Convergence of Simultaneous Perturbation Stochastic Approximation for Nondifferentiable Optimization

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***Abstract*—In this note, we consider simultaneous perturbation stochastic approximation for function minimization. The standard assumption for convergence is that the function be three times differentiable, although weaker assumptions have been used for special cases. However, all work that we are aware of at least requires differentiability. In this note, we relax the differentiability requirement and prove convergence using convex analysis.**

***Index Terms*—Convex analysis, simultaneous perturbation stochastic approximation (SPSA), subgradient.**

I. Introduction

Simultaneous perturbation stochastic approximation (SPSA), proposed by Spall [15], has been successfully applied to many optimization problems. Like other Kiefer–Wolfowitz-type stochastic approximation algorithms, such as the finite-difference based stochastic approximation algorithm, SPSA uses only objective function measurements. Furthermore, SPSA is especially efficient in high-dimensional problems in terms of providing a good solution for a relatively small number of measurements of the objective function [17].

Convergence of SPSA has been analyzed under various conditions. Much of the literature assumes the objective function be three times differentiable [3], [5], [8], [10], [15], [16], [18], though weaker assumptions are found as well, e.g., [1], [4], [12], [14], and [19]. However, all of them require that the function be at least differentiable. Among the weakest assumptions on the objective function, Fu and Hill [4] assume that the function is differentiable and convex; Chen *et al.* [1] assume that the function is differentiable and the gradient satisfies a Lipschitz condition. In a semiconductor fab-level decision making problem [7], we found that the one-step cost function is continuous and convex with respect to the decision variables, but nondifferentiable, so that the problem of finding the one-step optimal action requires minimizing a continuous and convex function. So the question is: does the SPSA algorithm converge in this setting? The answer is affirmative, and the details will be presented.

Gerencsér *et al.* [6] have discussed nonsmooth optimization. However, they approximate the nonsmooth function by a smooth enough function, and then optimize the smooth function by SPSA. Thus, they take an indirect approach.

In this note, we consider function minimization and show that the SPSA algorithm converges for nondifferentiable convex functions, which is especially important when the function is not differentiable at the minimizing point. First, similar to [19], we decompose the SPSA algorithm into four terms: a subgradient term, a bias term, a random direction noise term and an observation noise term. In our setting, the subgradient term replaces the gradient term in [19], since we assume that the function does not have to be differentiable. Hence, we need to show the asymptotic behavior of the algorithm follows a differential inclusion instead of an ordinary differentiable equation. Kushner and Yin [9] state a theorem (Theorem 5.6.2) for convergence of a Kiefer–Wolfowitz algorithm in a nondifferentiable setting. However, this theorem is not general enough to cover our SPSA algorithm. We will prove a more general theorem to establish convergence of SPSA.

The general approach for proving convergence for these types of algorithms requires showing that the bias term vanishes asymptotically. In the differentiable case, a Taylor series expansion or the mean value theorem is used to establish this. These tools are not applicable in our more general setting, but we are able to use convex analysis for this task, which is one new contribution of this note. For the random direction noise term, we use a similar argument as in [19] to show the noise goes to zero with probability 1 (w.p. 1), except that now the term is a function of the subgradient instead of the gradient. For the observation noise term, the conditions for general Kiefer–Wolfowitz algorithms given in [9, pp. 113–114] are used, and we also show it goes to zero w.p. 1.

To be more specific, we want to minimize the function $E[F(\theta,\chi)] = f(\theta)$ over the parameter $\theta \in H \subset R^r$, where $f(\cdot)$ is continuous and convex, χ is a random vector and H is a convex and compact set. Let θ_k denote the kth estimate of the minimum, and let $\{\Delta_k\}$ be a random sequence of column random vectors with $\Delta_k = [\Delta_{k,1},\ldots,\Delta_{k,r}]^T$. $\Delta_1,\Delta_2,\ldots$ are not necessary identically distributed. The two-sided SPSA algorithm to update θ_k is as follows:

$$\theta_{k+1} = \Pi_H\left(\theta_k - \alpha_k\left[\Delta_k^{-1}\right]\frac{F_k^+ - F_k^-}{2c_k}\right) \qquad (1)$$

where Π_H denotes a projection onto the set H, $F_k^{\pm}$ are observations taken at parameter values $\theta_k \pm c_k\Delta_k$, c_k is a positive sequence converging to zero, α_k is the step size multiplier, and $[\Delta_k^{-1}]$ is defined as $[\Delta_k^{-1}] := [\Delta_{k,1}^{-1},\ldots,\Delta_{k,r}^{-1}]^T$.

Write the observation in the form

$$F_k^{\pm} = f(\theta_k \pm c_k\Delta_k) + \phi_k^{\pm}$$

where $\phi_k^{\pm}$ are observation noises, and define

$$G_k := \frac{f(\theta_k + c_k\Delta_k) - f(\theta_k - c_k\Delta_k)}{2c_k}. \qquad (2)$$

Then, the algorithm (1) can be written as

$$\theta_{k+1} = \Pi_H\left(\theta_k - \alpha_k\left[\Delta_k^{-1}\right]G_k + \alpha_k\left[\Delta_k^{-1}\right]\frac{\phi_k^- - \phi_k^+}{2c_k}\right). \qquad (3)$$

The convergence of the SPSA algorithm (1) has been proved under various conditions. One of the weakest conditions on the objective function is that $f(\cdot)$ be differentiable and convex [4]. Under the differentiability condition, one generally invokes a Taylor series expansion or the mean value theorem to obtain $f(\theta_k \pm c_k\Delta_k) = f(\theta_k) \pm c_k\Delta_k^T\nabla f(\theta_k) + O(|c_k|^2|\Delta_k|^2)$. Therefore, $G_k = \Delta_k^T\nabla f(\theta_k) + O(|c_k||\Delta_k|^2)$, which means G_k can be

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approximated by $\Delta_k^T \nabla f(\theta_k)$. Then, suppose $H = R^r$, the algorithm (3) can be written as

$$\theta_{k+1} = \theta_k - \alpha_k \nabla f(\theta_k) + \alpha_k \left(I - \left[\Delta_k^{-1}\right] \Delta_k^T\right) \nabla f(\theta_k) + \alpha_k \left[\Delta_k^{-1}\right] \frac{\phi_k^- - \phi_k^+}{2c_k}$$

where a standard argument of the ordinary differential equation (ODE) method implies that the trajectory of θ_k follows the ODE

$$\dot{\theta} = -\nabla f(\theta).$$

In our context, however, we only assume that $f(\cdot)$ is continuous and convex $-\nabla f(\cdot)$ may not exist at some points, so a Taylor series expansion or the mean value theorem is not applicable. Instead, using convex analysis we show that G_k is close to the product of Δ_k^T and a *subgradient* of $f(\cdot)$.

II. Subgradient and Reformulation of the SPSA Algorithm

First, we introduce some definitions and preliminary results on convex analysis, with more details in [11].

Let h be a real-valued convex function on R^r; a vector $sg(x)$ is a *subgradient* of h at a point x if $h(z) \geq h(x) + (z-x)^T sg(x)$, $\forall z$. The set of all subgradients of h at x is called the *subdifferential of* h *at* x and is denoted by $\partial h(x)$ [11, p. 214]. If h is a convex function, the set $\partial h(x)$ is a convex set, which means that $\lambda z_1 + (1-\lambda) z_2 \in \partial h(x)$ if $z_1 \in \partial h(x)$, $z_2 \in \partial h(x)$ and $0 \leq \lambda \leq 1$.

The *one-sided directional derivative* of h at x with respect to a vector y is defined to be the limit

$$h'(x; y) = \lim_{\lambda \downarrow 0} \frac{h(x + \lambda y) - h(x)}{\lambda}. \tag{4}$$

According to [11, Th. 23.1, p. 213], if h is a convex function, $h'(x;y)$ exists for each y. Furthermore, according to [11, Th. 23.4, p. 217], at each point x, the subdifferential $\partial h(x)$ is a nonempty closed bounded convex set, and for each vector y the directional derivative $h'(x;y)$ is the maximum of the inner products $\langle sg(x), y\rangle$ as $sg(x)$ ranges over $\partial h(x)$. Denote the set of $sg(x)$ on which $h'(x;y)$ attains its maximum by $\partial h_y(x)$. Thus, for all $sg_y(x) \in \partial h_y(x)$ and $sg(x) \in \partial h(x)$

$$h'(x;y) = y^T sg_y(x) \geq y^T sg(x).$$

Now, let us discuss the relationship between G_k defined by (2) and subgradients.

Lemma 1: Consider the algorithm (1), assume $f(\cdot)$ is a continuous and convex function, $\lim_{k\to\infty} c_k = 0$, $\{\Delta_k\}$ has support on a finite discrete set. Then, $\forall \varepsilon > 0$, $\exists\ \widetilde{sg}(\theta_k) \in \partial f(\theta_k)$ and finite K such that

$$\left| G_k - \Delta_k^T \widetilde{sg}(\theta_k) \right| < \varepsilon \quad \text{w.p.1} \qquad \forall k \geq K.$$

Proof: Since $f(\cdot)$ is a continuous and convex function, for fixed $\Delta_k = z$, both $f'(\theta_k; z)$ and $f'(\theta_k; -z)$ exist. By (4) and $\lim_{k\to\infty} c_k = 0$, $\forall \varepsilon > 0$, $\exists\ K_1(z)$, $K_2(-z) < \infty$ s.t. $|f'(\theta_k; z) - (f(\theta_k + c_k z) - f(\theta_k))/c_k| < \varepsilon$, $\forall k \geq K_1(z)$, and $|f'(\theta_k; -z) - (f(\theta_k - c_k z) - f(\theta_k))/c_k| < \varepsilon$, $\forall k \geq K_2(-z)$. Let $K = \max_z \{K_1(z), K_2(-z)\}$. Since $\{\Delta_k\}$ has support on a finite discrete set, which implies it is bounded, K exists and is finite, $\forall k \geq K$

$$\left| f'(\theta_k; \Delta_k) - \frac{f(\theta_k + c_k\Delta_k) - f(\theta_k)}{c_k} \right| < \varepsilon \ \text{ w.p.1}$$
$$\left| f'(\theta_k; -\Delta_k) - \frac{f(\theta_k - c_k\Delta_k) - f(\theta_k)}{c_k} \right| < \varepsilon \ \text{ w.p.1.}$$

Since $G_k = (f(\theta_k + c_k\Delta_k) - f(\theta_k - c_k\Delta_k))/(2c_k)$

$$\left| G_k - \frac{1}{2}\left(f'(\theta_k; \Delta_k) - f'(\theta_k; -\Delta_k)\right) \right| < \varepsilon \ \text{ w.p.1.} \tag{5}$$

In addition, for $f'(\theta_k; \Delta_k)$, there exists $sg_{\Delta_k}(\theta_k) \in \partial f_{\Delta_k}(\theta_k)$ such that

$$f'(\theta_k; \Delta_k) = \Delta_k^T sg_{\Delta_k}(\theta_k). \tag{6}$$

Similarly, for $f'(\theta_k; -\Delta_k)$, there exists $sg_{-\Delta_k}(\theta_k) \in \partial f_{-\Delta_k}(\theta_k)$ such that

$$f'(\theta_k; -\Delta_k) = (-\Delta_k)^T sg_{-\Delta_k}(\theta_k). \tag{7}$$

Combining (5), (6), and (7), we conclude that $\forall \varepsilon > 0$, $\exists$ finite K and $sg_{\Delta_k}(\theta_k), sg_{-\Delta_k}(\theta_k) \in \partial f(\theta_k)$ such that

$$\left| G_k - \Delta_k^T \left(\frac{1}{2} sg_{\Delta_k}(\theta_k) + \frac{1}{2} sg_{-\Delta_k}(\theta_k) \right) \right| < \varepsilon \ \text{ w.p.1} \quad k \geq K. \tag{8}$$

Note that $\partial f(\theta_k)$ is a convex set, so $\widetilde{sg}(\theta_k) := 1/2 sg_{\Delta_k}(\theta_k) + 1/2 sg_{-\Delta_k}(\theta_k) \in \partial f(\theta_k)$, and $|G_k - \Delta_k^T \widetilde{sg}(\theta_k)| < \varepsilon$ w.p.1. □

Define $\delta_k = \Delta_k^T \widetilde{sg}(\theta_k) - G_k$. The SPSA algorithm (3) can be decomposed as

$$\theta_{k+1} = \Pi_H \left(\theta_k - \alpha_k \widetilde{sg}(\theta_k) + \alpha_k \left(I - \left[\Delta_k^{-1}\right] \Delta_k^T\right) \widetilde{sg}(\theta_k) + \alpha_k \left[\Delta_k^{-1}\right] \delta_k + \alpha_k \left[\Delta_k^{-1}\right] \frac{\phi_k^- - \phi_k^+}{2c_k} \right). \tag{9}$$

Suppose $H = R^r$, and if we can prove that the third, fourth, and fifth terms inside of the projection go to zero as k goes to infinity, the trajectory of θ_k would follow the differential inclusion [9, p. 16]

$$\dot{\theta} \in -\partial f(\theta).$$

According to [11, p. 264], the necessary and sufficient condition for a given x to belong to the minimum set of f (the set of points where the minimum of f is attained) is that $0 \in \partial f(x)$.

III. Basic Constrained Stochastic Approximation Algorithm

Kushner and Yin [9, p. 124] state a theorem (Theorem 5.6.2) for convergence of a Kiefer–Wolfowitz algorithm in a nondifferentiable setting. However, this theorem is not general enough to cover the SPSA algorithm given by (9). So, we establish a more general theorem.

Note that the SPSA algorithm given by (9) is a special case of the stochastic approximation algorithm

$$\theta_{k+1} = \Pi_H \left(\theta_k + \alpha_k\, \widetilde{sf}(\theta_k) + \alpha_k\, b_k + \alpha_k\, e_k \right) \tag{10}$$
$$= \theta_k + \alpha_k\, \widetilde{sf}(\theta_k) + \alpha_k\, b_k + \alpha_k\, e_k + \alpha_k\, Z_k \tag{11}$$

where Z_k is the reflection term, b_k is the bias term, e_k is the noise term, and $\widetilde{sf}(\theta_k)$ can be any element of $-\partial f(\theta_k)$. Similar to (9), we need to show that b_k, e_k and Z_k go to zero.

As in [9, p. 90], let $m(t)$ denote the unique value of k such that $t_k \leq t < t_{k+1}$ for $t \geq 0$, and set $m(t) = 0$ for $t < 0$, where the time scale t_k is defined as follows: $t_0 = 0$, $t_k = \sum_{i=0}^{k-1} \alpha_i$.

Define the shifted continuous-time interpolation $\theta^k(t)$ of θ_k as follows:

$$\theta^k(t) = \theta_k + \sum_{i=k}^{m(t+t_k)-1} \alpha_i \widetilde{sf}(\theta_i) + \sum_{i=k}^{m(t+t_k)-1} \alpha_i b_i + \sum_{i=k}^{m(t+t_k)-1} \alpha_i e_i + \sum_{i=k}^{m(t+t_k)-1} \alpha_i Z_i. \tag{12}$$

Define $B^k(t) = \sum_{i=k}^{m(t+t_k)-1} \alpha_i b_i$, and define $M^k(t)$ and $Z^k(t)$ similarly, with e_k and Z_k, respectively, in place of b_k. Since $\theta^k(t)$ is piecewise constant, we can rewrite (12) as

$$\theta^k(t) = \theta_k + \int_0^t \widetilde{sf}\left(\theta^k(s)\right) ds + B^k(t) + M^k(t) + Z^k(t) + \rho^k(t) \quad (13)$$

where

$$\rho^k(t) = \int_{t_{m(t+t_k)}}^{t} \widetilde{sf}\left(\theta^{m(t+t_k)}(s)\right) ds \le \alpha_{m(t+t_k)} \left|\widetilde{sf}\left(\theta_{m(t+t_k)}\right)\right|.$$

Note that $\rho^k(t)$ is due to the replacement of the first summation in (12) by an integral, and $\rho^k(t) = 0$ at the jump times t_k of the interpolated process, and $\rho^k(t) \to 0$, since α_k goes to zero as k goes to infinity.

We require the following conditions, similar to those of [9, Th. 5.3.1, pp. 88–108].

A.1) $\alpha_k \to 0, \sum \alpha_k = \infty$.

A.2) The feasible region H is a hyperrectangle. In other words, there are numbers $a_i < b_i$, $i = 1, \ldots, r$, such that $H = \{x : a_i \le x_i \le b_i\}$.

A.3) For some positive number T_1

$$\lim_{k\to\infty} \sup_{|t|\le T_1} \left|B^k(t)\right| = 0 \quad \text{w.p.1.}$$

A.4) For some positive number T_2

$$\lim_{k\to\infty} \sup_{|t|\le T_2} \left|M^k(t)\right| = 0 \quad \text{w.p.1.}$$

For $x \in H$ satisfying A.2), define the set $C(x)$ as follows. For $x \in H^0$, the interior of H, $C(x)$ contains only the zero element; for $x \in \partial H$, the boundary of H, let $C(x)$ be the infinite convex cone generated by the outer normals at x of the faces on which x lies [9, p. 77].

Proposition 1: For the algorithm given by (11), where $\widetilde{sf}(\theta_k) \in \partial f(\theta_k)$, assume $\partial f(\theta)$ is bounded $\forall \theta \in H$ and A.1)–A.4) hold. Suppose that $f(\cdot)$ is continuous and convex, but not constant. Consider the differential inclusion

$$\dot{\theta} \in -\partial f(\theta) + z, \quad z(t) \in -C(\theta(t)) \quad (14)$$

and let S_H denote the set of stationary points of (14), i.e., points in H where $0 \in -\partial f(\theta) + z$. Then, $\{\theta_k(\omega)\}$ converges to a point in S_H, which attains the minimum of f.

Proof: See the Appendix. □

IV. Convergence of the SPSA Algorithm

We now use Proposition 1 to prove convergence of the SPSA algorithm given by (9), which we first rewrite as follows:

$$\theta_{k+1} = \theta_k - \alpha_k \widetilde{sg}(\theta_k) + \alpha_k \left(I - \left[\Delta_k^{-1}\right] \Delta_k^T\right) \widetilde{sg}(\theta_k) + \alpha_k \left[\Delta_k^{-1}\right] \delta_k + \alpha_k \left[\Delta_k^{-1}\right] \frac{\phi_k^- - \phi_k^+}{2c_k} + \alpha_k \tilde{Z}_k \quad (15)$$

where $\tilde{Z}_k$ is the reflection term.

Note that the counterparts of b_k and e_k in (11) are $[\Delta_k^{-1}]\delta_k$ and $e_{r,k} + e_{o,k}$, respectively, where the latter quantity is decomposed into a random direction noise term $e_{r,k} := (I - [\Delta_k^{-1}]\Delta_k^T)\widetilde{sg}(\theta_k)$ and an observation noise term $e_{o,k} := ([\Delta_k^{-1}]/2c_k)(\phi_k^- - \phi_k^+)$.

Lemma 2: Assume that

$$E\left[\frac{\Delta_{k,i}}{\Delta_{k,j}} \mid \Delta_0, \ldots, \Delta_{k-1}, \theta_0, \ldots, \theta_{k-1}\right] = 0, i \ne j$$

and $\{\widetilde{sg}(\theta_k)\}$, $\{\Delta_k\}$ and $\{[\Delta_k^{-1}]\}$ are bounded. Then, $e_{r,k} = (I - [\Delta_k^{-1}]\Delta_k^T)\widetilde{sg}(\theta_k)$ is a martingale difference.

Proof: By the boundedness assumption, each element of the sequence $\{\widetilde{sg}(\theta_k)\}$ is bounded by a finite B_k. Using a similar argument as in [19], define $M_k = \sum_{l=0}^{k}(I - [\Delta_l^{-1}]\Delta_l^T)\widetilde{sg}(\theta_l)$, so $e_{r,k} = M_k - M_{k-1}$.

$$\begin{aligned} &E\left[M_{k+1} \mid M_l, l \le k\right] \\ &= E\left[M_k + \left(I - \left[\Delta_{k+1}^{-1}\right] \Delta_{k+1}^T\right) \widetilde{sg}\left(\theta_{k+1}\right) \mid M_l, l \le k\right] \\ &= M_k + E\left[e_{r,k+1} \mid M_l, l \le k\right] \end{aligned}$$

and the absolute value of $E[e_{r,k+1} \mid M_l, l \le k]$

$$\begin{aligned} &\left|E\left[e_{r,k+1} \mid M_l, l \le k\right]\right| \\ &\le |B_k| \cdot \left|E\left[\left(I - \left[\Delta_{k+1}^{-1}\right] \Delta_{k+1}^T\right) \bar{\mathbf{1}} \mid M_l, l \le k\right]\right| \\ &= |B_k| \cdot E\left|[\Lambda_k \bar{\mathbf{1}} \mid M_l, l \le k]\right| \\ &= 0 \end{aligned}$$

where $\bar{\mathbf{1}}$ is a column vector with each element being 1, and

$$\Lambda_k = \begin{bmatrix} 0 & \frac{\Delta_{k+1,1}}{\Delta_{k+1,2}} & \cdots & \frac{\Delta_{k+1,1}}{\Delta_{k+1,r}} \\ \frac{\Delta_{k+1,2}}{\Delta_{k+1,1}} & 0 & \cdots & \frac{\Delta_{k+1,2}}{\Delta_{k+1,r}} \\ \vdots & \vdots & & \vdots \\ \frac{\Delta_{k+1,r}}{\Delta_{k+1,}} & \cdots & \frac{\Delta_{k+1,r}}{\Delta_{k+1,2}} & 0 \end{bmatrix}.$$

$\{E|M_k|\}$ is bounded since $\{\widetilde{sg}(\theta_k)\}$, $\{\Delta_k\}$ and $\{[\Delta_k^{-1}]\}$ are bounded. Thus, $e_{r,k}$ is a martingale difference. □

For the observation noise term $e_{o,k}$, we assume the following conditions.

B.1) $\phi_k^- - \phi_k^+$ is a martingale difference.

B.2) There is a $K < \infty$ such that for a small γ, all k, and each component $(\phi_k^- - \phi_k^+)_j$ of $(\phi_k^- - \phi_k^+)$,

$$E\left[e^{\gamma(\phi_k^- - \phi_k^+)_j} \mid \Delta_0, \ldots, \Delta_{k-1}, \theta_0, \ldots, \theta_{k-1}\right] \le e^{\gamma^2 K/2}.$$

B.3) For each $\mu > 0$, $\sum_k e^{-\mu c_k^2/\alpha_k} < \infty$.

B.4) For some $T > 0$, there is a $c_1(T) < \infty$ such that for all k,

$$\sup_{k \le i < m(t_k+T)} \frac{\frac{\alpha_i}{c_i^2}}{\frac{\alpha_k}{c_k^2}} \le c_1(T).$$

Examples of condition B.2) and a discussion related to B.3) and B.4) can be found in [9, pp. 110–112]. Note that the moment condition on the observation noises $\phi_k^{\pm}$ is not required in [1]. For other alternative noise conditions, see [2].

Lemma 3: Assume that A.1) and B.1)–B.4) hold and $\{[\Delta_k^{-1}]\}$ is bounded. Then, for some positive T, $\lim_{k\to\infty} \sup_{|t|\le T} |M^{o,k}(t)| = 0$ w.p. 1, where $M^{o,k}(t) := \sum_{i=k}^{m(t+t_k)-1} \alpha_i e_{o,i}$.

Proof: Define $\tilde{M}^k(t) := \sum_{i=k}^{m(t+t_k)-1} (\alpha_i/2c_i)(\phi_i^- - \phi_i^+)$. By A.4) and B.1)–B.4), there exists a positive T such that $\lim_{k\to\infty} \sup_{|t|\le T} |\tilde{M}^k(t)| = 0$ w.p. 1, following the same argument as the one in the proof of [9, Ths. 5.3.2, 5.3.3, pp. 108–110].

$\{[\Delta_k^{-1}]\}$ is bounded, so $|M^{o,k}(t)| \le D_k |\tilde{M}^k(t)|$, where D_k is the bound of $[\Delta_k^{-1}]$.

Thus, $\lim_{k\to\infty} \sup_{|t|\le T_1} |M^{o,k}(t)| = 0$ w.p. 1. □

Proposition 2: Consider the SPSA algorithm (9), where $\widetilde{sg}(\theta_k) \in \partial f(\theta_k)$. Assume $\partial f(\theta_k)$ is bounded $\forall \theta_k \in H$, A.1)–A.2) and B.1)–B.4) hold, $\{\Delta_k\}$ has support on a finite discrete set, $\{[\Delta_k^{-1}]\}$ is bounded, and $E[\Delta_{k,i}/\Delta_{k,j} \mid \Delta_0, \ldots, \Delta_{k-1}, \theta_0, \ldots, \theta_{k-1}] = 0$ for $i \ne j$. Suppose that $f(\cdot)$ is continuous and convex, but not constant. Consider the differential inclusion

$$\dot{\theta} \in -\partial f(\theta) + z, \qquad z(t) \in -C(\theta(t)) \quad (16)$$

and let S_H denote the set of stationary points of (16), i.e., points in H where $0 \in -\partial f(\theta) + z$. Then, $\{\theta_k(\omega)\}$ converges to a point in S_H, which attains the minimum of f.

Proof: Since $\{[\Delta_k^{-1}]\}$ is bounded and $\lim_{k\to\infty}\delta_k = 0$ by Lemma 1, $\lim_{k\to\infty}[\Delta_k^{-1}]\delta_k = 0$ w.p. 1, which implies that A.3) holds.

By Lemma 3, $\lim_{k\to\infty}\sup_{|t|\le T}|M^{o,k}(t)| = 0$ w.p. 1. So, A.4) holds for $e_{o,k}$.

Since $\{\Delta_k\}$ has support on a finite discrete set, $\{\widetilde{sg}(\theta_k)\}$ and $\{[\Delta_k^{-1}]\}$ are bounded, $E|e_{r,k}|^2 < \infty$. Using the martingale convergence theorem and Lemma 2, we get

$$\lim_{k\to\infty}\sup_{|t|\le T}|M^{r,k}(t)| = 0 \quad w.p.1$$

where $M^{r,k}(t) := \sum_{i=k}^{m(t+t_k)-1}\alpha_i e_{r,i}$. So, A.4) holds for $e_{r,k}$.

So all conditions of Proposition 1 are satisfied, and all its conclusions hold. □

V. Conclusion

In this note, we use convex analysis to establish convergence of constrained SPSA for the setting in which the objective function is not necessarily differentiable. As alluded to in the introduction, we were motivated to consider this setting by a capacity allocation problem in manufacturing, in which nondifferentiability appeared for the case of period demands having discrete support rather than continuous. Similar phenomena arise in other contexts, e.g., in discrete event dynamic systems, as observed by Shapiro and Wardi [13]. Clearly, there are numerous avenues for further research in the nondifferentiable setting. We believe the analysis can be extended to SPSA algorithms with nondifferentiable constraints, as well as to other (non-SPSA) stochastic approximation algorithms for nondifferentiable function optimization. For example, the same basic analysis could be used for random directions stochastic approximation algorithms as well. We specifically intend to consider global optimization of nondifferentiable functions along the line of [10]. On the more technical side, it would be desirable to weaken conditions such as A.2) and B.4), which are not required in [1].

Appendix
Proof of Proposition 1

Define $G^k(t) = \int_0^t \widetilde{sf}(\theta^k(s))ds$, and let B be the bound of $\widetilde{sf}(\theta^k(s))$, which exists due to the boundedness of $\partial f(\theta)$. Then, for each T and $\varepsilon > 0$, there is a δ which satisfies $0 < \delta < \varepsilon/B$, such that for all k

$$\sup_{0<t-s<\delta, |t|<T}\left|G^k(t) - G^k(s)\right| \le \sup_{0<t-s<\delta, |t|>T}\int_t^s \widetilde{sf}\left(\theta^k(u)\right)du < \varepsilon$$

which means $G^k(\cdot)$ is equicontinuous.

By A.3) and A.4), there is a null set O such that for sample points $\omega \notin O$, $M^k(\omega,\cdot)$ and $B^k(\omega,\cdot)$ go to zero uniformly on each bounded interval in $(-\infty,\infty)$ as $k\to\infty$. Hence, $M^k(\cdot)$ and $B^k(\cdot)$ are equicontinuous and their limits are zero.

By the same argument as in the proof of [9, Th. 5.2.1, pp. 96–97], $Z^k(\cdot)$ is equicontinuous.

$\theta^k(\cdot)$ is equicontinuous since $M^k(\cdot)$, $B^k(\cdot)$, $G^k(\cdot)$ and $Z^k(\cdot)$ are equicontinuous.

Let $\omega \notin O$ and let k_j denote a subsequence such that

$$\left\{\theta^{k_j}(\omega,\cdot), G^{k_j}(\omega,\cdot)\right\}$$

converges, and denote the limit by $(\theta(\omega,\cdot), G(\omega,\cdot))$. The existence of such subsequences is guaranteed by Arzela–Ascoli theorem.

Since f is convex, according to [11, Cor. 24.5.1, p. 234], $\forall\varepsilon > 0$, $\exists\ \delta > 0$, if $|\theta^{k_j}(\omega,s) - \theta(\omega,s)| < \delta$, then $-\partial f(\theta^{k_j}(\omega,s)) \subset N_\varepsilon(-\partial f(\theta(\omega,s)))$, where $N_\varepsilon(\cdot)$ means ε-neighborhood.

Furthermore, since $\lim_{j\to\infty}\theta^{k_j}(\omega,s) = \theta(\omega,s)$ for fixed ω and s, $\forall\varepsilon > 0$, $\exists$ finite J, if $j \ge J$, $-\partial f(\theta^{k_j}(\omega,s)) \subset N_\varepsilon(-\partial f(\theta(\omega,s)))$, i.e., for each $\widetilde{sf}(\theta^{k_j}(\omega,s)) \in -\partial f(\theta^{k_j}(\omega,s))$ and $\varepsilon > 0$, there is finite J and $\tilde{g}(\omega,s) \in -\partial f(\theta(\omega,s))$ such that if $j \ge J$, $|\widetilde{sf}(\theta^{k_j}(\omega,s)) - \tilde{g}(\omega,s)| < \varepsilon$.

Since $\widetilde{sf}(\cdot)$ and $\tilde{g}(\omega,\cdot)$ are bounded functions on [0, t], by the Lebesgue dominated convergence theorem

$$\lim_{j\to\infty}\int_0^t \widetilde{sf}\left(\theta^{k_j}(\omega,s)\right)ds = \int_0^t \tilde{g}(\omega,s)ds$$

which means $G(\omega,t) = \int_0^t \tilde{g}(\omega,s)ds$.

Thus, we can write $\theta(\omega,t) = \theta(\omega,0) + \int_0^t \tilde{g}(\omega,s)ds + Z(\omega,t)$, where $\tilde{g}(\omega,s) \in -\partial f(\theta(\omega,s))$.

Using a similar argument as in [9, p. 97], $Z(\omega,t) = \int z(\omega,s)ds$, where $z(\omega,s) \in -C(\theta(\omega,s))$ for almost all s.

Hence, the limit $\theta(\omega,\cdot)$ of any convergent subsequence satisfies the differential inclusion (14).

Note that S_H is the set of stationary points of (14) in H. Following a similar argument as in the proof of [9, Th. 5.2.1, pp. 96–97], we can show that $\{\theta_k(\omega)\}$ visits S_H infinite often, S_H is asymptotically stable in the sense of Lyapunov due to the convexity of f, and thus $\{\theta_k(\omega)\}$ converges to S_H w.p. 1. Since f is a convex function and H is a nonempty convex set, by [11, Th. 27.4, pp. 270–271], any point in S_H attains the minimum of f relative to H. □

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Robust Adaptive Tracking for Time-Varying Uncertain Nonlinear Systems With Unknown Control Coefficients

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***Abstract*—This note presents a robust adaptive control approach for a class of time-varying uncertain nonlinear systems in the strict feedback form with completely unknown time-varying virtual control coefficients, uncertain time-varying parameters and unknown time-varying bounded disturbances. The proposed design method does not require any *a priori* knowledge of the unknown coefficients except for their bounds. It is proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals and disturbance attenuation.**

***Index Terms*—Robust adaptive control, time-varying nonlinear systems.**

I. Introduction

Adaptive nonlinear control has seen a significant development in the past decade with the appearance of recursive backstepping design [1]–[3]. A great deal of attention has been paid to tackle the uncertain nonlinear systems with unknown constant parameters [1], [4]–[6]. In this note, we consider a class of single-input–single-output (SISO) uncertain time-varying nonlinear systems with time-varying disturbances in the strict feedback form

$$\begin{aligned}\dot{x}_i &= g_i(t)x_{i+1} + \theta_i^T(t)\psi_i(\bar{x}_i) + d_i^T(t)\phi_i(\bar{x}_i)\\ \dot{x}_n &= g_n(t)u + \theta_n^T(t)\psi_n(x) + d_n^T(t)\phi_n(x)\\ y &= x_1\end{aligned} \tag{1}$$

where $i = 1,\ldots,n-1$, $x = [x_1,\ldots,x_n]^T \in R^n$ is the state vector, $\bar{x}_i = [x_1,\ldots,x_i]^T$, $i = 1,\ldots,n-1$, $u \in R$ is the control, $\theta_i(t) \in R^{pi}$ are vectors of uncertain and time-varying parameters belonging to known compact sets $\Omega_i \subset R^{pi}$, $d_i(t)$ are vectors of unknown time-varying bounded disturbance evolving in R^{qi}, ψ_i and ϕ_i, $i = 1,\ldots,n$ are known dimensionally compatible smooth nonlinear functions, $g_i(t) \neq 0, i = 1,\ldots,n$ are bounded uncertain time-varying piecewise continuous functions, and they are referred to as virtual control coefficients, in particular, $g_n(t)$ is referred to as the high-frequency gain. For simplicity, let Ω_i be a closed ball of known radius r_{Ω_i} centered in the origin.

Based on the cancellation backstepping design, as termed in [7], many well-known results have been developed for systems with constant virtual control coefficients by seeking for a cancellation of the coupling terms related to $z_i z_{i+1}$ in the next step of the cancellation based backstepping design. When virtual control coefficients $g_i = 1$ and $\theta_i(t)$ are unknown constants, robust adaptive control for a class of systems similar to system (1) have been developed in [8]–[10]. In the presence of time-varying parameters and time-varying disturbance, robust adaptive tracking control was presented in [11] and boundedness of all the signals and arbitrary disturbance attenuation can be achieved. When g_i's are unknown constants, several excellent adaptive control algorithms have also been developed in the literature for nonlinear systems. In [1], under the assumption of unknown constants g_i's but with known signs of g_i's, adaptive control was presented for strict feedback nonlinear systems without disturbances. With the aid of neural networks [12], [13] adaptive control is expanded to much larger class of systems, uncertain strict-feedback and pure-feedback nonlinear systems, where the unknown virtual control coefficients g_i's are functions of states and the signs of g_i as well as the upper bounds of g_i are assumed to be known.

When the signs of g_i are unknown, the adaptive control problem becomes much more difficult. The first solution was given in [14] for a class of first-order linear systems, where the Nussbaum-type gain was originally proposed. Using Nussbaum gains, adaptive control was given for first-order nonlinear systems in [15], for a class of strict feedback nonlinear systems with unknown constant parameters and without disturbances in [16] and [17], and nonlinear systems with completely unknown control coefficients, constant parametric uncertainties and uncertainties in [18] using decoupled backstepping (which, in contrast to cancellation based backstepping, decouples z_i from z_{i+1} using Young's inequality and seeks for the boundedness of z_{i+1} in the next step as said in [7]). Thus far, little attention has been paid to the robust adaptive control problem for systems in (1) in the literature, except for the work in [19]. Reference [19] studied the regulation problem for a class of time-varying uncertain nonlinear systems with time-varying unknown control coefficients under the assumption that uncertain system functions satisfy an additive bound condition. However, when nonlinear systems involve time-varying uncertain parameters and disturbances as well as unknown virtual control coefficients, the solution remains open, and the problem becomes much more difficult due to the presence of the time-varying uncertainties.

In this note, robust adaptive control is presented for system (1) with completely unknown time-varying control coefficients, uncertain time-varying parameters with known bounds, and unknown time-varying disturbances. It is proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals and disturbance attenuation. In addition, for systems with unknown constant parameters and without disturbance, asymptotical tracking of the output can be achieved. The main contributions of the note lie in the following aspects: 1) through introduction of a new technical lemma and the use of Nussbaum gain, stable adaptive control is presented for a class of strict feedback nonlinear system with time-varying uncertain parameters and unknown disturbance; 2) asymptotic output tracking control is achieved when the disturbances

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