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Derivative Estimation for Buffer Capacity of Continuous Transfer Lines Subject to Operation-Dependent Failures

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Abstract. We derive estimators of throughput sensitivity to changes in buffer capacity for continuous flow models of a transfer line comprising two machines separated by a buffer of finite capacity, where machines are subject to operation-dependent failures, i.e., a machine cannot fail when it is idle. Both repair times and failure times may be general, i.e., they need not be exponentially distributed. The system is hybrid in the sense that it has both continuous dynamics—as a result of continuous material flow—and discrete events: failures and repairs. The combination of operation-dependent failures and buffer capacity as the parameter of interest make the derivative estimation problem difficult, in that unlike previous work on continuous flow models, careful use of conditional expectation (i.e., smoothed perturbation analysis) is required to obtain unbiased estimators.

Keywords: transfer lines, continuous flow, perturbation analysis, operation-dependent failures

1. Introduction

A transfer line consists of a set of machines arranged in a serial configuration and separated by buffers. A part to be processed arrives to the first machine as raw material from outside the system. After being processed by the first machine, it queues in the first buffer, waiting to be processed by the second machine. It continues in this manner through all machines and reaches the inventory of finished products after being processed by the last machine. The rate at which a machine processes a part is called the machine's production rate. The performance of a transfer line is adversely affected by machine failures. While a machine is being repaired, it is unable to process parts, thus disrupting the flow of the transfer line. During this down time, the level of the machine's downstream buffer decreases while the level of its upstream buffer increases. If the repair takes a long time, then the downstream buffer may empty out—starving the downstream machine, and/or the upstream buffer may fill to capacity—blocking the upstream machine. In either case, the affected machine is said to be forced down.

Machine failures may be either operation dependent or time dependent. Operation-dependent failures can only occur while a machine is processing a part, whereas time-dependent failures can occur even if it is forced down. Both types of failures have been considered in the literature. Operation-dependent failures are commonly considered in performance analysis of production lines, and time-dependent failures are usually assumed in the flow control of failure-prone manufacturing systems. It is commonly believed that operation-dependent failures better model most real-life production systems, since equipment failures are usually related to usage. However, operation-dependent failures often lead to intractable models, so time-dependent failure are often used to simplify the analysis. Excellent literature surveys on performance evaluation of production lines can be found in Buzacott and Shanthikumar (1992) and Dallery and Gershwin (1992), though here only closely related work will be reviewed. We note here that for the most part little has been done for systems with generally distributed failure and repair times.

In this paper, we consider a continuous flow model of a two-machine transfer line subject to operation-dependent failures (see Figure 1). Because the distributions for failure and repair times are assumed to be general, the resulting model is not analytically tractable, so that performance evaluation requires simulation. The goal of this paper is to estimate the derivative of the throughput rate with respect to the buffer capacity. Such derivatives are important in transfer line design, which involves the allocation of buffer capacity. In closely related work, Suri and Fu (1994) proposed a generalized semi-Markov process (GSMP) model for representing the underlying stochastic process of a continuous production line (see also Plambeck et al., 1996), and derivative estimators with respect to maximal production rates were derived using infinitesimal perturbation analysis (IPA) by Shi et al. (1999). However, they do not consider the more difficult problem of derivative estimation with respect to buffer levels, where IPA fails, because certain event order changes cause significant jumps in the performance of the line. Using smoothed perturbation analysis (SPA), we derive unbiased estimators, providing detailed derivation and rigorous proofs of their correctness. For certain special cases, we provide simplification of the estimators that allow their implementation on a single sample path. It is worth noting that IPA does work in the setting presented here if failures are instead time dependent (see Xie, 1998).

The transfer line model considered in this paper is a piecewise deterministic control system (PDCA). Related work on PDCAs have been addressed by many authors, motivated primarily by the optimal control of manufacturing flow, but these all assume time-dependent failures, e.g., IPA-based stochastic approximation algorithms in a fairly general framework by Haurie et al. (1994); IPA-based flow controller design of manufacturing systems by Caramanis and Liberopoulos (1992); IPA derivative estimators for inventory cost with respect to control parameters for a two-machine production line with constant demand rate by Yan et al. (1994); and IPA derivative estimators for a single-machine/single-item production system having multiple machine states by Brémaud et al. (1997). IPA derivative estimators for loss measures in continuous flow models (without any explicit failure mechanism) of a single-queue system were derived by Wardi and Melamed (1996). Thus, there has been a large body of work in derivative estimation for various

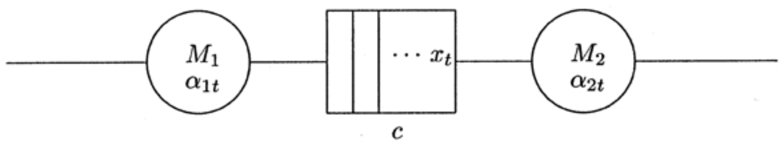

Figure 1. Two-machine transfer line with finite buffer.

continuous flow models of manufacturing systems, but this work all assumes time-dependent failures, for which IPA is applicable. Operation-dependent failures provide a more realistic model for transfer lines, and the only paper that considers derivative estimation for these types of models (Shi et al., 1999) considers only derivatives with respect to processing rates, where again IPA can be applied. In sum, our work is the first to handle the case of derivative estimation for buffer capacities for operation-dependent failures, and the difficulty of the problem necessitates the use of SPA. Somewhat related, in terms of the derivative estimation, is the use of SPA by Fu (1994) and Bashyam and Fu (1994) for inventory control models.

The rest of the paper is organized as follows. Notation and basic relations for continuous production lines are presented in Section 2. Section 3 presents the (biased) IPA estimators, whereas Section 4 presents the unbiased SPA estimators. In Section 5, numerical results for two simple examples are presented. Derivation of the estimators and sketches of the proofs are provided in the Appendix.

2. Notation and Basic Relations

We consider a production line composed of two machines M_1 and M_2 separated by a buffer of capacity c. The following events are possible: the failure of M_1, the repair of M_1, the failure of M_2, the repair of M_2, buffer full, and buffer empty, denoted respectively by $F1$, $R1$, $F2$, $R2$, BF, BE. We assume the synchronous case where the maximal production rate of both machines is the same, and without loss of generality assumed to be 1. As usual in the performance evaluation of production lines, we assume that the input buffer of the first machine is never empty and the output buffer of the second machine is never full. The following notation will be used throughout the paper (with $i = 1, 2$):

$X_{ik} =$ k-th time to failure of machine $M_i, \lambda_i = 1/E[X_{ik}]$,

$Y_{ik} =$ k-th time to repair of machine $M_i, \mu_i = 1/E[Y_{ik}]$,

$F_i(\text{resp. } G_i) =$ distribution function of X_{ik} (resp. Y_{ik}),

$f_i(\text{resp. } g_i) =$ density function of X_{ik} (resp. Y_{ik}),

$\alpha_{it} =$ state of machine M_i at time t; 1 if up and 0 otherwise, $\alpha_t = (\alpha_{1t}, \alpha_{2t})$,

$r_{it} =$ remaining lifetime (until failure or repair) of machine M_i in state α_{it} at time t^+,

$a_{it} =$ age (since last failure or repair) of machine M_i in state α_{it} at time t^+,

$P_{it} =$ cumulative production of M_i up to time t,

$x_t =$ buffer level at time t,

$e_k =$ k-th event $\in \{F1, R1, F2, R2, BF, BE\}$,

$t_k =$ epoch of event e_k,

$s_k =$ state of the system at time $t_k^+ = (\alpha_{1k}, \alpha_{2k}, r_{1k}, r_{2k}, x_k)$,

$\tau_k =$ time from e_{k-1} to e_k, i.e., $\tau_k = t_k - t_{k-1}$,

$\#(k, e) =$ number of occurrences of event e in $e_1 e_2 \ldots e_k$.

In the above notation, for the sake of simplicity, x_k is used to denote x_{t_k}. This will not lead to confusion, since throughout the paper only x_t denotes the inventory level at time t and, in all other cases, $x_\bullet$ denotes in fact $x_{t_\bullet}$. The same abuse of notation is often followed for α_{ik}, r_{ik}, a_{ik}, τ_k, and s_k.

We take the initial condition $\alpha_0 = (1, 1)$ and $x_0 = 0$, i.e., the system begins empty with both machines up. The performance measure considered in this paper is the throughput rate of the system:

$$L = \lim_{t\to\infty} \frac{P_{2t}}{t}$$

assumed to exist w.p. 1. Three finite-time estimators will be considered, all of which converge to L as $t \longrightarrow \infty$:

$$L_t = \frac{P_{2t}}{t}$$

$$L_n = \frac{P_{2t_{w(n)}}}{t_{w(n)}}$$

$$L_n = \frac{P_{2t_n}}{t_n}$$

where $e_{w(n)}$ is the n-th repair of M_2.

The dynamics of the system can be characterized similar to a GSMP model: starting from s_0, the next event e_1 is determined and the state of the system is updated as to be described. Then the system evolves in the same way starting from its new state s_1.

Machine M_1 is blocked in state $s_k = (\alpha_{1k}, \alpha_{2k}, r_{1k}, r_{2k}, x_k)$ if $\alpha_k = (1, 0)$ and $x_k = c$. Machine M_2 is starved in state s_k if $\alpha_k = (0, 1)$ and $x_k = 0$. In either case, the machine is said to be forced down, in which case the machine ceases production, it cannot break down, and its remaining time to failure r_{ik} remains unchanged as long as it is forced down.

The dynamics of the system are governed by the following equations.

- Time to state change of machine $M_i (i = 1, 2)$:

$$T_{ik} = \begin{cases} r_{ik}, & \text{if } M_i \text{ is not forced down in } s_k \\ \infty, & \text{otherwise} \end{cases}$$

- Time to buffer full event:

$$T_{Fk} = \begin{cases} c - x_k, & \text{if } \alpha_k = (1, 0) \wedge x_k < c \\ \infty, & \text{otherwise} \end{cases}$$

- Time to buffer empty event:

$$T_{Ek} = \begin{cases} x_k, & \text{if } \alpha_k = (0, 1) \wedge x_k > 0 \\ \infty, & \text{otherwise} \end{cases}$$

- Next event epoch $t_{k+1} = t_k + \tau_k$, with

$$\tau_k = \min\{T_{1k}, T_{2k}, T_{Fk}, T_{Ek}\}$$

- Next event:

$$e_k = \begin{cases} Fi, & \text{if } \tau_k = T_{ik} \wedge \alpha_{ik} = 1 \\ Ri, & \text{if } \tau_k = T_{ik} \wedge \alpha_{ik} = 0 \\ BF, & \text{if } \tau_k = T_{Fk} \\ BE, & \text{if } \tau_k = T_{Ek} \end{cases}$$

- Next state:

$$x_{k+1} = \begin{cases} x_{k,} & \text{if } M_1 \text{ or } M_2 \text{ is forced down in } s_k \\ x_k + (\alpha_{1k} - \alpha_{2k})\tau_k, & \text{otherwise} \end{cases}$$

$$\alpha_{1k+1} = \begin{cases} \alpha_{1k,} & \text{if } e_{k+1} \notin \{F1, R1\} \\ 1 - \alpha_{1k}, & \text{otherwise} \end{cases}$$

$$\alpha_{2k+1} = \begin{cases} \alpha_{2k,} & \text{if } e_{k+1} \notin \{F2, R2\} \\ 1 - \alpha_{2k}, & \text{otherwise} \end{cases}$$

$$r_{1k+1} = \begin{cases} Y_{1,\#(k+1,F1)}, & \text{if } e_{k+1} = F1 \\ X_{1,\#(k+1,R1)}, & \text{if } e_{k+1} = R1 \\ r_{1k}, & \text{if } M_1 \text{ is blocked in } s_k \\ r_{1k} - \tau_k, & \text{otherwise} \end{cases}$$

$$r_{2k+1} = \begin{cases} Y_{2,\#(k+1,F2)}, & \text{if } e_{k+1} = F2 \\ X_{2,\#(k+1,R2)}, & \text{if } e_{k+1} = R2 \\ r_{2k}, & \text{if } M_2 \text{ is starved in } s_k \\ r_{2k} - \tau_k, & \text{otherwise} \end{cases}$$

From the above equations, $e_{k+1} = R2$ if M_1 is blocked, and $e_{k+1} = R1$ if M_2 is starved. A typical sample path is shown in Figure 2.

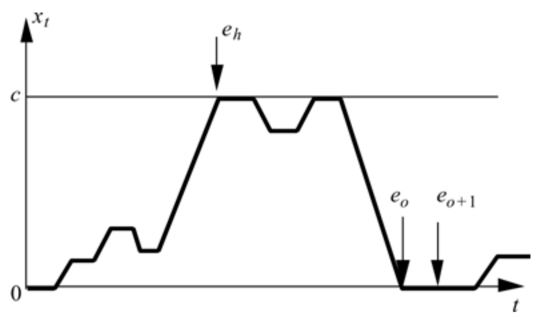

Figure 2. A typical sample path.

3. IPA Estimators

The IPA estimators corresponding to the three finite-horizon performance measures introduced in the previous section will be presented here, with the derivations provided in the Appendix. IPA estimators assume that the event sequence is unchanged with the introduction of a perturbation of size Δ, i.e.,

(A1) $e_1(c+\Delta) = e_1(c), \ldots, e_k(c+\Delta) = e_k(c)$.

For the estimator $L_n = P_{2t_{w(n)}}/t_{w(n)}$, assumption (A1) leads to

$$L_n(c+\Delta) - L_n(c) = \frac{L_n(c)\Delta}{t_{w(n)}(c)/Q_{w(n)} - \Delta}$$

where $w(n)$ is the n-th occurrence of $R2$ and Q_k is the number of cycles completed by the k-th event.

For the estimator $L_t = P_{2t}/t$, assume (A1) holds $1 \le k \le N(t)+1$, where $N(t)$ is the number of events up to time t, i.e., $N(t) = \sup\{k : t_k \le t\}$. Furthermore assume that the perturbation Δ also leaves $N(t)$ unchanged. Then the IPA estimator is given by

$$L_t(c+\Delta) - L_t(c) = \frac{\alpha_{2t}\gamma_{2t}}{t/Q_{N(t)}}\Delta + \frac{v_t}{t}\Delta$$

where $\gamma_{2t} = 1$ if machine M_2 is not starved at time t and $\gamma_{2t} = 0$ otherwise, and v_t is a random variable independent of Δ such that $-2 \le v_t \le 2$.

For the estimator $L_n = P_{2t_n}/t_n$, assumption (A1) leads to

$$L_n(c+\Delta) - L_n(c) = \frac{L_n(c)Q_n + v_t}{t_n(c+\Delta)}\Delta$$

where $t_n(c+\Delta) = t_n(c) - (Q_n + u_n)\Delta$, v_n and u_n are random variables such that $-2 \le v_n \le 2, -1 \le u_n \le 1$.

By taking the limit of the above derivative estimators, we obtain:

$$IPA1 \triangleq \lim_{n\to\infty}\lim_{\Delta\to 0}\frac{L_n(c+\Delta) - L_n(c)}{\Delta} = \frac{L(c)}{C}\,\text{w.p.1}$$

$$IPA2 \triangleq \lim_{t\to\infty}\lim_{\Delta\to 0}\frac{L_t(c+\Delta) - L_t(c)}{\Delta} = \frac{\alpha_{2\infty}\gamma_{2\infty}}{C}\quad\text{in distribution}$$

$$IPA3 \triangleq \lim_{n\to\infty}\lim_{\Delta\to 0}\frac{L_n(c+\Delta) - L_n(c)}{\Delta} = \frac{L(c)}{C}\,\text{w.p.1}$$

where C is the average length of the cycles and $\alpha_{2\infty}$ and $\gamma_{2\infty}$ correspond to the obvious limits of α_{2n} and γ_{2n}, respectively. Note that $IPA1$ and $IPA3$ are constants, whereas $IPA2$ is a random variable. Since $P(\alpha_{2\infty}\gamma_{2\infty} = 0) > 0$, if a long simulation is performed to estimate *IPA2*, we can expect that *IPA2* is sometimes equal to 0 and sometimes equal to $1/C$ and the related estimator does not converge to a constant. However, since $\alpha_{2t}\gamma_{2t} = 1$ if and only if machine M_2 is producing, $P(\alpha_{2\infty}\gamma_{2\infty} = 1)$ is equal to the throughput rate of the system, so

$$E[IPA2] = \frac{P(\alpha_{2\infty}\gamma_{2\infty} = 1)}{C} = \frac{L}{C} = IPA1 = IPA3$$

Unfortunately, because event changes that occur when assumption (A1) is violated can lead to large (non-infinitesimal) jumps in the performance measure (as confirmed by the numerical simulation experiments described in Section 5), *IPA1* and *IPA3* are not strongly consistent, i.e., $IPA1 = IPA3 \neq dL/dc$. This means that event changes need to be considered in order to obtain unbiased derivative estimators, which leads to the SPA estimators considered in the following section.

4. SPA Estimators

We present SPA estimators for the performance measure $L_t = P_{2t}/t$. Estimators for the other performance measures can be found in an analogous manner. In the Appendix, the following finite-horizon SPA estimator is derived:

$$\frac{\alpha_{2t}\gamma_{2t}}{t/Q_{N(t)}} + \frac{v_t}{t} + \sum_{k \in BFS} \left(L^{PP,k} - L^{DNP,k}\right) \frac{f_1(a_{1k})}{1 - F_1(a_{1k})} \tag{1}$$

where $BFS = \{k : \alpha_k = (1,0) \wedge x_k = c\}$ is the set of blocking states, L^{DNP} is the performance measure of a so-called degenerated nominal path denoted by DNP, L^{PP} is the performance measure of a so-called perturbed path, denoted by PP. PP is identical to DNP up to time, say t_k, where an event change occurs. At this instant, an event occurs on both nominal and perturbed paths; however, the event on the nominal path differs from the one on the perturbed path. Note that $t/Q_{N(t)}$ is the average length of an operation cycle defined in Section 3, and $\alpha_{2t}\gamma_{2t} = 1$ if and only if machine M_2 is producing at time t.

To simplify the estimator, we next consider the infinite horizon case.

4.1 Regenerative Case

We assume that the steady state exists; more precisely, we make the following assumptions:

(A2) There exists a finite $C > 0$ such that $\lim_{t\to\infty} t/Q_{N(t)} = C$ w.p.1.

(A3) $\lim_{t\to\infty} E[\alpha_{2t}\gamma_{2t}] = L$ with $L = \lim_{t\to\infty} P_{2t}/t$ w.p.1.

LEMMA 1. *Under assumptions (A2)–(A3),* $\lim_{t\to\infty} E[dL_t(c)/dc] = L/C$.

Proof: From the IPA term in (1),

$$E\left[\frac{dL_t(c)}{dc}\right] = \frac{E[\alpha_{2t}\gamma_{2t}]}{C} + E\left[\alpha_{2t}\gamma_{2t}\left(\frac{Q_{N(t)}}{t} - \frac{1}{C}\right)\right] + E\left[\frac{v_t}{t}\right]$$

Since $|v_t| \leq 2, E[v_t/t]\to 0$. From assumption (A2), $w_t = \alpha_{2t}\gamma_{2t}(Q_{N(t)}/t - 1/C)\to 0$ with probability 1. Since $\alpha_{2t}\gamma_{2t} \leq 1$, $|w_t| \leq Q_{N(t)}/t + 1/C$. From the definition of a cycle, it takes at least c time units to go from empty to full and c time units to go from full to empty, $Q_{N(t)}/t \leq 1/2c$, which leads to $|w_t| \leq 1/2c + 1/C$. Applying the dominated convergence theorem yields $E[w_t]\to 0$. The lemma is then established by combining the above results and assumption (A3). ■

From this result, $L_t/(t/Q_{N(t)})$ is a strongly consistent estimator of $\lim_{t\to\infty} E[dL_t(c)/dc]$. This leads to the following strongly consistent estimator of $dL(c)/dc$:

$$\frac{L_t}{t/Q_{N(t)}} + \sum_{k\in BFS}\left(L^{PP,k} - L^{DNP,k}\right)\frac{f_1(a_{1k})}{1 - F_1(a_{1k})} \tag{2}$$

A rigorous proof of strong consistency, though not carried out explicitly here, can be established under the regenerative assumption below along the lines used in Fu and Hu (1997), since we will show that

$$\lim_{t\to\infty}\left(L^{PP,k} - L^{DNP,k}\right)$$

is a function of r_{2k} only, so the estimator (asymptotically) depends only on a_{1k} and r_{2k} within a regenerative cycle. This regenerative property of the estimator, along with some technical assumptions, are the main conditions required in such a proof.

The main difficulty in estimating (2) using simulation is the computation of $L^{PP,k} - L^{DNP,k}$. Its evaluation for the general case is still an open issue. In the following we consider the regenerative case and make the following assumption:

(A4) The underlying stochastic process of the two-machine line has a regeneration point **sr**.

The points A, B, C and D of Figure 3 are possible regeneration points depending on the distribution of times to failure. If the time to failure X_1 of machine M_1 has a phase-type distribution, the points A and D can be used to define regeneration points by extending the definition of a machine state to include the phase. Similarly, if the time to failure X_2 of machine M_2 has a phase-type distribution, the points B and C can be used to defined regeneration points. Of course, A and D are regeneration points if X_1 is exponentially distributed, and B and C are regeneration points if X_2 is exponentially distributed.

The estimation of the derivative estimator requires the estimation of $L^{PP,k} - L^{DNP,k}$. We recall the relationship between PP_k and DNP_k. PP_k is identical to DNP_k up to time t_{k-} where an event change occurs. At this instant, machine M_1 is blocked in the nominal

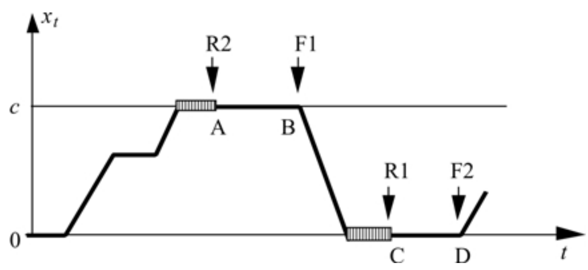

Figure 3. Possible regeneration points.

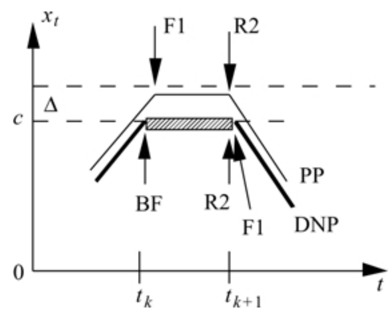

Figure 4. M_1 still fails at t_{k+1} in PP.

system, whereas it breaks down in the perturbed system. Clearly, M_1 immediately fails in DNP_k following the repair of M_2 at time t_{k+1}. Furthermore, $x_t = c$ at time $t = t_{k+1}$ for both PP_k and DNP_k. Two cases are possible concerning the state of machine M_1 in PP at t_{k+1} (from DNP): (i) M_1 still failed at t_{k+1} or (ii) it is repaired at t_{k+1} (see Figures 4 and 5). Case (i) occurs w.p. $\overline{G_1}(r_{2k})$ and case (ii) occurs w.p. $G_1(r_{2k})$. Finally, since $e_k = BF$ in both DNP_k and NP, then $e_{k+1} = R2$ in both DNP_k and NP. As a result, t_{k+1} is identical for both DNP_k and NP, hence r_{2k} can be taken form NP.

To summarize, the inventory trajectory and the cumulative production are identical for

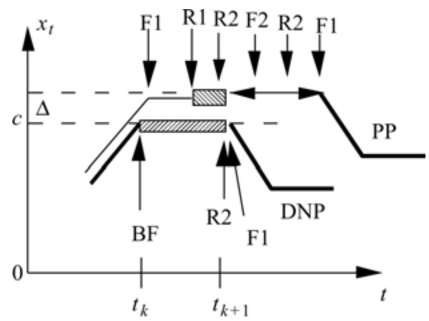

Figure 5. M_1 is repaired at t_{k+1} in PP.

Figure 6. Construction of PP and DNP.

both PP_k and DNP_k up to time $t = t_{k+1}$ (from NP). At this point, $x_t = c$ and M_2 is just repaired in both PP_k and DNP_k. M_1 breaks down in DNP_k while it is under repair with age r_{2k} with probability $\overline{G_1}(r_{2k})$ and is just repaired with probability $G_1(r_{2k})$ in PP_k.

In order to estimate $L^{PP,k} - L^{DNP,k}$, let us consider the following construction of PP_k and DNP_k (see Figure 6). For DNP_k, a piece of sample path I is inserted at time t_{k+1} by starting with appropriate initial state defined above and this piece of sample path stops when the regeneration point **sr** is met. Similarly, for PP_k, a piece of sample path II is inserted at time t_{k+1} until the regeneration point **sr** is met. From there on, both DNP_k and PP_k pursue the same sample path III.

Let

$t_I^{PP,k}, P_I^{PP,k}$: length and cumulative production of sample path I,
$t_{II}^{DNP,k}, P_{II}^{DNP,k}$: length and cumulative production of sample path II,
$t_{III} = t - t_{k+1}$, P_{III}: length and cumulative production of sample path III.

Under this construction, we have for large t,

$$L_t^{PP,k} \approx L_{t+t_{II}^{PP,k}}^{PP,k} = \frac{P_{2,t+t_{II}^{PP,k}}^{PP,k}}{t + t_{II}^{PP,k}} = \frac{P_{2,t_{k+1}} + P_{II}^{PP,k} + P_{III}}{t + t_{II}^{PP,k}}$$
$$L_t^{DNP,k} \approx L_{t+t_I^{DNP,k}}^{DNP,k} = \frac{P_{2,t+t_I^{DNP,k}}^{DNP,k}}{t + t_I^{DNP,k}} = \frac{P_{2,t_{k+1}} + P_I^{DNP,k} + P_{III}}{t + t_I^{DNP,k}}$$

leading to

$$\begin{aligned} L_t^{PP,k} - L_t^{DNP,k} &\approx \frac{t_I^{DNP,k} P_{2,t+t_{II}^{PP,k}}^{PP,k} - t_{II}^{PP,k} P_{2,t+t_I^{DNP,k}}^{DNP,k} + t\left(P_{II}^{PP,k} - P_I^{DNP,k}\right)}{\left(t + t_{II}^{PP,k}\right)\left(t + t_I^{DNP,k}\right)} \\ &= \frac{t_I^{DNP,k} L_{t+t_{II}^{PP,k}}^{PP,k}}{t + t_I^{DNP,k}} - \frac{t_{II}^{PP,k} L_{t+t_I^{DNP,k}}^{DNP,k}}{t + t_{II}^{PP,k}} + t\frac{P_{II}^{PP,k} - P_I^{DNP,k}}{\left(t + t_{II}^{PP,k}\right)\left(t + t_I^{DNP,k}\right)} \\ &\approx \frac{L_t}{t}\left(t_I^{DNP,k} + t_{II}^{PP,k}\right) + \frac{1}{t}\left(P_{II}^{PP,k} - P_I^{DNP,k}\right) \end{aligned}$$

Therefore, we take the following as our long-run derivative estimator:

$$\mathrm{SPA0} = \frac{L_t}{t/Q_{N(t)}} + \frac{L_t}{t} \sum_{k \in BFS} \frac{f_1(a_{1k})}{1 - F_1(a_{1k})} \left(t_I^{DNP,k} - t_{II}^{PP,k} \right) + \frac{1}{t} \sum_{k \in BFS} \frac{f_1(a_{1k})}{1 - F_1(a_{1k})} \left(P_{II}^{PP,k} - P_I^{DNP,k} \right) \tag{3}$$

When computing the above estimator using simulation, all terms except the two summations can be evaluated easily. Whenever an event e_k leading to a blocking state, extra simulation is performed to construct sample paths I and II as described above. We then update these two summations. The estimator (3) can be obtained at the end of the simulation accordingly.

4.2. A Particular Case

In this subsection, we assume that Y_{1k} and X_{2k} are exponentially distributed and derive strongly consistent derivative estimators computable without extra simulation. As shown above, the estimation of $L^{PP,k} - L^{DNP,k}$ requires the consideration of two cases : (i) M_1 still fails at t_{k+1} or (ii) it is repaired at t_{k+1}. Case (i) occurs w.p. $\overline{G_1}(r_{2k})$ and case (ii) occurs w.p. $G_1(r_{2k})$. By using different construction of sample paths, we prove in the following that

$$L^{PP,k} - L^{DNP,k} = 0$$

for case (i) and

$$L^{PP,k} - L^{DNP,k} \approx \frac{1 - L_t/E_2}{\lambda_1 t}$$

for case (ii), where E_2 is the isolated average throughput rate of M_2 given by $E_2 = \mu_2/(\lambda_2 + \mu_2)$. Our final derivative estimator for dL/dc is the following:

$$\mathrm{SPA1} = \frac{L_t}{t/Q_{N(t)}} + \frac{1 - L_t/E_2}{\lambda_1 t} \sum_{k \in BFS} G_1(r_{2k}) \frac{f_1(a_{1k})}{1 - F_1(a_{1k})} \tag{4}$$

Of course, if X_{1k} are exponentially distributed as well, then $f_1(x)/(1 - F_1(x)) = \lambda_1$ for all $x \geq 0$, and the derivative estimator becomes:

$$\frac{L_t}{t/Q_{N(t)}} + \frac{1 - L_t/E_2}{t} \sum_{k \in BFS} G_1(r_{2k})$$

If all random variables are exponentially distributed, r_{2k} are exponentially distributed with mean equal to $E[Y_{2k}]$. Hence, case (ii) occurs w.p. $\mu_1/(\mu_1 + \mu_2)$ and the derivative estimator becomes:

$$\mathrm{SPA2} = \frac{L_t}{t/Q_{N(t)}} + (1 - L_t/E_2)\frac{\mu_1}{\mu_1 + \mu_2}\frac{\#(t, BF)}{t} \tag{5}$$

where $\#(t, BF)$ denotes the number of blocking states up to time t.

Let us consider now the estimation of $L^{PP,k} - L^{DNP,k}$. Appropriate sample path constructions will be used, and Figures 4 and 5 are helpful for understanding what follows. If M_1 still fails at t_{k+1} in PP_k, the states of the machines and the buffer level are the same at t_{k+1} in both DNP_k and PP_k. We construct the portion of sample path following t_{k+1} as follows. At time t_{k+1}, new samples of Y_{1k} and X_{2k} is generated for setting the time to repair of M_1 and the time to failure of M_2 in DNP_k. At this point, we also use the same samples to reset the time to repair of M_1 and to set the time to failure of M_2 in PP_k. As a result, the state of the system at time t^+_{k+1} is the same in PP_k and DNP_k. Let us notice that resetting the time to repair of M_1 in PP_k is possible due to the exponential distribution of Y_{1k}. As a result of above construction, we have: $L^{PP,k} - L^{DNP,k} = 0$.

If M_1 is repaired at t_{k+1} in PP_k, then an independent portion of sample path is inserted in PP_k until machine M_1 fails. At this point, the portion of DNP_k following t^+_{k+1} is added to construct PP_k. Clearly the correctness of this construction is due to the exponential distribution of X_{2k}. For the inserted portion, let $\tilde{X}_1$ be time to failure of M_1, N be the number of failures of M_2 and $\tilde{Y}_{2k}$ the related times to repair. Let T be the length of the inserted portion. Since in the inserted portion, M_1 is blocked whenever M_2 fails. Thus,

$$T = \tilde{X}_1 + \sum_{k=1}^{N} \tilde{Y}_{2k}$$

and the production of M_2 during the inserted portion is equal to $\tilde{X}_1$. As a result, for large t,

$$L_t^{PP} - L_t^{DNP} \approx L_{t+T}^{PP} - L_t^{DNP} = \frac{P_{2t}^{DNP} + \tilde{X}_1}{t+T} - \frac{P_{2t}^{DNP}}{t} = \frac{\tilde{X}_1 - TL_t^{DNP}}{t+T} \approx \frac{\tilde{X}_1 - TL_t}{t}$$

By taking expectation with respect to random variables of the inserted portion, we have

$$E_{\tilde{X},\tilde{Y}}\left[L_t^{PP} - L_t^{DNP}\right] \approx \frac{E[\tilde{X}_1] - E[T]L_t}{t}$$

Notice that $\{\tilde{Y}_{2k}\}$ are independent of N. Hence,

$$E[T] = E\left[\tilde{X}_1\right] + E\left[\sum_{k=1}^{N} \tilde{Y}_{2k}\right] = E\left[\tilde{X}_1\right] + E[N]E\left[\tilde{Y}_{2k}\right]$$

Since $\{X_{2k}\}$ are exponentially distributed, N follows a Poisson distribution, so:

$$E[N] = E\left[E\left[N|\widetilde{X}_1\right]\right] = E\left[\lambda_2\widetilde{X}_1\right] = \lambda_2 E\left[\widetilde{X}_1\right]$$

The combination of the above results gives:

$$E_{\widetilde{X},\widetilde{Y}}\left[L_t^{PP} - L_t^{DNP}\right] \approx \frac{1 - L_t/E_2}{\lambda_1 t}$$

5. Numerical Results

We compared the numerical properties of the various estimators by performing simulation experiments on two examples. The biased IPA estimator and the three SPA estimators—SPA0 given by (3), SPA1 given by (4), SPA2 given by (5)—are compared with symmetric difference (SD) estimates using common random numbers and $\Delta c = 0.05$. In all three cases, sample means and 95% confidence half-widths based on 20 independent replications are calculated.

Example 1: All random variables are exponentially distributed with $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. Analytical results are available, which can be used to assess the convergence of the various estimators. The throughput rate can be found in Dallery and Gershwin (1992) and is equal to

$$L = \left(2 + (1 + c)^{-1}\right)^{-1}$$

leading to $dL/dc = (3 + 2c)^{-2}$. The simulation results are summarized in Table 1, and they confirm that the IPA estimator is biased, whereas all three SPA estimators converge to the correct value. SPA2 and SPA1 seem to have a similar convergence rate that is substantially faster than SPA0, the estimator based on regenerative analysis. The SD estimates fare worse than SPA1/SPA2, but slightly better than SPA0.

Example 2: In this example, Y_{1k} and X_{2k} are exponentially distributed, X_{1k} and Y_{2k} have two-stage Erlang distributions. All random variables have mean equal to one, i.e., $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. Only the case $c = 1$ is considered, and the results are shown in Table 2. For this example, an analytical solution is not available. SPA0 and SPA1 appear to converge to the same limiting value that is consistent with the SD estimates, whereas the results indicate that SPA2 is biased for this example, which is not surprising, since not all the random variables are exponentially distributed. Again, the convergence rate of SPA0 appears to be substantially slower than that of SPA1, and the SD estimates fall between the two.

In both of these examples, it appears that the value of the IPA estimator is approximately half of the SPA estimators. Unfortunately, this is not true in general. For the line of Example 2, with $E[X_{1k}] = 0.5$, results for t = 100,000 yielded an IPA estimate of 0.01078 (0.00006) and an SPA1 estimate of 0.02691 (0.00019), which is considerably more than twice the value of the IPA estimate.

Table 1. Simulation results for Example 1 (95% confidence half-widths in parentheses).

c	t	L_t	L	dL/dc	IPA	$SPA2$	$SPA1$	$SPA0$	SD
0.5	1,000	0.3722	0.375	0.0625	0.0304	0.0623	0.0625	0.0658	0.0573
		(0.0045)			(0.0010)	(0.0017)	(0.0018)	(0.0071)	(0.0075)
	10,000	0.3740	0.375	0.0625	0.031	0.0624	0.0625	0.0638	0.0625
		(0.0014)			(0.0003)	(0.0004)	(0.0005)	(0.0034)	(0.0028)
	100,000	0.3749	0.375	0.0625	0.0312	0.0625	0.0626	0.0629	0.0620
		(0.0005)			(0.0001)	(0.0001)	(0.0001)	(0.0009)	(0.0010)
	1,000,000	0.3751	0.375	0.0625	0.0313	0.0625	0.0625	0.0624	0.0625
		(0.0001)			(0.00003)	(0.00005)	(0.00004)	(0.0003)	(0.0002)
1	1,000	0.4000	0.4	0.04	0.0197	0.0390	0.0393	0.0432	0.0366
		(0.0050)			(0.0008)	(0.0015)	(0.0015)	(0.0105)	(0.0059)
	10,000	0.4001	0.4	0.04	0.0200	0.0399	0.0399	0.0387	0.0386
		(0.0017)			(0.0003)	(0.0004)	(0.0004)	(0.0033)	(0.0024)
	100,000	0.4002	0.4	0.04	0.0200	0.0399	0.0399	0.0402	0.0404
		(0.0005)			(0.00006)	(0.0002)	(0.00017)	(0.0008)	(0.0006)
	1,000,000	0.3999	0.4	0.04	0.0200	0.0400	0.0400	0.0399	0.0400
		(0.0001)			(0.00003)	(0.00004)	(0.00004)	(0.0002)	(0.0002)
2	1000	0.4276	0.4286	0.02041	0.01028	0.02081	0.02090	0.02582	0.02114
		(0.0048)			(0.00064)	(0.00110)	(0.00117)	(0.01366)	(0.00581)
	10,000	0.4274	0.4286	0.02041	0.01022	0.02076	0.02079	0.01980	0.02030
		(0.0016)			(0.00020)	(0.00041)	(0.00043)	(0.00309)	(0.00153)
	100,000	0.4284	0.4286	0.02041	0.01018	0.02040	0.02039	0.02016	0.02031
		(0.0005)			(0.00005)	(0.00010)	(0.00010)	(0.00124)	(0.00042)
	1,000,000	0.4285	0.4286	0.02041	0.01021	0.02043	0.02043	0.02063	0.02038
		(0.0002)			(0.00001)	(0.00003)	(0.00003)	(0.00047)	(0.00019)

Table 2. Simulation results for Example 2 (95% confidence half-widths in parentheses).

t	L_t	IPA	$SPA2$	$SPA1$	$SPA0$	SD
1,000	0.4090	0.02120	0.04047	0.04308	0.04748	0.04046
	(0.0040)	(0.00076)	(0.00119)	(0.00146)	(0.01112)	(0.00512)
10,000	0.4104	0.02148	0.04021	0.04295	0.04228	0.04447
	(0.0010)	(0.00015)	(0.00026)	(0.00028)	(0.00271)	(0.00200)
100,000	0.4101	0.02155	0.04041	0.04308	0.04290	0.04318
	(0.0004)	(0.00008)	(0.00008)	(0.00009)	(0.00092)	(0.00048)
1,000,000	0.4100	0.02153	0.04036	0.04304	0.04319	0.04303
	(0.0001)	(0.00003)	(0.00005)	(0.00006)	(0.00030)	(0.00017)

Appendix

IPA

For the purpose of perturbation analysis, we compare the sample path of the system having buffer capacity c, called the nominal system, with that of the system having buffer capacity $c + \Delta$, $\Delta > 0$, called the perturbed system. As usual in IPA, we assume that the event sequence is identical for both systems up to the k-th event, i.e., assumption (A1), under which the following holds:

$$\alpha_k(c+\Delta) = \alpha_k(c),$$
$$a_{ik}(c+\Delta) + r_{ik}(c+\Delta) = a_{ik}(c) + r_{ik}(c), \quad \forall i = 1,2 \tag{6}$$

More relations between the nominal and perturbed systems can be obtained by detailed sample path analysis. For this purpose, consider the sample path of Figure 2 and let

$$H = \min\{n \geq 1 : e_n = BF\}$$
$$O = \min\{n \geq H : e_n = BE\}$$

Clearly H and O define an operation cycle of the system. At event e_{O+1}, both machines are up and the buffer is empty, and the system starts a similar cycle as at time 0. Hence it is natural to first conduct the perturbation analysis for the first cycle, and then extend the results to the other cycles. However, we note that except for the case where times to failure of machine M_2 are exponential, the points are not in fact regenerative points. Five cases are considered for the first cycle.

THEOREM 1 *Under assumption (A1), the following hold:*

- *If* $k < H : t_k(c+\Delta) = t_k(c), x_k(c+\Delta) = x_k(c), r_{ik}(c+\Delta) = r_{ik}(c), \forall i = 1,2.$
- *If* $k = H : t_k(c+\Delta) = t_k(c) + \Delta, x_k(c+\Delta) = x_k(c) + \Delta, r_{ik}(c+\Delta) = r_{ik}(c) - \Delta, \forall i = 1,2.$
- *If* $H < k < O$ *and* $e_k \in \{F2, R2\} : t_k(c+\Delta) = t_k(c), x_k(c+\Delta) = x_k(c) + \alpha_{1k}\Delta, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c).$
- *If* $H < k < O$ *and* $e_k = BF : t_k(c+\Delta) = t_k(c), x_k(c+\Delta) = x_k(c) + \Delta, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c).$
- *If* $H < k < O$ *and* $e_k \in \{F1, R1\} : t_k(c+\Delta) = t_k(c) - \Delta, x_k(c+\Delta) = x_k(c) + \alpha_{2k}\Delta, r_{1k}(c+\Delta) = r_{1k}(c), r_{2k}(c+\Delta) = r_{2k}(c) + \Delta.$
- *If* $k = O : t_k(c+\Delta) = t_k(c), x_k(c+\Delta) = x_k(c) = 0, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c).$
- *If* $k = O+1 : t_k(c+\Delta) = t_k(c) - \Delta, x_k(c+\Delta) = x_k(c) = 0, r_{1k}(c+\Delta) = r_{1k}(c), r_{2k}(c+\Delta) = r_{2k}(c).$

Proof: Case $0 < k < H$ is obvious, since from $t = 0$ to t_H, the buffer level is always below H, so the sample paths of the nominal and perturbed systems are identical. Case $k = H$ is a trivial consequence of case $0 < k < H$. Consider the case $k = O$. In this case, $\alpha_{k-1} = \alpha_k = (0,1)$ and $e_{k-1} \in \{F1, R2\}$. Only the proof for $e_{k-1} = F1$ is given, since that for $e_{k-1} = R2$ is similar. If $e_{k-1} = F1$, then $t_{k-1}(c+\Delta) = t_{k-1}(c) - \Delta$ and $x_{k-1}(c+\Delta) = x_{k-1}(c) + \Delta$. As a result, $t_k(c+\Delta) = t_{k-1}(c+\Delta) + x_{k-1}(c+\Delta) =$

$t_{k-1}(c) + x_{k-1}(c) = t_k(c)$, i.e., $t_k(c+\Delta) - t_{k-1}(c+\Delta) = t_k(c) - t_{k-1}(c) + \Delta$. Hence, $r_{1k}(c+\Delta) = r_{1k}(c) - \Delta$, $r_{2k}(c+\Delta) = r_{2k}(c)$. Finally, the result for the case $k = O+1$ is a trivial consequence of the case $k = O$, since for $k = O+1$, under assumption (A1), $e_k = R1$, and machine M_2 is starved from e_{k-1} to e_k. The proof of results concerning cases $H < k < O$ is similar but more complex. Details can be found in Fu and Xie (1998). ■

It should be noted that at the occurrence of e_{O+1}, the sample path of the perturbed system can be derived from the one of the nominal system as above, and the only difference is that the event epochs $t_k(c+\Delta)$ are further shifted leftward by Δ. To generalize the results, we decompose the sample path into cycles as follows:

$$H_m = \min\{n = O_{m-1} : e_n = BF\}$$
$$O_m = \min\{n = H_m : e_n = BE\}$$

where $O_0 = 0$. Clearly, $H_m > O_{m-1} + 1$ and $O_m > H_m + 1$.

THEOREM 2 *Assume that assumption (A1) holds and that e_k is an event of cycle $m+1$ with $m \geq 0$, i.e., $O_m + 1 < k \leq O_{m+1} + 1$. Then the following hold:*

- *If* $k < H_{m+1} : t_k(c+\Delta) = t_k(c) - m\Delta, x_k(c+\Delta) = x_k(c), r_{ik}(c+\Delta) = r_{ik}(c), \forall i = 1, 2$.
- *If* $k = H_{m+1}$ *and* $e_k = BF : t_k(c+\Delta) = t_k(c) - (m-1)\Delta, x_k(c+\Delta) = x_k(c) + \Delta, r_{ik}(c+\Delta) = r_{ik}(c) - \Delta, \forall i = 1, 2$.
- *If* $H_{m+1} < k < O_{m+1}$ *and* $e_k \in \{F2, R2\} : t_k(c+\Delta) = t_k(c) - m\Delta, x_k(c+\Delta) = x_k(c) + \alpha_{1k}\Delta, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c)$.
- *If* $H_{m+1} < k < O_{m+1}$ *and* $e_k = BF : t_k(c+\Delta) = t_k(c) - m\Delta, x_k(c+\Delta) = x_k(c) + \Delta, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c)$,
- *If* $H_{m+1} < k < O_{m+1}$ *and* $e_k \in \{F1, R1\} : t_k(c+\Delta) = t_k(c) - (m+1)\Delta, x_k(c+\Delta) = x_k(c) + \alpha_{2k}\Delta, r_{1k}(c+\Delta) = r_{1k}(c), r_{2k}(c+\Delta) = r_{2k}(c) + \Delta$.
- *If* $k = O_{m+1}$ *and* $e_k = BE : t_k(c+\Delta) = t_k(c) - m\Delta, x_k(c+\Delta) = x_k(c) = 0, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c)$.
- *If* $k = O_{m+1} + 1$ *and* $e_k = R1 : t_k(c+\Delta) = t_k(c) - (m+1)\Delta, x_k(c+\Delta) = x_k(c) = 0, r_{1k}(c+\Delta) = r_{1k}(c), r_{2k}(c+\Delta) = r_{2k}(c)$.

From the above results, we can derive the sensitivities of the performance measures.

THEOREM 3 *Consider the estimator $L_n = P_{2t_{w(n)}}/t_{w(n)}$. Assume that assumption (A1) holds $1 \leq k \leq w(n)$, where $w(n)$ is the n-th occurrence of R2. Then,*

$$L_n(c+\Delta) - L_n(c) = \frac{L_n(c)\Delta}{t_{w(n)}(c)/Q_{w(n)} - \Delta}$$

where Q_k is the number of cycles defined above completed by the k-th event.

Proof: From the definition of $w(n)$, we have:

$$P_{2t_{w(n)}}(c+\Delta) = P_{2t_{w(n)}}(c) = \sum_{k=1}^{n} X_{2k}$$

where X_{2k} for $1 \le k \le n$ are times to failure of machine M_2. From Theorem 2,

$$t_{w(n)}(c+\Delta) = t_{w(n)}(c) - Q_n\Delta$$

The combination of the two relations gives:

$$L_n(c+\Delta) - L_n(c) = \frac{P_{2t_{w(n)}}(c)}{t_{w(n)}(c) - Q_n\Delta} - \frac{P_{2t_{w(n)}}(c)}{t_{w(n)}(c)} = \frac{P_{2t_{w(n)}}(c)Q_n\Delta}{t_{w(n)}(c)(t_{w(n)}(c) - Q_n\Delta)}$$
$$= \frac{L_n(c)\Delta}{t_{w(n)}(c)/Q_{w(n)} - \Delta}$$

THEOREM 4 *Consider the estimator $L_t = P_{2t}/t$. Assume that assumption (A1) holds $1 \le k \le N(t)+1$, where $N(t)$ is the number of events up to time t, i.e., $N(t) = \sup\{k : t_k \le t\}$. Further, assume that $N(t)$ is the same for both perturbed and nominal systems. Then,*

$$L_t(c+\Delta) - L_t(c) = \frac{\alpha_{2t}\gamma_{2t}}{t/Q_{N(t)}}\Delta + \frac{v_t}{t}\Delta$$

where $\gamma_{2t} = 1$ if machine M_2 is not starved at time t and $\gamma_{2t} = 0$ otherwise, and v_t is a random variable independent of Δ such that $-2 \le v_t \le 2$.

Proof: Clearly,

$$P_{2t} = \sum_{k=1}^{\#(N(t),F2)} X_{2k} + \alpha_{2t}(a_{2N(t)} + \gamma_{2t}(t - t_{N(t)}))$$

We note that γ_{2t} can be derived from the event sequence up to $e_{N(t)}$ as follows:

$$\gamma_{2t} = \begin{cases} \alpha_{1k} & \text{if } \exists n \le N(t) \text{ such that } e_n = BE \text{ and } e_k \in \{F1, R1\}, n < k \le N(t) \\ 1 & \text{otherwise} \end{cases}$$

As a result, under the condition of the theorem, $\alpha_{2t}(c+\Delta) = \alpha_{2t}(c)$ and $\gamma_{2t}(c+\Delta) = \gamma_{2t}(c)$. Therefore,

$$P_{2t}(c+\Delta) - P_{2t}(c) = \alpha_{2t}(a_{2N(t)}(c+\Delta) - a_{2N(t)}(c)) - \alpha_{2t}\gamma_{2t}(t_{N(t)}(c+\Delta) - t_{N(t)}(c))$$

Let

$$\begin{aligned} t_k(c+\Delta) &= t_k(c) - Q_k\Delta + \xi_{1k}\Delta \\ r_{2k}(c+\Delta) &= r_2(c) + \xi_{2k}\Delta \end{aligned}$$

From Theorem 2, $\xi_{1k}, \xi_{2k} \in \{1, 0, -1\}$ depend only on the sequence of events up to e_k. Hence $a_{2N(t)}(c+\Delta) = a_{2N(t)}(c) - \xi_{2N(t)}\Delta$ and $t_{N(t)}(c+\Delta) = t_{N(t)}(c) - Q_{N(t)}\Delta - \xi_{1N(t)}\Delta$. Let $v_t = -\alpha_{2t}\xi_{2N(t)} - \alpha_{2t}\gamma_{2t}\xi_{1N(t)}$. Clearly v_t does not depend on Δ and

$$L_t(c+\Delta) - L_t(c) = \frac{\alpha_{2t}\gamma_{2t}}{t/Q_{N(t)}}\Delta + \frac{v_t}{t}\Delta$$

Similarly, it can be proved that:

THEOREM 5 *Consider the estimator $L_n = P_{2t_n}/t_n$. Assume that assumption (A1) holds $1 \le k \le n$. Then,*

$$L_n(c+\Delta) - L_n(c) = \frac{L_n(c)Q_n + v_t}{t_n(c+\Delta)}\Delta$$

where $t_n(c+\Delta) = t_n(c) - (Q_n + u_n)\Delta$, v_n and u_n are random variables such that $-2 \le v_n \le 2, -1 \le u_n \le 1$.

SPA Analysis: Finite Horizon

We derive SPA estimators for the performance measure $L_t = P_{2t}/t$. Estimators for the other performance measures can be derived in an analogous manner. Following the framework of Fu and Hu (1997), the sample path space is partitioned into sets (probability events) $\mathscr{A}(\Delta)$ and $\mathscr{A}^c(\Delta)$, where $\mathscr{A}(\Delta)$ contains the sample paths that experience no event changes due to a perturbation of size Δ, and the complement set $\mathscr{A}^c$ contains the sample paths on which event changes do occur as a result of the perturbation. Using this partition,

$$\frac{dE[L_t]}{dc} = \lim_{\Delta\to 0} E\left[\frac{L_t(c+\Delta) - L_t(c)}{\Delta}|\mathscr{A}\right]P(\mathscr{A})$$
$$+ \lim_{\Delta\to 0} E\left[\frac{L_t(c+\Delta) - L_t(c)}{\Delta}|\mathscr{A}^c\right]P(\mathscr{A}^c)$$
$$= E\left[\frac{dL_t}{dc}\right] + E\left[\lim_{\Delta\to 0} E[(L_t(c+\Delta) - L_t(c))|z, \mathscr{A}^c] \lim_{\Delta\to 0}\frac{P(\mathscr{A}^c|z)}{\Delta}\right]$$
$$= E\left[\frac{dL_t}{dc}\right] + E\left[\left(E_z\left[L^{PP}\right] - E_z\left[L^{DNP}\right]\right)\frac{dP_z}{dc}\right]$$

where $E_z\left[L^{PP}\right] = \lim_{\Delta\to 0} E[L_t(c+\Delta)|z, \mathscr{A}^c]$

$$E_z\left[L^{DNP}\right] = \lim_{\Delta\to 0} E[L_t(c)|z, \mathscr{A}^c]$$
$$\frac{dP_z}{dc} = \lim_{\Delta\to 0}\frac{P(\mathscr{A}^c|z)}{\Delta}$$

where dP_z/dc is the probability rate of event changes, z is a set of sample path quantities selected to smooth the effect of event changes, L^{DNP} is the performance measure of a so-called degenerated nominal path denoted by DNP, L^{PP} is the performance measure of a so-called perturbed path, denoted by PP. PP is identical to DNP up to time, say t_k, where an event change occurs. At this instant, an event occurs on both nominal and perturbed paths; however, the event on the nominal path differs from the one on the perturbed path.

In the following, we examine the possible event changes, select the characterization and determine the effect of event changes, i.e., $L^{PP} - L^{DNP}$. Assume that the k-th event changes, i.e.,

$$e_1(c+\Delta) = e_1(c), \ldots, e_{k-1}(c+\Delta) = e_{k-1}(c), e_k(c+\Delta) \neq e_k(c)$$

Clearly the above assumption implies that the state of the machines α_{k-1} is identical for both nominal and perturbed systems. Consider as well the following indicator of the buffer state $\beta_k = (\beta_{1k}, \beta_{2k})$ with $\beta_{1k} = \mathbf{1}\{x_k = c\}$ and $\beta_{2k} = \mathbf{1}\{x_k = 0\}$. β_{k-1} is also identical for both nominal and perturbed systems since it is totally determined by the sequence of events up to e_{k-1} as follows:

$$\beta_{1k-1} = \begin{cases} 1 & \text{if } \exists n \leq k-1 \text{ such that } e_n = BF \text{ and } e_i \in \{F2, R2\}, n < i \leq k-1 \\ 0 & \text{otherwise} \end{cases}$$
$$\beta_{2k-1} = \begin{cases} 1 & \text{if } \exists n \leq k-1 \text{ such that } e_n = BE \text{ and } e_i \in \{F1, R1\}, n < i \leq k-1 \\ 0 & \text{otherwise} \end{cases}$$

A change of the k-th event with the next event is not possible if machine M_1 is blocked in state s_{k-1} or M_2 is starved in state s_{k-1}, i.e., $\alpha_{k-1} = (1,0) \wedge \beta_{k-1} = (1,0)$ or $\alpha_{k-1} = (0,1) \wedge \beta_{k-1} = (0,1)$, because only one event ($R2$ in the first case and $R1$ in the second case) is feasible. For a third case with $\alpha_{k-1} = (1,1)$ and $\beta_{k-1} = (0,1)$, an event change is not possible, because there are only two competing events, and the relative remaining lifetimes of both are unchanged by a change in c.

Case B: $\alpha_{k-1} = (1,1) \wedge \beta_{k-1} = (0,0)$. According to Theorem 2, an event change is possible only if $H_{m+1} < k-1 < O_{m+1}$ and $e_{k-1} = R2$ or $e_{k-1} = R1$. As a result, either $r_{1k-1}(c+\Delta) = r_{1k-1}(c) - \Delta \wedge r_{2k-1}(c+\Delta) = r_{2k-1}(c)$ or $r_{1k-1}(c+\Delta) = r_{1k-1}(c) \wedge r_{2k-1}(c+\Delta) = r_{2k-1}(c) + \Delta$. The only event change is: $e_k(c+\Delta) = F1 \wedge e_k(c) = F2$ under conditions $r_{1k}(c) - \Delta < r_{2k}(c) < r_{1k}(c)$ or $r_{2k}(c) < r_{1k}(c) < r_{2k}(c) + \Delta$. Since $\beta_{k-1} = (0,0)$, i.e., $0 < x_k < c$ and, for small enough Δ, $e_{k+1}(c+\Delta) = F2 \wedge e_{k+1}(c) = F1$. In the limiting case, i.e., $\Delta \to 0$, $r_{ik-1}(c+\Delta) = r_{ik-1}(c)$ and the two sample paths PP and DNP become identical everywhere except at time t_k where $F1, F2$ occurs in PP and $F2$, $F1$ occurs in DNP. As a result, $L^{PP} - L^{DNP} = 0$.

By similar reasoning, results for the remaining cases can be established, and we summarize all the cases here:

- *Case A:* $\alpha_{k-1} \wedge \beta_{k-1} = (1,0) \wedge (1,0), (0,1) \wedge (0,1)$:
  No event change possible.

- *Case B:* $\alpha_{k-1} = (1,1) \wedge \beta_{k-1} = (0,0)$:
  $e_k(c+\Delta) = F1 \wedge e_k(c) = F2$ and $L^{PP} - L^{DNP} = 0$.

- *Case C:* $\alpha_{k-1} = (1,1) \wedge \beta_{k-1} = (1,0)$:
  $e_k(c+\Delta) = F1 \wedge e_k(c) = F2$ and $L^{PP} - L^{DNP} \neq 0$.

- *Case D:* $\alpha_{k-1} = (0,0) \wedge \beta_{k-1} = (x,x), x \in \{0,1\}$:
  $e_k(c+\Delta) = R1 \wedge e_k(c) = R2$ and $L^{PP} - L^{DNP} = 0$.

- *Case E:* $\alpha_{k-1} = (0,1) \wedge \beta_{k-1} = (0,0)$ or $\alpha_{k-1} = (0,1) \wedge \beta_{k-1} = (1,0)$:
  (i) $e_k(c+\Delta) = R1 \wedge e_k(c) = F2$ and $L^{PP} - L^{DNP} = 0$;
  (ii) $e_k(c+\Delta) = R1 \wedge e_k(c) = BE$ and $L^{PP} - L^{DNP} = 0$.

- *Case F:* $\alpha_{k-1} = (1,0) \wedge \beta_{k-1} = (0,x), x \in \{0,1\}$:
  (i) $e_k(c+\Delta) = R2 \wedge e_k(c) = BF$ and $L^{PP} - L^{DNP} = 0$;
  (ii) If $x = 0$, $e_k(c+\Delta) = F1 \wedge e_k(c) = R2$ and $L^{PP} - L^{DNP} = 0$ (only $x = 0$ is possible);
  (iii) $e_k(c+\Delta) = F1 \wedge e_k(c) = BF$ and $L^{PP} - L^{DNP} \neq 0$.

- *Case G:* Interchange of the last event and the end of horizon:
  $e_{N(t)}(c+\Delta) > t \wedge e_{N(t)}(c) < t$ and $L^{PP} - L^{DNP} = 0$.

Thus, only event changes of Case C and Case F(iii) need to be further considered, i.e., these correspond to the critical event changes. These event changes are important, since machine M_1 is blocked in state s_k, and the event $F1$ is suspended until the repair of M_2 in DNP, whereas it is under repair in PP (see Figure 7).

The characterization for smoothing the discrete changes is the set of all random variables except X_{1k*}:

$$z_k = \{X_{in}, i = 1,2; n \geq 1\} \backslash \{X_{1k*}\}$$

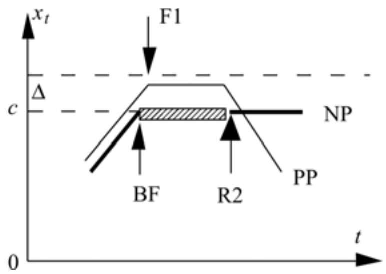

Figure 7. A critical event change.

where X_{1k*} is the newest time to failure of M_1 sampled prior to e_k. As a result, the probability rate for event change can be determined as follows:

Case C: $e_k(c+\Delta) = F1 \wedge e_k(c) = F2$ implies that $r_{1k-1}(c+\Delta) = r_{2k-1}(c+\Delta)$ and $r_{2k-1}(c) < r_{1k-1}(c)$. From Theorem 2, $e_k(c+\Delta) = F1 \Leftrightarrow r_{1k-1}(c) - \Delta \leq r_{2k-1}(c)$ and $e_k(c) = F2 \Leftrightarrow r_{2k-1}(c) < r_{1k-1}(c)$. Since $X_{ik} = a_{ik} + r_{ik}$, $e_k(c+\Delta) = F1 \Leftrightarrow X_{1k-1}(c) = a_{1k-1}(c) + r_{2k-1}(c) + \Delta$ and $e_k(c) = F2 \Leftrightarrow X_{1k}(c) > a_{1k-1}(c) + r_{2k-1}(c)$. As a result, the rate of event change is

$$\begin{aligned}\frac{dP_{z_k}}{dc} &= \lim_{\Delta\to 0} \frac{P(X_{1k} \leq a_{1k-1} + r_{2k-1} + \Delta | X_{1k} > a_{1k-1} + r_{2k-1})}{\Delta} \\ &= \frac{f_1(a_{1k-1} + r_{2k-1})}{1 - F_1(a_{1k-1} + r_{2k-1})}\end{aligned}$$

Case F(iii): By similar arguments as in Case C,

$$\begin{aligned}\frac{dP_{z_k}}{dc} &= \lim_{\Delta\to 0} \frac{P(X_{1k} \leq a_{1k-1} + c + \Delta - x_{k-1} | X_{1k} > a_{1k-1} + c + x_{k-1})}{\Delta} \\ &= \frac{f_1(a_{1k-1} + c - x_{k-1})}{1 - F_1(a_{1k-1} + c - x_{k-1})}\end{aligned}$$

Since NP and DNP_k are identical up to t_k with $e_k = BF$ for Case F(iii) and $e_k = F2$ for Case C, then it holds for DNP_k and NP that $a_{1k} = a_{1k-1} + c - x_{k-1}$ in Case F(iii) and $a_{1k} = a_{1k-1} + r_{2k-1}$ in Case C. As a result, it holds in both cases that:

$$\frac{dP_{z_k}}{dc} = \frac{f_1(a_{1k})}{1 - F_1(a_{1k})}$$

Combining the above results leads to the derivative estimator given by (1).

In order to establish unbiasedness of the estimator we define some sets of sample paths that are characterized by their behavior when a perturbation of size Δ is introduced. Denote a sample path by ω. For $k = 1, 2, \ldots, n = N(t)$, let

$$\begin{aligned}\mathcal{U}_k(\Delta) &= \{\omega : e_1(c+\Delta) = e_1(c), \ldots, e_k(c+\Delta) = e_k(c)\} \\ \mathcal{V}_k(\Delta) &= \{\omega : e_1(c+\Delta) = e_1(c), \ldots, e_k(c+\Delta) \neq e_k(c)\}\end{aligned}$$

By definition, $\mathscr{V}_k = \mathscr{U}_{k-1}/\mathscr{U}_k$. The set $\mathscr{U}_k$ contains sample paths that experience no change in their event sequence through the k-th transition, due to the introduction of a perturbation of size Δ. In particular, the set $\mathscr{U}_n$ contains sample paths that experience no change in their *entire* event sequence due to the introduction of a perturbation of size Δ. On the other hand, the set $\mathscr{V}_k$ contains sample paths in which the first change occurs in the event sequence at the k-th transition. Since ω is usually understood, its explicit display will henceforth be omitted except when used in defining new sets of sample paths. Also, the dependence of $\mathscr{U}_k$ and $\mathscr{V}_k$ on Δ is usually omitted for notational brevity. To take into account the time horizon, we further partition the set of sample paths as follows:

$$\begin{aligned}\mathscr{A} &= \{\omega \in \mathscr{U}_{N(t,c)} : N(t, c+\Delta) = N(t,c)\}\\ \mathscr{B}_k &= \{\omega \in \mathscr{V}_k : k \le N(t,c)\}\\ \mathscr{C} &= \{\omega \in \mathscr{U}_{N(t,c)} : N(t, c+\Delta) > N(t,c)\}\end{aligned}$$

Since $\mathscr{A}$, $\mathscr{B}_k$, $k = 1, 2, \ldots$, and $\mathscr{C}$ partition the set of possible sample paths, by conditioning on whether or not Δ causes a change in the event sequence, we write

$$\begin{aligned}E[L_t(c+\Delta)] - E[L_t(c)] =& E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathscr{A})] + E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathscr{C})]\\ &+ \sum_{k=1}^{\infty}(E[L_t(c+\Delta)\mathbf{1}(\mathscr{B}_k)] - E[L_t(c)\mathbf{1}(\mathscr{B}_k)])\end{aligned}$$

Dividing by Δ and then taking the limit, we have

$$\begin{aligned}\frac{dE[L_t(c)]}{dc} =& \lim_{\Delta\to 0}\frac{E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathscr{A})]}{\Delta}\\ &+ \lim_{\Delta\to 0}\frac{E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathscr{C})]}{\Delta}\\ &+ \sum_{k=1}^{\infty}\lim_{\Delta\to 0}\frac{E[L_t(c+\Delta)\mathbf{1}(\mathscr{B}_k)] - E[L_t(c)\mathbf{1}(\mathscr{B}_k)]}{\Delta}\end{aligned}$$

We now establish

THEOREM 6 *If $F_1(\cdot)$ is Lipschitz continuous with Lipschitz constant K and density $f_1(\cdot)$, then the estimator given by (1) is an unbiased estimator for $dE[L_t(c)]/dc$.*

Proof: The proof of this theorem follows the general approach outlined in Fu and Hu (1997) via the dominated convergence theorem. Details of the proof can be found in Fu and Xie (1998). ■

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