leq E(\sum_{i=1}^{\eta}\sum_{j=1}^{i} B)^2 = B^2 E(\sum_{i=1}^{\eta} i)^2 = B^2 E[\frac{\eta(\eta+1)}{2}]^2 = \frac{B^2}{4}[E\eta^4 + 2E\eta^3 + E\eta^2] \\
&\leq \frac{B^2}{4}[k_4 + 2k_3 + k_2] \triangleq \sigma_1^2. \\
E(\eta\frac{dY}{d\theta}) &\leq \sqrt{E\eta^2 E(\frac{dY}{d\theta})^2} \\
&\leq \sqrt{k_2\sigma_1^2} \triangleq \sigma_2^2. \\
E[g^2(\theta)] &= E[\frac{dY}{d\theta} + \eta C'(\theta)]^2 = E(\frac{dY}{d\theta})^2 + 2C'(\theta)E(\eta\frac{dY}{d\theta}) + [C'(\theta)]^2 E\eta^2 \\
&\leq \sigma_1^2 + 2K\sigma_2^2 + K^2 k_2 \triangleq \sigma^2. \qquad \square
\end{aligned}
$$

Combining the above with our previous results, we have our convergence theorem:

Theorem 4.1. Assume $\sum a_n = \infty,\ \sum a_n^2 < \infty$, and on $\theta \in (a, b)$,

(i) the IPA estimator for $\bar{T}'$ is strongly consistent,

(ii) (a) C is bounded at $\theta = b$, and $C \to +\infty$ as $\theta \downarrow a$,

(b) C is convex w.r.t.θ,

(iii) (a) $\bar{T}$ is bounded at $\theta = a$, and $\bar{T} \to +\infty$ as $\theta \uparrow b$,

(b) $\bar{T}$ is convex w.r.t.θ,

(c) $\bar{T}'$ is continuous w.r.t.θ,

(iv) $E\eta^l$ is continuous w.r.t.θ for $l = 1, 2, 3, 4$,

(v) X_j continuously differentiable w.r.t.θ.

Then, $\exists$ unique θ_* s.t. $J(\theta)$ is minimized there, and $\forall [a', b']$ s.t. $\theta_* \in [a', b'], \theta_n \to \theta_*$ w.p.1 for the projection algorithm on $[a', b']$.

Proof. The conditions give existence of a unique minimum $\theta_* \in (a, b)$ s.t. $J'(\theta_*) = 0$, or equivalently, $g(\theta_*) = 0$, upon application of the corollary from the last section, and it is obvious that $\exists [a', b'] \subset (a, b)$ s.t. $\theta_* \in [a', b']$ (e.g., $[\frac{\theta_*+a}{2}, \frac{\theta_*+b}{2}]$). Since θ_* is a unique minimum, $g(\theta) < 0\ \forall \theta < \theta_*$. By construction, $E[\tilde{g}] = g(\theta)$, and Lemma 3 gives the requisite bounding on $E[\tilde{g}^2(\theta)]$, so that the Theorem from the last section can be applied. $\square$

Note that this results gives existence of $[a', b'] \subset (a, b)$ on which the algorithm converges, but gives no means of determining $[a', b']$. Thus, in practice, one must "guess" $[a', b']$, resulting in a tradeoff between choosing the region large enough so as to be confident it includes θ_*, but not so large as to approach the regions $\theta = a, \theta = b$, where convergence rate is slowed considerably due to the severity of the gradient at these points ($g(\theta) \longrightarrow \infty$). There are, of course, many different ways the theorem could have been formulated, e.g., using boundedness on a closed interval instead of continuity; also, requirements 2a and 3a are tailored to the GI/G/1 problem.

If we specialize to the problem studied in (Refs.1,2,5), where the controllable parameter, θ, is the mean service time, and $C(\theta) = c_1/\theta$, we have the following:

Corollary 4.1. Assume $\sum a_n = \infty$, $\sum a_n^2 < \infty$, and on $\theta \in (a, b)$,

(i) strong consistency of IPA estimator for derivative of $\bar{T}$,

(ii) $\bar{T}$ is convex and has a continuous derivative,

(iii) $E\eta^l$ is continuous w.r.t.θ for $l = 1, 2, 3, 4$,

(iv) X_j continuously differentiable w.r.t.θ.

Then, $\exists [a', b'] \subset (a, b)$ s.t. $\theta_n \to \theta_*$ w.p.1 for the projection algorithm on $[a', b']$.

If we take the analytically tractable M/G/1 queue for illustrative purposes, standard queueing theory results can be used to show that conditions (2) and (3) are satisfied, while Suri & Zazanis show that (1) holds, with (4) being one of the requisite conditions. Thus, the theorem follows easily for the M/G/1 queue. Furthermore, we believe that all the conditions are in fact satisfied for most "reasonable" GI/G/1 queues. Numerical studies for three specific examples can be found in Fu (Ref.9).

5. Conclusions

Although the focus of this note has been the GI/G/1 queue, we emphasize that in principle the straightforward proof techniques demonstrated here can be easily extended for optimization of any regenerative discrete-event system for which IPA yields a strongly consistent estimator for the gradient of the performance measure of interest. Experimental results in (Ref.9) show convergence rates comparable to those of fixed length observation period algorithms for which no theoretical convergence proofs presently exist. Since the latter algorithm does not require regenerative structure, it is more widely applicable, and it is the belief of this author that a convergence proof should be possible. But because the estimator used in the algorithm is not unbiased when the system is not in steady-state (which is the case during optimization until the algorithm "settles down" near the optimum), more difficult proof techniques are necessary. The purpose of this note was to formulate an IPA estimator-based algorithm for which theoretical convergence could be proved using "standard" stochastic approximation results.

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