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A Note on
Perturbation Analysis Estimators for American-Style Options[1]

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Abstract

In this note, we correct an error in the paper by Fu and Hu (1995) for the perturbation analysis estimator given for the gradient of an American call option payoff on an underlying asset paying multiple dividends. We then introduce a different asset price model that is more straightforward than the previous model, and derive the corresponding gradient estimators. We conclude with a brief discussion of extensions of the estimator to other American-style options.

Keywords: Monte Carlo simulation, American options, perturbation analysis, computational finance, gradient estimation

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1 Introduction

Section 3 of Fu and Hu (1995) considers an American call option defined on a stock paying dividends at discrete time points, and gradient estimators are derived for the expected payoff of the option with respect to various parameters (see also Fu and Hu 1997, and Gürkan, Özge, and Robinson 1996). The one-dividend case is analyzed in detail, and a rigorous proof provided for the correctness of the estimator. The estimator is then extended to an arbitrary number of dividends without derivation. It turns out that the estimator is in fact incorrect, and the main purpose of this note is to provide the correct estimator. We provide a detailed derivation for the two-dividend case, and then show how it generalizes to any number of dividends.

A secondary purpose of this note is to consider an alternative model for the dynamics of the underlying stock price process. The model considered in Fu and Hu (1995) follows Stoll and Whaley (1993) and others by discounting dividends back to time 0 and assuming that the process modeling the stock price minus the discounted dividends follows a geometric Brownian motion. This makes the model more analytically tractable in many cases, but implicitly assumes that all future dividends are known at time 0, which may not be realistic, for example in the case of uncertain dividend payouts. Here, we consider a model where the stock price process and dividend process are generated separately (but not necessarily independently), and provide the corresponding gradient estimators. We show that this estimator can also be used with slight modifications for another American-style option: a Bermudan call option defined on an underlying asset that pays dividends at a continuous rate.

2 The Corrected Estimator

We first briefly recall the model with the associated notation. An American call option is defined on a stock that distributes dividend D_j at time $t_j = \sum_{i=1}^{j} \tau_i$ $(\tau_j > 0)$, $j = 1, \ldots, \eta(T)$, where $\eta(T)$ is the number of dividends distributed during the lifetime of the call contract, taking $\tau_{\eta(T)+1} = T - \sum_{i=1}^{\eta(T)} \tau_i, t_0 = 0, t_{\eta(T)+1} = T$ for notational convenience. Following standard models, we assume that the stock price drops after ex-dividend by the amount of the dividend, i.e., $S_{t_j^+} = S_{t_j^-} - D_j$. The American feature allows exercise of the option at any time up to and including the expiration date, but under the assumption of a frictionless market, the call option should only be exercised – if at all – right before an ex-dividend date or at the expiration date, resulting in the following threshold exercise policy: exercise at t_j if $S_{t_j^-} > s_j (\geq K)$.

The sample performance is the net present (at time 0) value of the option payoff (not the option value), and can be written as

$$L = \sum_{i=1}^{\eta(T)+1} \mathbf{1}\left\{ \bigcap_{j=1}^{i-1} S_{t_j^-} \leq s_j, S_{t_i^-} > s_i \right\} \left(S_{t_i^-} - K \right) e^{-rt_i}, \tag{1}$$

where we define $s_{\eta(T)+1} = K$ for notational convenience, and we are interested in estimating $\partial E[L]/\partial\theta$. The indicator functions simply indicate whether or not the option has been exercised at each potential exercise point.

The stock price model in Fu and Hu (1995) followed that of Stoll and Whaley (1993) and others, in considering the dynamics of the stock price net of the present value of escrowed dividends, given by

$$\tilde{S}_t = h(Z; \tilde{S}_0, t, \mu, \sigma),$$

for some random vector Z independent of the parameters. In particular, they took

$$\tilde{S}_{t_j} = h(Z_j; \tilde{S}_{t_{j-1}}, \tau_j, \mu, \sigma), \quad j = 1, ..., \eta(T) + 1,$$

with independent $Z_j \sim f_j$, probability density functions. The actual stock price process was then given by

$$S_t = \tilde{S}_t + \sum_{i=j+1}^{\eta(T)} D_i e^{-r(t_i - t)}, \quad \text{for } t_j \le t < t_{j+1}, \quad j = 0, 1, ..., \eta(T).$$

In particular, at exercise decision points,

$$S_{t_j^-} = \tilde{S}_{t_j} + \sum_{i=j}^{\eta(T)} D_i \exp\left(-r \sum_{k=j+1}^{i} \tau_k\right), \quad j = 1, ..., \eta(T).$$

The estimator for $\partial E[L]/\partial\theta$ given in Fu and Hu (1995), Equations (13)-(16) on page 433, was

$$\begin{aligned}
&\sum_{i=1}^{\eta(T)} \frac{\partial h^{-1}(y_i^*)}{\partial \theta} f_i(h^{-1}(y_i^*)) \left\{ E\left[L|S_{t_i^-} = s_i^-\right] - E\left[L|S_{t_i^-} = s_i^+\right] \right\} \\
&\qquad + \sum_{i=1}^{\eta(T)} \left[\prod_{j=1}^{i-1} \mathbf{1}\{S_{t_j^-} \le s_j\}\right] \mathbf{1}\{S_{t_i^-} > s_i\} \frac{\partial}{\partial\theta}\left[(S_{t_i^-} - K)e^{-rt_i}\right] \\
&\qquad + \prod_{j=1}^{\eta(T)} \mathbf{1}\{S_{t_j^-} \le s_j\} \frac{\partial}{\partial\theta}\left[(S_T - K)^+ e^{-rT}\right],
\end{aligned} \tag{2}$$

where

$$E[L|S_{t_i^-} = s_i^-] = E\left[(S_T - K)^+ e^{-rT} | S_{t_i^-} = s_i^-\right], \tag{3}$$

$$E[L|S_{t_i^-} = s_i^+] = (s_i - K)e^{-rt_i}, \tag{4}$$

$$y_i^* = (s_i - D_i; \tilde{S}_{t_{i-1}^-}, \tau_i, \mu, \sigma), \quad i = 1, ..., \eta(T), \tag{5}$$

where we have corrected a typographical error of a missing tilde over $S_{t_{i-1}^-}$ in the definition of y_i^*.

The correct estimator is the following:

$$\begin{aligned}
&\sum_{i=1}^{\eta(T)} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j\right\} \frac{\partial h^{-1}(y_i^*)}{\partial\theta} f_i(h^{-1}(y_i^*)) \left\{ E\left[L \middle| \bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} = s_i^-\right] - (s_i - K)e^{-rt_i} \right\} \\
&\qquad + \sum_{i=1}^{\eta(T)} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} > s_i\right\} \frac{\partial}{\partial\theta}\left[(S_{t_i^-} - K)e^{-rt_i}\right] \\
&\qquad + \mathbf{1}\{S_{t_1^-} \le s_1, ..., S_{t_{\eta(T)}^-} \le s_{\eta(T)}\} \frac{\partial}{\partial\theta}\left[(S_T - K)^+ e^{-rT}\right],
\end{aligned} \tag{6}$$

where

$$y_i^* = (s_i - \sum_{j=i}^{\eta(T)} D_j \exp\left(-r \sum_{k=i+1}^{j} \tau_k\right); \tilde{S}_{t_{i-1}}, \tau_i, \mu, \sigma), \quad i = 1, ..., \eta(T). \tag{7}$$

Comparison of the two expressions reveals three errors in the previous estimator (2)-(5):

1. an incorrect expression for y_i^*, (5), was given;
2. an incorrect expression for $E[L|S_{t_i^-} = s_i^-]$, (3), was given;
3. missing conditions involving $\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j$ in the first summation of (2).

In addition, there is an implicit, but unstated, assumption of independence that is generally satisfied for asset models of practical interest, including the derivations to be provided in the next section. Note that otherwise the two estimators match for $\eta(T) = 1$, the one-dividend estimator in Fu and Hu (1995) being correct.

3 An Alternative Stock Price Model

In the alternative stock price model, we separate the stock price and dividend processes in a natural way, as follows:

$$S_{t_j^-} = h(Z_j; S_{t_{j-1}^+}, \tau_j, r, \sigma), \quad j = 1, ..., \eta(T), \quad S_{t_0^+} = S_0,$$

with, as before, independent $Z_j \sim f_j$, probability density functions, where at exercise decision points

$$S_{t_j^+} = S_{t_j^-} - D_j, \quad j = 1, ..., \eta(T).$$

For this model, the same estimator given by Equation (6) applies with one minor change in the definition of y_i^* given by (7):

$$y_i^* = (s_i; S_{t_{i-1}^+}, \tau_i, \mu, \sigma), \quad i = 1, ..., \eta(T). \tag{8}$$

One caution to heed in using this model is that by separating the dividends from the stock price process there is the possibility of obtaining negative stock prices if the dividend process is not properly gauged (correlated) with the stock price process, e.g., if the dividends are specified a priori. Of course, in general, the probability is infinitesimal, but the corresponding binomial tree model may have the undesirable properties of non-recombining and negative nodes.

We now give a derivation for $\eta(T) = 2$, for which the sample performance is given by

$$L = \mathbf{1}\{S_{t_1^-} > s_1\} \left(S_{t_1^-} - K\right) e^{-rt_1} + \mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\} \left(S_{t_2^-} - K\right) e^{-rt_2}$$
$$+\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}(S_T - K)^+ e^{-rT}.$$

The derivative of the first term with respect to θ is given similar to the term in the one-dividend derivation of Fu and Hu (1995) by

$$\frac{\partial}{\partial\theta} E\left[\mathbf{1}\{S_{t_1^-} > s_1\}(S_{t_1^-} - K)e^{-rt_1)}\right]$$
$$= -E\left[\frac{\partial h^{-1}(y_1^*)}{\partial\theta}(s_1 - K)e^{-rt_1} f_1(h^{-1}(y_1^*))\right]$$
$$+E\left[\mathbf{1}\{S_{t_1^-} > s_1\}\frac{\partial}{\partial\theta}\left(\left(S_{t_1^-} - K\right)e^{-rt_1}\right)\right],$$

where $y_1^* = (s_1; S_0, \tau_1, \mu, \sigma)$. Taking the expectation of the second term of L, and using the independence of Z_1 and Z_2, we have

$$E\left[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\}(S_{t_2^-} - K)e^{-rt_2}\right]$$
$$= E\left[\mathbf{1}\{h(Z_1; S_0, \tau_1) \le s_1, h(Z_2; h(Z_1; S_0, \tau_1) - D_1, \tau_2) > s_2\}(h(Z_2; h(Z_1; S_0, \tau_1) - D_1, \tau_2) - K)e^{-rt_2}\right]$$
$$= \int_{x_1=-\infty}^{h^{-1}(y_1^*)} \int_{x_2=h^{-1}(y_2^*(x_1))}^{\infty} (h(x_2; h(x_1; S_0, \tau_1) - D_1, \tau_2) - K)e^{-rt_2}) f_1(x_1) f_2(x_2) dx_2 dx_1,$$

where we have defined $y_2^*(x_1) = (s_2; h(x_1; S_0, \tau_1) - D_1, \tau_2)$. Differentiating and assuming an interchange of differentiation and expectation, we have

$$\frac{\partial}{\partial\theta} E\left[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\}(S_{t_2^-} - K)e^{-rt_2}\right]$$
$$= E\left[\frac{\partial h^{-1}(y_1^*)}{\partial\theta} f_1(h^{-1}(y_1^*)) E\left[\mathbf{1}\{S_{t_2^-} > s_2\}(S_{t_2^-} - K)e^{-rt_2} \middle| S_{t_1^-} = s_1^-\right]\right]$$

$$
\begin{aligned}
&-E\left[\frac{\partial h^{-1}(y_2^*)}{\partial\theta} f_2(h^{-1}(y_2^*))\mathbf{1}\{S_{t_1^-} \le s_1\}(s_2-K)e^{-rt_2}\right]\\
&+E\left[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\}\frac{\partial}{\partial\theta}\left((S_{t_2^-}-K)e^{-rt_2}\right)\right],
\end{aligned}
$$

where we have defined $y_2^* = (s_2; S_{t_1^+}, \tau_2, \mu, \sigma)$. Considering the third term, we have

$$
\begin{aligned}
&E\left[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}(S_T-K)^+e^{-rT}\right]\\
&= E\left[\mathbf{1}\{h(Z_1;S_0,\tau_1) \le s_1, h(Z_2;S_{t_1^-}-D_1,\tau_2) \le s_2\}e^{-rT}(h(Z_3;S_{t_2^-}-D_2,\tau_3)-K)^+\right]\\
&= E\left[\int_{-\infty}^{h^{-1}(y_1^*)}\int_{-\infty}^{h^{-1}(y_2^*(x_1))}(h(Z_3;h(x_2;h(x_1;S_0,\tau_1)-D_1,\tau_2)-D_2,\tau_3)-K)^+e^{-rT}f_1(x_1)f_2(x_2)dx_2dx_1\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{\partial}{\partial\theta}E\left[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}(S_T-K)^+e^{-rT}\right]\\
&= E\left[\frac{\partial h^{-1}(y_1^*)}{\partial\theta} f_1(h^{-1}(y_1^*))E\left[\mathbf{1}\{S_{t_2^-} \le s_2\}(S_T-K)^+e^{-rT}\middle|S_{t_1^-}=s_1^-\right]\right]\\
&\quad+E\left[\frac{\partial h^{-1}(y_2^*)}{\partial\theta} f_2(h^{-1}(y_2^*))E\left[\mathbf{1}\{S_{t_1^-} \le s_1\}(S_T-K)^+e^{-rT}\middle|S_{t_2^-}=s_2^-\right]\right]\\
&\quad+E\left[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}\frac{\partial}{\partial\theta}\left((S_T-K)^+e^{-rT}\right)\right].
\end{aligned}
$$

Combining all these results, we obtain the PA estimator for the two dividends case:

$$
\begin{aligned}
&\frac{\partial h^{-1}(y_1^*)}{\partial\theta} f_1(h^{-1}(y_1^*))\left(E\left[L\middle|S_{t_1^-}=s_1^-\right]-(s_1-K)e^{-rt_1}\right)\\
&\quad+\mathbf{1}\{S_{t_1^-} \le s_1\}\frac{\partial h^{-1}(y_2^*)}{\partial\theta} f_2(h^{-1}(y_2^*))\\
&\qquad\times\left(E\left[(S_T-K)^+e^{-rT}\middle|S_{t_1^-} \le s_1, S_{t_2^-}=s_2^-\right]-(s_2-K)e^{-rt_2}\right)\\
&\qquad+\mathbf{1}\left\{S_{t_1^-} > s_1\right\}\frac{\partial}{\partial\theta}\left[(S_{t_1^-}-K)e^{-rt_1}\right]\\
&\qquad+\mathbf{1}\left\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\right\}\frac{\partial}{\partial\theta}\left[(S_{t_2^-}-K)e^{-rt_2}\right]\\
&\qquad+\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}\frac{\partial}{\partial\theta}\left[(S_T-K)^+e^{-rT}\right],
\end{aligned}
$$

which matches Equation (6) for $\eta(T)=2$, with the modification on y_i^* from (7) to (8).

For the general $\eta(T)$ dividends case, we have

$$
\begin{aligned}
&\frac{\partial}{\partial\theta}E\left[\mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} > s_i\right\}\left(S_{t_i^-}-K\right)e^{-rt_i}\right]\\
&= \sum_{j=1}^{i-1}E\left[\mathbf{1}\left\{\bigcap_{l=1}^{j-1} S_{t_l^-} \le s_l\right\}\frac{\partial h^{-1}(y_j^*)}{\partial\theta} f_j(h^{-1}(y_j^*))E\left[\mathbf{1}\left\{\bigcap_{m=j+1}^{i-1} S_{t_m^-} \le s_m, S_{t_i^-} > s_i\right\}(S_{t_i^-}-K)e^{-rt_i}\middle|S_{t_j^-}=s_j^-\right]\right]\\
&\quad-E\left[\mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j\right\}\frac{\partial h^{-1}(y_i^*)}{\partial\theta} f_i(h^{-1}(y_i^*))(s_i-K)e^{-rt_i}\right]
\end{aligned}
$$

$$+E\left[\mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} > s_i\right\}\frac{\partial}{\partial\theta}\left((S_{t_i^-} - K)e^{-rt_i}\right)\right], \quad i = 1, 2, ..., \eta(T),$$

and

$$\frac{\partial}{\partial\theta}E\left[\mathbf{1}\left\{\bigcap_{j=1}^{\eta(T)} S_{t_j^-} \le s_j\right\}(S_T - K)^+ e^{-rT}\right]$$
$$= \sum_{j=1}^{\eta(T)} E\left[\mathbf{1}\left\{\bigcap_{l=1}^{j-1} S_{t_l^-} \le s_l\right\}\frac{\partial h^{-1}(y_j^*)}{\partial\theta} f_j(h^{-1}(y_j^*))E\left[\mathbf{1}\left\{\bigcap_{m=j+1}^{\eta(T)} S_{t_m^-} \le s_m\right\}(S_T - K)^+ e^{-rT}\middle| S_{t_j^-} = s_j^-\right]\right]$$
$$+E\left[\mathbf{1}\left\{\bigcap_{j=1}^{\eta(T)} S_{t_j^-} \le s_j\right\}\frac{\partial}{\partial\theta}\left((S_T - K)^+ e^{-rT}\right)\right].$$

Combining these results leads to the PA estimator given by (6) for general $\eta(T)$, with the modification on y_i^* from (7) to (8).

To obtain the correct estimator for the original stock price model, the above derivations can be repeated with the following slight modifications. Define for notational convenience $D^{(i)} = \sum_{j=i}^{\eta(T)} D_j \exp\left(-r\sum_{k=i+1}^{j}\tau_k\right)$, and observing the following:

$$S_{t_i^-} > s_i \Longleftrightarrow \tilde{S}_{t_i} + D^{(i)} > s_i \Longleftrightarrow h(Z_i; \tilde{S}_{t_i}, \tau_i) > s_i - D^{(i)} \Longleftrightarrow Z_i > h^{-1}(s_i - D^{(i)}; \tilde{S}_{t_i}, \tau_i).$$

4 A Different Option

Another extension is the case where the stock pays continuous dividends at a rate δ, but early exercise is restricted to discrete points, say $\{t_i, i = 1, ..., \eta(T)\}$. The latter restriction makes this a Bermudan option. In this case, assume that the stock price net of the present value of escrowed dividends changes continuously according to

$$S_{t_j} = S_{t_j^-} = S_{t_j^+} = h(Z; S_{t_{j-1}}, \tau_j, \mu - \delta, \sigma), \quad j = 1, ..., \eta(T).$$

For this option, again the same estimator given by Equation (6) is applicable with the change in the definition of y_i^* given by (7) to the following:

$$y_i^* = (s_i; S_{t_{i-1}}, \tau_i, \mu - \delta, \sigma), \quad i = 1, ..., \eta(T).$$

Note that this stock model corresponds to a generalization of the first model, since the dividends are incorporated directly into the stock price dynamics.

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