OPTIMAL EXERCISE POLICIES AND SIMULATION-BASED VALUATION FOR AMERICAN-ASIAN OPTIONS

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American-Asian options are average-price options that allow early exercise. In this paper, we derive structural properties for the optimal exercise policy, which are then used to develop an efficient numerical algorithm for pricing such options. In particular, we show that the optimal policy is a threshold policy: The option should be exercised as soon as the average asset price reaches a characterized threshold, which can be written as a function of the asset price at that time. By exploiting this and other structural properties, we are able to parameterize the exercise boundary, and derive gradient estimators for the option payoff with respect to the parameters of the model. These estimators are then incorporated into a simulation-based algorithm to price American-Asian options. Computational experiments carried out indicate that the algorithm is very competitive with other recently proposed numerical algorithms.

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1. INTRODUCTION

Asian options are derivative securities with payoffs that depend on the average of an underlying asset price over some specified period. Because of their relatively small exposure to risk, they have become one of the most popular exotic options traded over the counter. The purposes of this paper are to rigorously establish a characterization of the optimal exercise policy for American-Asian options[1] and to develop a Monte Carlo simulation-based method that exploits the established structural properties to efficiently price such options.

Sections 2.5–2.7 of Karatzas and Shreve (1998) provide a fairly comprehensive survey on the properties of the early exercise boundaries for ordinary "vanilla" American options; however, "exotic" American-Asian options are not considered there. These options differ from ordinary American options in many aspects. First, since their payoff is a function of the average asset price, the payoff upon exercise depends on the price path of the asset, rather than only the asset price at the exercise date. Second, at any exercisable date, the asset price remains influential in determining the early exercise decision. This interplay between the current asset price and the average stock price makes the analysis of American-Asian options more complicated.

Prior work by Grant et al. (1997) provides plausible heuristic arguments for the form of the optimal exercise boundary. Our work provides rigorous mathematical proofs establishing the structure of the optimal exercise policy for American-Asian options. Assuming the asset price evolves according to a Markovian model in a quite general setting, we rigorously show that the optimal exercise policy for a fixed strike American-Asian call option is a threshold policy: The option should be exercised as soon as the average asset price reaches a characterized threshold, which can be written as a function of the asset price at that time. Furthermore, we prove that the threshold level is unbounded, and under a mild condition, nondecreasing in the asset price at that time, and for a large class of models, the threshold level is also convex.

A closely-related purpose of this paper is to price American-Asian options. Asian options have proven to be much more difficult to value than regular asset options. Because of their path dependencies, standard techniques tend to be impractical or inaccurate. There are a few approximation methods for European-style (i.e., without early exercise features) Asian options appearing in the literature (e.g., Turnbull and Wakeman 1991, Vorst 1992, Levy 1992, Levy and Turnbull 1992, Geman and Yor 1993). Monte Carlo simulation seems to be a popular approach, especially for practitioners, to price European-style Asian options (e.g., Fu et al. 1999). As for American-Asian options, there are even fewer alternatives. Hull and White (1993) propose a modification of the binomial method, but provide no proof of convergence. Neave (1994) provides a frequency distribution approach based on a binomial tree, but his method still requires $O(N^4)$ computation time, where N is the number of time steps in the lattice. Hansen and Jorgenson (2000) provide analysis of the floating strike case that leads to a closed-form solution for geometric averaging and an approximation for arithmetic averaging, but their methodology cannot be applied to the (more common) fixed strike case. There are recently developed partial differential equation (PDE) approaches (Barraquand

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and Pudet 1996, Zvan et al. 1997), for which special care needs to be taken in order to get an accurate option value. Also, the computational requirements for the binomial and PDE approaches become impractical for models incorporating stochastic volatility and stochastic interest rates.

Monte Carlo simulation was first introduced to finance in Boyle (1977). Since that time, simulation has been successfully applied to a wide range of asset pricing problems (Boyle et al. 1997). However, until recently the technique has not been applied to the valuation of American-style options. The major difficulty lies in the need to estimate an optimal exercise policy, which is usually obtained via a backward induction algorithm, whereas simulation is a forward-based process. In the past decade, a number of Monte Carlo simulation-based approaches have been proposed to address the problem of pricing American-style options. For an overview of the approaches, see Broadie and Glasserman (1997b) or Fu et al. (2001). Of the work surveyed there, only Grant et al. (1997) address specifically the pricing of American-Asian options. Their procedure mimics the backward induction solution method of stochastic dynamic programming. At every exercisable date, the optimal threshold parameters are estimated by testing all possible values from a preselected finite parameter grid. The algorithms proposed by Broadie and Glasserman (1997a, 1997c) are based on simulated paths and lead to biased high estimators and biased low estimators that converge to the true value in the appropriate limit. Unfortunately, since there is no proper transition probability density function for American-Asian options, the stochastic mesh method (Broadie and Glasserman 1997c) does not appear applicable to American-Asian options. Although one can extend the simulated tree method (Broadie and Glasserman 1997a) to American-Asian options, a large number of simulated trees need to be generated in order to get an accurate option value, which is impractical from the perspective of computation costs.

Over the last decade, there has been a lot of research on simulation-based approximate dynamic programming (see Bertsekas and Tsitsiklis 1995). Recently, Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) applied this approach to the pricing of American-style options by approximating the holding value function at each time step using a linear combination of basis functions fitted to the simulated data via least square regression. In particular, Longstaff and Schwartz (2001) demonstrate the efficiency of their least square approach through several numerical examples, and Tsitsiklis and Van Roy (2001) rigorously establish the general convergence properties of the method.

An alternative to approximating the value function is to approximate the exercise boundary, i.e., the boundary at which the holding value equals the exercise value. However, in order for this approach to be effective, some knowledge on the structure of the optimal policy is crucial, and thus the theoretical results on the form of the policy are exploited to this end. Our simulation-based approach to value American-Asian options parameterizes the exercise boundary and maximizes the expected discounted payoff with respect to the early exercise threshold parameters. Similar ideas are also used in Fu and Hu (1995) to price ordinary American call options. Once a parameterization is assumed, the most difficult and challenging part of our approach is to find a good gradient estimator for American-Asian options, a task that is more complicated than that for ordinary American options (Fu et al. 2000). We derive the gradients with respect to associated parameters via perturbation analysis (PA)(Ho and Cao 1991, Glasserman 1991, Fu and Hu 1997), a sample path method for gradient estimation. Then we incorporate the PA estimators into a stochastic approximation algorithm to estimate the optimal threshold parameters, and consequently obtain an estimate for the option price. Using examples from Grant et al. (1997), we compare our algorithm with their algorithm and with the algorithm from Longstaff and Schwartz (2001), and find that our approach is quite competive, if not superior, for the testbed of problems considered.

In sum, our work contributes to the research stream on pricing American-Asian options in significant ways:

- We provide rigorous proofs establishing various structural properties of the optimal exercise policy (in a Markovian setting more general than geometric Brownian motion).
- We derive gradient estimators for the option payoff with respect to model parameters.
- By exploiting the structural properties, we apply the gradient estimates to a parameterized exercise boundary in order to provide a computationally efficient simulation-based pricing method.

During final preparation of the initial version of our paper, we were made aware of related work by Ben Ameur et al. (2002), who also develop a numerical method for pricing American-Asian options based on dynamic programming combined with finite-element piecewise-polynomial approximation of the value function. Independently from us, they also establish some similiar theoretical properties for the optimal exercise strategy for American-style Asian options in the Black-Scholes setting. Many of our results hold in a more general setting than theirs, because our proof techniques differ from theirs, in that the stopping times are carried out throughout our proofs, whereas their proofs proceed by backward induction on time steps of the dynamic programming optimality equation.

The rest of the paper is organized as follows. Section 2 introduces the problem setting. Section 3 describes the various structural properties for the optimal exercise policy. All proofs of the results can be found in the Appendix. The perturbation analysis estimators are presented in §4, with the detailed derivations in the Appendix. In §5, we parameterize the exercise boundary, simplify the estimators derived in the previous section, and provide the simulation-based valuation algorithm, which is tested on some numerical examples in §6. Section 7 contains concluding remarks.

2. PROBLEM SETTING

We begin by introducing the following notation to be used throughout:

$S_t =$ asset price at time t,
$r =$ annualized riskless interest rate (compounded continuously),
$\sigma =$ volatility of the underlying asset,
$K =$ strike price of the option contract,
$T =$ expiration date of the option contract.

For ease of exposition, r, σ, and K will be assumed constant. If r or σ is stochastic, it can also be easily incorporated into our context. Without loss of generality, we designate the present time as time 0.

We consider a discrete arithmetic American-Asian option, where the averaging starts at time t_0 and Δt is the equally-spaced interval between the averaging dates. Let N be the number of price average dates if held to expiration T (so $T = t_0 + (N-1)\Delta t$), and $\eta(< N)$ be the number of early exercisable dates for the American-Asian option (see Figure 1), denoted as $t_i, i = 1, 2, \ldots, \eta$. For notational convenience we also denote $T = t_{\eta+1}$. Define $\mathscr{A}$ as the set of all average dates and $\mathscr{E}$ as the set of all exercisable dates, i.e., $\mathscr{A} = \{t_0, t_0+\Delta t, t_0+2\Delta t, \ldots, t_0+(N-2)\Delta t, T\}$ and $\mathscr{E} = \{t_i, i = 1, 2, \ldots, \eta+1\}$. Note that $\mathscr{E} \subseteq \mathscr{A}$. If $\mathscr{E} = \mathscr{A}$, then the option can be exercised at any price averaging date. For any $t \in \mathscr{A}$, define n_t as the number of averaging dates up to and including time t.

Let $\{S_t\}$ denote the asset price process, which we assume throughout to be Markovian. We define the stopping time to be a random variable τ that takes values in $\mathscr{E}$ such that each event of the form $\{\tau = t\}, t \in \mathscr{E}$ is an element of the algebra $\mathscr{F}_t$, the filtration generated by the relevant price processes up to time t in the economy. Write $\psi(x) = (x-K)^+ \triangleq \max(x-K, 0)$. Then the immediate exercise value of the American-Asian call option at time $t \in \mathscr{E}$ is given by $\psi(\overline{S_t})$, where $\overline{S_t}$ is the average price up to and including time t, i.e.,

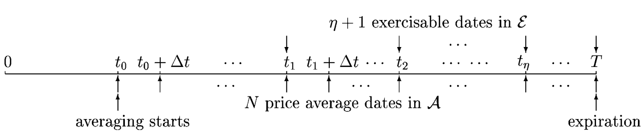
$$\overline{S_t} = \frac{S_{t_0} + S_{t_0+\Delta t} + \cdots + S_t}{n_t}.$$

We assume that the financial market is arbitrage free, so that by the fundamental theorem in asset pricing (e.g., Harrison and Pliska 1981), there exists an equivalent risk-neutral pricing measure Q such that $e^{-rt}S_t$ is a martingale under measure Q. Since the stock price process $\{S_t\}$ is Markovian, the future stock price path $\{S_{t'}\}_{t'>t}$ only depends on the current stock price S_t. Arbitrage-free valuation theory implies that the value of the American-Asian call option at time $t \in \mathscr{E}$ is given by taking the supremum over all stopping times $\tau \geqslant t$ of the expected discounted payoff of the option under risk-neutral pricing measure Q:

$$\sup_{\tau \geqslant t} E^Q\left[e^{-r(\tau-t)}\psi(\overline{S_\tau}) | \overline{S_t}, S_t\right]. \tag{1}$$

Throughout, all expectations will be taken under the Q measure, so for ease of notation, the superscript Q will be omitted.

Figure 1. Averaging and exercise dates for the discrete American-Asian option.



3. STRUCTURE OF THE OPTIMAL EXERCISE POLICY

In this section, we characterize the structure of the optimal exercise policy for the American-Asian call option. Analogous ideas can be used for a discussion of the American-Asian put option. Specifically, we first show that the optimal exercise strategy is a threshold policy, i.e., the option should be exercised as soon as the average asset price reaches a characterized threshold, which can be written as a function of the asset price at that time. Then we try to explore further the properties of the threshold, i.e., the shape of the exercise boundary.

Note that at each exercisable time $t \in \mathscr{E}$, the option holder must choose whether to exercise immediately or to continue the life of the option and revisit the exercise decision at the next exercisable date. The payoff upon immediate exercise at time t with average asset price $\overline{S_t}$ is given by $\psi(\overline{S_t})$. We introduce the notation $c(x, y, t)$ to denote the continuation value of the option, i.e., the value of the option conditional on the option not being exercised at or prior to time t, with current average asset price $\overline{S_t} = x$ and current asset price $S_t = y$. Then we have

$$c(x, y, t) = \sup_{\tau > t} E\left[e^{-r(\tau-t)}\psi(\overline{S_\tau}) | \overline{S_t} = x, S_t = y\right],$$

where the supremum is taken over all stopping times $\tau > t$, instead of $\tau \geqslant t$. Since the value of an American option at any time is the maximum of the payoff upon immediate exercise and the continuation value, one may also write the option value at time t, given by (1), as $\max(c(\overline{S_t}, S_t, t), \psi(\overline{S_t}))$. Since the option holder exercises as soon as the immediate exercise value is greater than or equal to the value of continuation, the exercise region at exercise point $t \in \mathscr{E}$, denoted as R^*, can be characterized by

$$R^*(t) = \{(x, y) : c(x, y, t) \leqslant \psi(x)\}.$$

The threshold policy is based on the following observation: For the same current asset price, a higher current running average will have a higher continuation value, but the difference in continuation value is not greater than that in the current running average.

Lemma 1. *For any $x, y, \epsilon > 0, t \in \mathscr{E}$, we have*

$$0 \leqslant c(x+\epsilon, y, t) - c(x, y, t) \leqslant \epsilon.$$

REMARK 1. It is easy to infer from Lemma 1 that for any fixed y and $t \in \mathscr{E}$, $c(\cdot, y, t)$ as a function of its first variable is nondecreasing and uniformly continuous.

THEOREM 1. *The optimal exercise policy is a threshold policy, i.e., there is a function $F_t^*(\cdot)$ at time $t \in \mathscr{E}$ such that it is optimal to exercise the option whenever $\overline{S}_t \geqslant F_t^*(S_t) \geqslant K$, where*

$$F_t^*(y) = \inf\{x : c(x, y, t) \leqslant \psi(x)\}.$$

By establishing the existence of a threshold policy, exercise decisions are completely determined by the function $F_t^*(s), s \in (0, +\infty)$ at each early exercisable date $t = t_i$. The remainder of the results in this section characterize the shape of this function, which will be useful in formulating an effective parameterization for developing numerical pricing algorithms. The first result establishes that the boundary goes off to positive infinity with increasing values of the current stock price.

THEOREM 2. *For any $t \in \mathscr{E}(t \neq T)$, $F_t^*(\cdot)$ is unbounded, i.e., if $y \to \infty$, then $F_t^*(y) \to \infty$.*

The next result establishes the monotonicity of the boundary. In order to show this, we need to make a mild assumption on the model of the underlying asset.

ASSUMPTION 1. *If $y_1 > y_2$, then $c(x, y_1, t) \geqslant c(x, y_2, t)$ for all $t \in \mathscr{E}$.*

Intuitively, Assumption 1 implies that for the same current running average at any fixed time, a higher current asset price cannot lead to a lower call option value. One can see that most of the models in practice satisfy the assumption.

DEFINITION 1. An asset price model is multiplicative if it can be represented by the form

$$S_{t'} = S_t X_{t,t'}$$

for any $t' > t$, where $X_{t,t'}(> 0)$ is a random variable independent of all $\{S_u, u \leqslant t\}$ and only a function of quantities defined on $[t, t']$.

Intuitively, if the price of the stock at time t doubles, then the stock price at time t' would double. For example, geometric Brownian motion falls into this category:

$$S_{t'} = S_t e^{(r-\sigma^2/2)(t'-t)+\sigma\sqrt{t'-t}Z}, \tag{2}$$

where Z is a $N(0,1)$ random variable. The general jump diffusion model (Merton 1976) is also multiplicative:

$$S_{t'} = S_t e^{(r-\sigma^2/2)(t'-t)+\sigma\sqrt{t'-t}Z_0+\sum_{i=1}^{q(t'-t)}(\delta Z_i-\delta^2/2)}, \tag{3}$$

where $Z_i \sim N(0,1)$ iid, $q(t) \sim Poisson(\lambda t)$ with λ being the jump arrival rate, and the jump sizes are i.i.d. lognormally distributed $LN(\mu_i, \sigma_i^2)$, with $\mu_i = -\delta^2/2$ and $\sigma_i = \delta(\delta > 0)$.

DEFINITION 2. An asset price model is additive if it can be represented by the form

$$S_{t'} = S_t + Y_{t,t'}$$

for any $t' > t$, where $Y_{t,t'}$ is a random variable independent of all $\{S_u, u \leqslant t\}$ and only a function of quantities defined on $[t, t']$. Note that $Y_{t,t'}$ is allowed to be negative (so our model could also handle negative stock prices).

It is easy to check that both multiplicative and additive models—which include a very general set of stochastic processes with stationary independent increments called Lévy processes—will satisfy Assumption 1. For example, consider the multiplicative model. For any stopping time $\tau > t$, we have

$$\begin{aligned}
&E\big[e^{-r(\tau-t)}(\overline{S}_\tau - K)^+ | \overline{S}_t = x, S_t = y_1)\big] \\
&= E\Bigg[e^{-r(\tau-t)}\Big(\big((n_t x + y_1(X_{t,t+\Delta t} + X_{t,t+\Delta t}X_{t+\Delta t,t+2\Delta t} \\
&\qquad + \cdots + X_{t,t+\Delta t}\cdots X_{\tau-\Delta t,\tau}))/n_\tau\big) - K\Big)^+\Bigg] \\
&\geqslant E\Bigg[e^{-r(\tau-t)}\Big(\big((n_t x + y_2(X_{t,t+\Delta t} + X_{t,t+\Delta t}X_{t+\Delta t,t+2\Delta t} \\
&\qquad + \cdots + X_{t,t+\Delta t}\cdots X_{\tau-\Delta t,\tau}))/n_\tau\big) - K\Big)^+\Bigg] \\
&= E[e^{-r(\tau-t)}(\overline{S}_\tau - K)^+ | \overline{S}_t = x, S_t = y_2],
\end{aligned}$$

where the inequality follows from $y_1 > y_2$ and the fact that $a^+ \geqslant b^+$ if $a > b$. Taking the supremum with respect to all stopping times $\tau > t$ yields the result of Assumption 1. Note that the same proof goes through for the additive model.

Actually, the class of asset price models that satisfy Assumption 1 extend far beyond just the multiplicative and additive models illustrated here. Bergman et al. (1996) demonstrate that as long as a certain no-crossing property holds, the price of a call option is nondecreasing in the underlying asset price. For example, they show that all the one-dimensional diffusion processes,

$$dS_t = \alpha(S_t, t)dt + \sigma(S_t, t)S_t dW_t,$$

where the instantaneous volatility $\sigma(\cdot)$ can be a function of the concurrent asset price, satisfy the no-crossing property. Note that the processes are generally not multiplicative. The Constant Elasticity of Variance (CEV) (Cox and Ross 1976) and the local volatility model (Dupire 1994, Derman and Kani 1994), two of the most widespread models among practioners, fall into this category.

Under Assumption 1, the monotonicity result states that the exercise boundary is increasing (in the nonstrict sense) as a function of the current asset price.

THEOREM 3. *If Assumption 1 holds, $F_t^*(\cdot)$ is nondecreasing.*

Lastly, we are able to establish convexity of the exercise boundary under a further restriction from Assumption 1, satisfied by, for example, the geometric Brownian motion model (2).

Theorem 4. *If the asset price model is multiplicative or additive, then* $F_t^*(\cdot)$ *is convex.*

4. PERTURBATION ANALYSIS ESTIMATORS

4.1. Motivation

Armed with knowledge on the structure of the optimal policy, we now try to price the American-Asian options by parameterizing the early exercise boundary and then formulating the optimal stopping problem as the following optimization problem:

$$\max_{\theta\in\Theta} E[\mathscr{L}(\theta,\omega)], \tag{4}$$

where $\theta \in \Theta \subset R^p$ is the p-dimensional vector of interest, e.g., the parameters of the early exercise boundary to be estimated, $\mathscr{L}(\theta,\omega)$ is the (sample) discounted payoff of an American-Asian option, Θ a compact set in R^p, and ω an element in the probability space of interest, e.g., a sample path in simulation. We will apply stochastic approximation (e.g., Kushner and Yin 1997) to the above optimization problem. Basically, we attempt to find the solution to (4) by mimicking steepest-decent algorithms from the deterministic domain of nonlinear programming using the following iterative search scheme:

$$\theta_{n+1} = \Pi_\Theta(\theta_n + a_n \hat{g}_n), \tag{5}$$

where $\theta_n = ((\theta_n)_1, \ldots, (\theta_n)_p)$ represents the nth iterate, $\hat{g}_n$ represents an estimate of the gradient of $E[\mathscr{L}]$ with respect to the parameter vector θ at θ_n, $\{a_n\}$ is a positive sequence of numbers converging to 0, and Π_Θ denotes a projection on Θ. In order to implement the algorithm, the key feature is the availability of a gradient estimate, which could either be a direct estimate or a finite difference estimate. However, a direct estimate generally will provide a superior convergence rate. Next, we derive such a direct gradient estimator via perturbation analysis (PA) (Ho and Cao 1991, Glasserman 1991, Fu and Hu 1997).

4.2. Derivation

In §3, we established that the optimal policy follows a threshold policy, i.e., at any time $t \in \mathscr{E}$, the option holder exercises the option whenever $\overline{S}_t \geqslant F_t^*(S_t)$. We will let $F_t(\cdot)$ denote an approximate form of $F_t^*(\cdot)$. For ease of notation, for the rest of the paper we write $F_i(\cdot) \triangleq F_{t_i}(\cdot)$ and $n_i \triangleq n_{t_i}$ for $t_i \in \mathscr{E}$. Therefore, the value of the American-Asian call option can be written as (4), with the sample performance $\mathscr{L}$ given by

$$\mathscr{L} = \sum_{i=1}^{\eta} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i} \geqslant F_i(S_{t_i})\right\}(\overline{S}_{t_i} - K)e^{-rt_i} + \mathbf{1}\left\{\bigcap_{j=1}^{\eta} \overline{S}_{t_j} < F_j(S_{t_j})\right\}(\overline{S}_T - K)^+ e^{-rT}, \tag{6}$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Expression (6) simply represents the discounted option payoff as a sum of payoffs at each exercisable date, where exercise can occur at most once over the life of the contract.

In order to derive the PA estimators, we assume that $F_t(\cdot)$ is convex for any $t \in \mathscr{E}$. As demonstrated in the previous section, convexity holds for a large class of stock price models, including geometric Brownian motion. In fact, we can also use the ideas presented here to derive the PA estimators for the case where $F_t(\cdot)$ is concave. Note that

$$\overline{S}_{t_i} = \frac{n_i - 1}{n_i}\overline{S}_{t_i-\Delta t} + \frac{S_{t_i}}{n_i}.$$

First, we need the following result in order to derive the PA estimator.

Lemma 2. *If* $F_i(\cdot)$ *is convex, assuming that* $\overline{S}_{t_i-\Delta t} = z$ *fixed and*

$$\left\{y : \frac{(n_i - 1)z + y}{n_i} \geqslant F_i(y)\right\} \neq \varnothing, \tag{7}$$

then we can always find $L_i(\cdot)$ *and* $U_i(\cdot)$ *such that*

$$\frac{(n_i - 1)z + y}{n_i} \geqslant F_i(y) \Longleftrightarrow L_i(z) \leqslant y \leqslant U_i(z),$$

where L_i *and* U_i *may take on the values* 0_+ *and* $+\infty$, *respectively, with the subscript notation* x_- *and* x_+ *denoting the corresponding respective left-hand and right-hand limits.*

Remark 2. In Lemma 2, we assume (7) holds. Actually, if it is empty, we may choose $L_i(z) = U_i(z) = constant$. It will not affect our derivation of the PA estimator, since the integral on any set with measure zero is zero.

Now we will derive the PA estimator. Since $\{S_t\}$ is Markovian, we assume the asset price dynamics follow the form

$$S_t = h(Z; S_0, t, r, \sigma)$$

for some random variable Z independent of the parameters and initial stock price S_0. In particular,

$$S_{t+\Delta t} = h(Z_{t+\Delta t}; S_t, \Delta t, r, \sigma)$$

for any $t \in \mathscr{A}$, with independent $Z_{t+\Delta t} \sim f$, the appropriate probability density function. We also assume that h is monotone in the first variable. Note that our derivation could admit different forms of h and f for different t; however, for ease of exposition, we assume the same form for all t. Also, for notational convenience, we will henceforth omit explicit dependence on r and σ in the display of function h. For the geometric Brownian motion model (2), h is given by

$$h(Z; S, t) = Se^{(r-\sigma^2/2)t + \sigma\sqrt{t}Z}, \tag{8}$$

where Z is a standard $N(0,1)$ random variable with density function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

Our goal is to find an estimate for $\frac{\partial E[\mathscr{L}]}{\partial \theta}$, where θ can be any parameter of the model, although our interest in the next section focuses on parameters of the early exercise boundary. The detailed derivation of the PA estimator for $\eta = 2$ is given in the Appendix. The extension of the PA estimator to the general η case is given by the following:

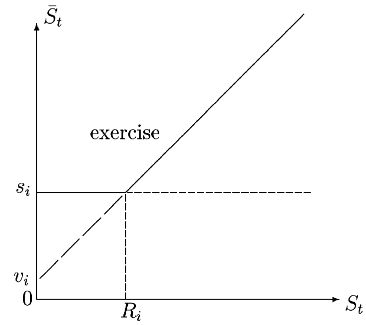
$$\begin{aligned}
&\sum_{i=1}^{\eta} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j})\right\} \\
&\times \Bigg\{ \frac{\partial h^{-1}(L_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)}{\partial \theta} \\
&\qquad \times f(h^{-1}(L_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)) \\
&\qquad \times \Bigg[E\Bigg[\mathscr{L} \Bigg| \bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t}, S_{t_i} = L_i(\overline{S}_{t_i-\Delta t})_- \Bigg] \\
&\qquad\qquad - E((\overline{S}_{t_i} - K)e^{-rt_i} | \overline{S}_{t_i-\Delta t}, S_{t_i} = L_i(\overline{S}_{t_i-\Delta t})) \Bigg] \\
&\qquad - \frac{\partial h^{-1}(U_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)}{\partial \theta} \\
&\qquad \times f(h^{-1}(U_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)) \\
&\qquad \times \Bigg[E\Bigg[\mathscr{L} \Bigg| \bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t}, S_{t_i} = U_i(\overline{S}_{t_i-\Delta t})_+ \Bigg] \\
&\qquad\qquad - E((\overline{S}_{t_i} - K)e^{-rt_i} | \overline{S}_{t_i-\Delta t}, S_{t_i} = U_i(\overline{S}_{t_i-\Delta t})) \Bigg] \Bigg\} \\
&+ \sum_{i=1}^{\eta} \mathbf{1} \left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i} \geqslant F_i(S_{t_i})\right\} \\
&\times \frac{\partial}{\partial \theta}[(\overline{S}_{t_i} - K)e^{-rt_i}] \\
&+ \mathbf{1}\left\{\bigcap_{j=1}^{\eta} \overline{S}_{t_j} < F_j(S_{t_j})\right\} \frac{\partial}{\partial \theta}[(\overline{S}_T - K)^+ e^{-rT}]. \qquad (9)
\end{aligned}$$

Although expression (9) for the estimator appears quite complicated, the implementation is fairly straightforward once the $L_i(\cdot)$ and $U_i(\cdot)$ functions defined in Lemma 2 are known, as we see in the next section. The terms involving $\partial/\partial\theta$ and $f(\cdot)$ are all readily available on the original simulated sample path, whereas the conditional expectation quantities require evaluating the expected payoff on paths generated by special starting conditions.

5. PARAMETERIZATION OF EARLY EXERCISE BOUNDARY

In order to develop a numerical pricing algorithm, we consider the PA estimator specifically applied to parameters of the early exercise boundary. There are many ways to parameterize the exercise boundary. Here we consider a linear approximation of the exercise boundary, as in Grant et al. (1997). For other forms of the exercise boundary, similar ideas could be followed to simplify the PA estimators derived in §4. For the call option considered, the exercise region is taken as follows (see Figure 2):

Figure 2. Early exercise boundary for the Asian option.



$$\{\overline{S}_t \geqslant s_i \text{ and } \overline{S}_t \geqslant S_t + v_i\}, \qquad (10)$$

i.e., we approximate the exercise boundary at t_i by a piecewise linear function:

$$F_i(y) = \begin{cases} s_i & \text{if } y \leqslant s_i - v_i, \\ y + v_i & \text{if } y > s_i - v_i, \end{cases} \qquad (11)$$

where s_i, v_i are parameters to be estimated such that they maximize the expected payoff of the option. Now we will proceed to find $L_i(\cdot)$ and $U_i(\cdot)$ from the exercise condition $\overline{S}_{t_i} \geqslant F_i(S_{t_i})$. First, we rewrite the exercise condition that compares $\overline{S}_{t_i}$ with $F(S_{t_i})$ defined by (11) as follows:

(i) If $S_{t_i} \leqslant s_i - v_i$, then $S_{t_i} \geqslant n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t}$, i.e., $n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t} \leqslant S_{t_i} \leqslant s_i - v_i$.

(ii) If $S_{t_i} > s_i - v_i$, then $S_{t_i} \leqslant \overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i - 1}$, i.e., $s_i - v_i < S_{t_i} \leqslant \overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i - 1}$.

Therefore, the option is exercised at time t_i if and only if

$$n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t} \leqslant S_{t_i} \leqslant s_i - v_i \text{ or}$$
$$s_i - v_i < S_{t_i} \leqslant \overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i - 1}.$$

Note that:

(i) If $\overline{S}_{t_i-\Delta t} > s_i + \frac{v_i}{n_i - 1}$, then we have $n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t} < s_i - v_i < \overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i - 1}$. So the above exercise condition can be simplified as $n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t} \leqslant S_{t_i} \leqslant \overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i - 1}$. In this case, L_i and U_i can be taken as

$$L_i(\overline{S}_{t_i-\Delta t}) = n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t} \text{ and}$$
$$U_i(\overline{S}_{t_i-\Delta t}) = \overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i - 1}.$$

(ii) Conversely, if $\overline{S}_{t_i-\Delta t} \leqslant s_i + \frac{v_i}{n_i - 1}$, then $n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t} \geqslant s_i - v_i \geqslant \overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i - 1}$. In this case, the option can be exercised if and only if $S_{t_i} = s_i - v_i$. This is a trivial case, where L_i and U_i can be simply chosen as $L_i(\cdot) = U_i(\cdot) = s_i - v_i$.

Note that in the second case, those terms in the PA estimator (9) that are directly related to $L_i(\cdot)$ and $U_i(\cdot)$ cancel each other. Furthermore, as indicated in the proof of Lemma 2, it is necessary that $L_i(\cdot) \geqslant 0_+$. Therefore, the PA estimator (9) with respect to the threshold parameter θ (θ can be s_i or v_i) can be simplified to

$$\sum_{i=1}^{\eta} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t} > s_i + \frac{v_i}{n_i - 1}\right\}$$
$$\times \left\{\mathbf{1}\{L_i(\overline{S}_{t_i-\Delta t}) > 0\} \frac{\partial h^{-1}(L_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)}{\partial \theta}\right.$$
$$\times f(h^{-1}(L_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t))$$
$$\times \left[E\left[\mathscr{L}\left|\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t}, S_{t_i} = L_i(\overline{S}_{t_i-\Delta t})_-\right]\right.\right.$$
$$\left. - (s_i - K)e^{-rt_i}\right] - \frac{\partial h^{-1}(U_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)}{\partial \theta}$$
$$\times f(h^{-1}(U_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t))$$
$$\times \left[E\left[\mathscr{L}\left|\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t}, S_{t_i} = U_i(\overline{S}_{t_i-\Delta t})_+\right]\right.\right.$$
$$\left.\left. - \left(\overline{S}_{t_i-\Delta t} - \frac{v_i}{n_i - 1} - K\right)e^{-rt_i}\right]\right\},$$

where $L_i(\overline{S}_{t_i-\Delta t}) = n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t}$ and $U_i(\overline{S}_{t_i-\Delta t}) = \overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i-1}$. The last two terms in (9) are zero, because the underlying asset price process is independent of the threshold parameters.

In particular, for the geometric Brownian Motion model (8), the inverse of h is given by

$$h^{-1}(y; S, t) = (\ln(y/S) - (r - \sigma^2/2)t)/(\sigma\sqrt{t}),$$

so we have

$$\frac{\partial h^{-1}(L_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)}{\partial s_i} = \frac{n_i}{\sigma\sqrt{\Delta t}(n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t})},$$
$$\frac{\partial h^{-1}(U_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)}{\partial s_i} = 0,$$
$$\frac{\partial h^{-1}(L_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)}{\partial v_i} = 0,$$
$$\frac{\partial h^{-1}(U_i(\overline{S}_{t_i-\Delta t}); S_{t_i-\Delta t}, \Delta t)}{\partial v_i} = \frac{n_i}{\sigma\sqrt{\Delta t}(n_i v_i - (n_i - 1)\overline{S}_{t_i-\Delta t})}.$$

The PA estimator with respect to s_i is given by

$$\mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t} > s_i + \frac{v_i}{n_i - 1}\right\}$$
$$\times \left\{\mathbf{1}\{n_i s_i > (n_i - 1)\overline{S}_{t_i-\Delta t}\}\right.$$
$$\times \frac{n_i e^{-((\ln((n_i s_i - (n_i-1)\overline{S}_{t_i-\Delta t})/S_{t_i-\Delta t}) - (r-\sigma^2/2)\Delta t)/(\sigma\sqrt{\Delta t}))^2/2}}{\sigma\sqrt{2\pi\Delta t}(n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t})}$$
$$\times \left[E\left[\mathscr{L}\left|\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t},\right.\right.\right.$$
$$\left.\left.\left. S_{t_i} = (n_i s_i - (n_i - 1)\overline{S}_{t_i-\Delta t})_-\right] - (s_i - K)e^{-rt_i}\right]\right\},$$

and that with respect to v_i becomes

$$\mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t} > s_i + \frac{v_i}{n_i - 1}\right\}$$
$$\times \left\{\frac{-n_i e^{-((\ln(((n_i-1)\overline{S}_{t_i-\Delta t} - n_i v_i)/((n_i-1)S_{t_i-\Delta t}))) - (r-\sigma^2/2)\Delta t)/(\sigma\sqrt{\Delta t}))^2/2}}{\sigma\sqrt{2\pi\Delta t}(n_i v_i - (n_i - 1)\overline{S}_{t_i-\Delta t})}\right.$$
$$\times \left[E\left[\mathscr{L}\left|\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\Delta t}, S_{t_i} = \left(\overline{S}_{t_i-\Delta t} - \frac{n_i v_i}{n_i - 1}\right)_+\right]\right.\right.$$
$$\left.\left. - \left(\overline{S}_{t_i-\Delta t} - \frac{v_i}{n_i - 1} - K\right)e^{-rt_i}\right]\right\}.$$

Thus, the PA estimators for the derivative w.r.t. parameters at the ith early exercise date have three types of terms: An indicator function which is based on whether or not the average stock price exceeds a certain level at every exercise date up to the ith, a density "rate" quantity which involves the stock price and the average stock price one averaging date (Δt) *prior* to the ith early exercise date and a number of model parameters, and the difference of two discounted expected payoffs at the ith early exercise date—one with the average stock price just below a certain level (e.g., s_i in the estimator w.r.t. s_i) and the other with the stock price just above the same level. The latter payoff is simply the (discounted) immediate exercise value, and is thus available on the simulated sample path, but the former payoff is a continuation value and requires some additional simulation to estimate.

6. NUMERICAL RESULTS

We now report numerical results on pricing American-Asian options by incorporating the perturbation analysis estimators into a stochastic approximation algorithm according to (5). We consider examples from Grant et al. (1997), using the following settings: initial stock price $S_0 = 100$, strike price $K = 90, 95, 100, 105, 110$, expiration date $T = 120$ days, interest rate $r = 0.09$, and volatility $\sigma = 0.20, 0.30$. Averaging starts at day $t_0 = 91$, and the averaging interval is one day, i.e., $\Delta t = \frac{1}{365}$. The earliest time t_1 for exercise is the end of day 105. In other words, the average includes at least 15 observations of the asset price. We consider three values for the number of early exercise opportunities: $\eta = 1, 3$, and 5. For the step-size sequence, we choose the harmonic series, i.e., $a_n = a/n$ with $a = 50$, and decrease the step size only if the gradient direction has changed from the previous iteration, i.e., the inner product of the current and previous gradient directions is negative. The starting values are $s_i = K$ and $v_i = 0$, for $i = 1, 2, \ldots, \eta$, with constraint condition $s_i \geqslant K$ (v_i unconstrained) and the projection operator defined by simply taking $s_i = K$ for any violated constraints. We take observation lengths of 50 for $\eta = 3$ and 5 and 40 for $\eta = 1$, where the observation length is the number of paths of the asset price generated for the gradient estimation in each

Table 1. $\sigma = 0.2$, $r = 0.09$, $S_0 = 100$. CPU seconds are for computing the threshold parameters only.

	$t_i = 105, 120$			$t_i = 105, 110, 115, 120$			$t_i = 105, 108, \ldots, 120$		
	Price	Std. Error	CPU	Price	Std. Error	CPU	Price	Std. Error	CPU
				$K = 90$					
PASA	13.091	0.007	0.12	13.179	0.007	0.17	13.197	0.007	0.18
DP	13.078	0.007	0.15	13.169	0.007	0.22	13.189	0.007	0.32
DIFF	0.013			0.010			0.008		
				$K = 95$					
PASA	9.019	0.006	0.11	9.108	0.006	0.17	9.123	0.006	0.18
DP	9.021	0.006	0.15	9.101	0.006	0.21	9.122	0.006	0.33
DIFF	−0.002			0.007			0.001		
				$K = 100$					
PASA	5.707	0.005	0.10	5.772	0.005	0.13	5.788	0.005	0.15
DP	5.702	0.005	0.15	5.768	0.005	0.22	5.786	0.005	0.32
DIFF	0.005			0.004			0.002		
				$K = 105$					
PASA	3.287	0.004	0.07	3.329	0.004	0.10	3.338	0.004	0.12
DP	3.280	0.004	0.15	3.329	0.004	0.22	3.337	0.004	0.33
DIFF	0.007			0.000			0.001		
				$K = 110$					
PASA	1.720	0.003	0.06	1.748	0.003	0.08	1.747	0.003	0.10
DP	1.716	0.003	0.15	1.745	0.003	0.22	1.751	0.003	0.32
DIFF	0.004			0.003			−0.004		

iteration. For each price path, we use 10 replications to estimate the conditional expectation portions in the PA estimators.

First, we compare our results with the simulation algorithm of Grant et al. (1997), taking their recommended parameter settings in implementing their algorithm. In both our and their procedures, option valuation is formulated as a maximization problem with respect to the associated threshold parameters. Therefore, comparison of the algorithms is carried out by estimating the expected discounted payoff at the parameter settings obtained by the corresponding algorithm, where *a higher estimate of the option price implies superior performance*. To make the comparisons more precise, we run 2,000,000 simulations after the parameter settings are obtained for each algorithm, in order to accurately estimate the expected option payoff. The results are provided in Tables 1 and 2, where all

Table 2. $\sigma = 0.3$, $r = 0.09$, $S_0 = 100$. CPU seconds are for computing the threshold parameters only.

	$t_i = 105, 120$			$t_i = 105, 110, 115, 120$			$t_i = 105, 108, \ldots, 120$		
	Price	Std. Error	CPU	Price	Std. Error	CPU	Price	Std. Error	CPU
				$K = 90$					
PASA	14.376	0.010	0.11	14.502	0.010	0.13	14.534	0.010	0.15
DP	14.368	0.010	0.15	14.490	0.010	0.21	14.526	0.010	0.33
DIFF	0.008			0.012			0.008		
				$K = 95$					
PASA	10.815	0.009	0.09	10.913	0.009	0.14	10.944	0.009	0.16
DP	10.797	0.009	0.14	10.911	0.009	0.20	10.942	0.009	0.33
DIFF	0.018			0.002			0.002		
				$K = 100$					
PASA	7.820	0.008	0.08	7.920	0.008	0.11	7.933	0.008	0.12
DP	7.814	0.008	0.15	7.916	0.008	0.21	7.935	0.008	0.32
DIFF	0.006			0.004			−0.002		
				$K = 105$					
PASA	5.450	0.007	0.07	5.526	0.007	0.09	5.544	0.007	0.10
DP	5.447	0.007	0.15	5.526	0.007	0.21	5.538	0.007	0.32
DIFF	0.003			0.000			0.006		
				$K = 110$					
PASA	3.665	0.006	0.06	3.723	0.006	0.07	3.730	0.006	0.09
DP	3.661	0.006	0.15	3.725	0.006	0.20	3.738	0.006	0.31
DIFF	0.004			−0.002			−0.008		

standard errors—indicated in the table column "StdErr"—are no more than one cent. The results for the stochastic approximation method based on the perturbation analysis estimators are indicated by "PASA" and those based on the (simulation-based) dynamic programming of Grant et al. (1997) by "DP," with "DIFF" the difference in the option prices. CPU times indicated are for approximating the threshold parameters only, since the final price estimation requires the same computational burden for both algorithms. All cases are run on the same platform: a Sun Ultra 60 Unix workstation.

We find that the stochastic approximation algorithm based on the perturbation analysis estimators converges very quickly. With just ten iterations for each case to compute the associated parameters, we obtain rapid convergence. In most cases, the data in the row of DIFF are positive, which means that the option values based on our method are higher than those based on Grant et al. (1997), i.e., our approach outperforms theirs. Furthermore, we use less CPU time to compute the associated threshold levels. In the five early exercise opportunity case, PASA typically needs about 0.15 seconds, while DP needs about 0.32 seconds. Figures 3 and 4 provide a typical graphical comparison of the two approaches for the case $\sigma = 0.2$, $K = 100$. In these examples, PASA finds better early exercise boundaries with less computational cost, indicating that our simulation-based approach is very promising.

We also compare our approach with the least squares (LS) simulation algorithm of Longstaff and Schwartz (2001) (cf., Tsitsiklis and Van Roy 2001) using the same testbed. We do 20,000 simulations for each approach. For the LS algorithm, all polynomial terms on the current asset price and the running average, up to third order, are used as basis functions. Thus, we use a total of ten basis functions in the regressions. The results are reported in Tables 3 and 4, where "LS" indicates the LS algorithm. Again, both approaches provide lower bounds for the option value (subject to statistical error). Most of the option values obtained from LS are smaller than those from PASA, although the differences are mostly within one standard error. Since CPU times for PASA are smaller than those for LS, our approach is at least competitive with, if not superior to, the approximate dynamic programming approach for this small testbed.

Figure 3. Comparison of option value for two methods.

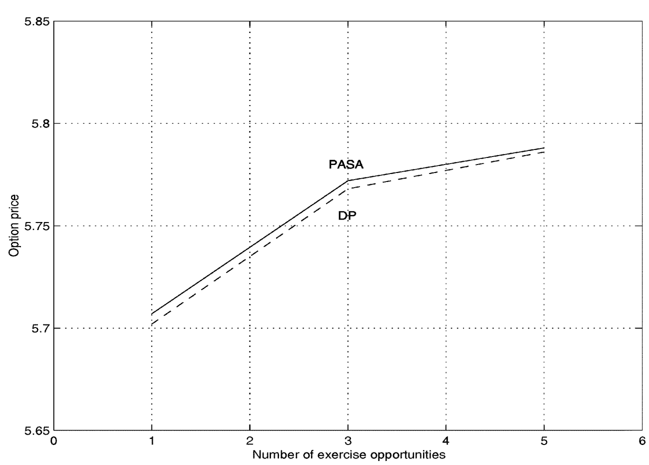
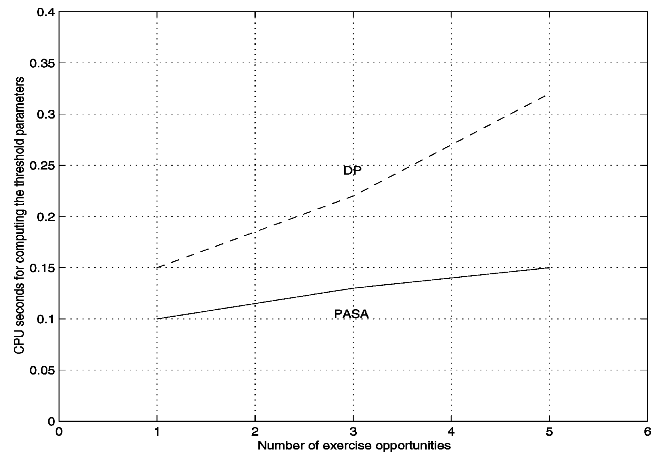


Figure 4. Comparison of CPU time for two methods.



7. CONCLUSION

Our work has illustrated the practical benefits of interplay between theoretical analysis and computational implementation. First, we rigorously established various structural properties of optimal exercise policies for American-Asian options. These properties provide a basis for characterizing the form of the exercise boundary at each potential exercise point, so that by parameterizing the exercise boundary, the option valuation can be cast as a parameterized optimization problem. By deriving stochastic gradient estimators, we provide an efficient simulation-based algorithm for pricing these types of options. This algorithm can be used in settings for which Monte Carlo simulation becomes the preferred numerical technique, e.g., problems involving multiple stochastic processes such as interest rates and volatilities. Furthermore, the results hold for a broad class of underlying Markovian asset price models, that of Lévy processes (generated by independent increments) and general one-dimensional diffusion processes. Future directions include more general optimal stopping problems that frequently occur in financial engineering.

APPENDIX

Proof of Lemma 1. For any stopping time $\tau > t$, we have

$$E\left[e^{-r(\tau-t)}(\overline{S}_\tau - K)^+ | \overline{S}_t = x, S_t = y\right]$$
$$= E\left[e^{-r(\tau-t)}\left(\frac{n_t x + S_{t+\Delta t} + \cdots + S_\tau}{n_\tau} - K\right)^+ \middle| S_t = y\right]$$

Table 3. $\sigma = 0.2$, $r = 0.09$, $S_0 = 100$. CPU seconds include time for computing the option value.

	$t_i = 105, 120$			$t_i = 105, 110, 115, 120$			$t_i = 105, 108, \ldots, 120$		
	Price	Std. Error	CPU	Price	Std. Error	CPU	Price	Std. Error	CPU
				$K = 90$					
PASA	13.085	0.068	0.89	13.188	0.069	0.93	13.200	0.069	0.94
LS	13.057	0.069	1.21	13.147	0.069	1.50	13.167	0.069	1.81
DIFF	0.028			0.041			0.033		
				$K = 95$					
PASA	8.989	0.062	0.93	9.092	0.063	0.95	9.113	0.062	0.98
LS	9.001	0.062	1.15	9.076	0.062	1.40	9.094	0.062	1.63
DIFF	−0.002			0.016			0.019		
				$K = 100$					
PASA	5.744	0.053	0.94	5.740	0.052	0.97	5.792	0.053	0.99
LS	5.685	0.052	1.11	5.744	0.053	1.31	5.756	0.053	1.49
DIFF	0.059			−0.004			0.036		
				$K = 105$					
PASA	3.248	0.041	0.97	3.315	0.041	0.97	3.378	0.041	1.00
LS	3.272	0.041	1.09	3.312	0.041	1.23	3.322	0.041	1.35
DIFF	−0.024			0.003			0.056		
				$K = 110$					
PASA	1.728	0.030	0.98	1.742	0.030	0.98	1.750	0.030	1.01
LS	1.712	0.030	1.06	1.736	0.030	1.15	1.743	0.030	1.24
DIFF	0.016			0.006			0.007		

$$\leqslant E\left[e^{-r(\tau-t)}\left(\frac{n_t(x+\epsilon)+S_{t+\Delta t}+\cdots+S_\tau}{n_\tau}-K\right)^+ \middle| S_t=y\right]$$

$$=E[e^{-r(\tau-t)}(\overline{S}_\tau-K)^+|\overline{S}_t=x+\epsilon, S_t=y]\leqslant c(x+\epsilon, y, t),$$

where we have used the fact that $\{S_t\}$ is a Markov process. Taking the supremum over all stopping times $\tau > t$ yields

$c(x, y, t) \leqslant c(x+\epsilon, y, t)$.

Conversely, for any stopping time $\tau > t$, we also have

$$E\big[e^{-r(\tau-t)}(\overline{S}_\tau - K)^+|\overline{S}_t = x+\epsilon, S_t = y\big]$$

$$= E\left[e^{-r(\tau-t)}\left(\frac{n_t(x+\epsilon)+S_{t+\Delta t}+\cdots+S_\tau}{n_\tau}-K\right)^+ \middle| \overline{S}_t = x+\epsilon, S_t = y\right]$$

Table 4. $\sigma = 0.3$, $r = 0.09$, $S_0 = 100$. CPU seconds include time for computing the option value.

	$t_i = 105, 120$			$t_i = 105, 110, 115, 120$			$t_i = 105, 108, \ldots, 120$		
	Price	Std. Error	CPU	Price	Std. Error	CPU	Price	Std. Error	CPU
				$K = 90$					
PASA	14.347	0.098	0.91	14.464	0.097	0.91	14.522	0.098	0.94
LS	14.341	0.097	1.18	14.462	0.097	1.45	14.489	0.097	1.71
DIFF	0.006			0.002			0.033		
				$K = 95$					
PASA	10.789	0.087	0.92	10.919	0.089	0.94	10.964	0.089	0.97
LS	10.769	0.088	1.13	10.873	0.089	1.36	10.897	0.089	1.58
DIFF	0.020			0.046			0.067		
				$K = 100$					
PASA	7.818	0.079	0.94	7.925	0.078	0.94	7.959	0.078	0.98
LS	7.794	0.078	1.11	7.876	0.079	1.31	7.896	0.079	1.50
DIFF	0.024			0.049			0.063		
				$K = 105$					
PASA	5.436	0.067	0.96	5.443	0.066	0.97	5.507	0.067	0.98
LS	5.434	0.067	1.09	5.500	0.067	1.24	5.516	0.067	1.39
DIFF	0.002			−0.057			−0.009		
				$K = 110$					
PASA	3.621	0.055	0.98	3.733	0.056	0.97	3.726	0.056	1.00
LS	3.654	0.056	1.08	3.702	0.056	1.19	3.714	0.056	1.28
DIFF	−0.033			0.031			0.012		

$$= E\left[e^{-r(\tau-t)}\left((\overline{S}_\tau - K) + \frac{n_t\epsilon}{n_\tau}\right)^+ \middle| \overline{S}_t = x, S_t = y\right]$$
$$\leqslant E\left[e^{-r(\tau-t)}\left((\overline{S}_\tau - K)^+ + \frac{n_t\epsilon}{n_\tau}\right) \middle| \overline{S}_t = x, S_t = y\right]$$
$$\leqslant E\left[e^{-r(\tau-t)}(\overline{S}_\tau - K)^+ \middle| \overline{S}_t = x, S_t = y\right] + \epsilon,$$

where the first inequality follows from $(a+b)^+ \leqslant a^+ + b$ for any $b > 0$, and the second inequality results from the fact that $\frac{n_t}{n_\tau} < 1$ and so $e^{-r(\tau-t)}\frac{n_t\epsilon}{n_\tau} < \epsilon$. Again, taking the supremum over all stopping times $\tau > t$ yields $c(x+\epsilon, y, t) \leqslant c(x, y, t) + \epsilon$. □

Proof of Theorem 1. At expiration date T, the option will be exercised as long as $\overline{S}_T \geqslant K$, so the threshold policy follows for the terminal case. Now we consider the case $t < T$. For any fixed $S_t = y$ at time $t \in \mathscr{E}(t \neq T)$, we can always find a value x such that

$$c(x, y, t) \leqslant \psi(x).$$

To see this, first we note that for any time $t' \in \mathscr{E}$ with $t' > t$, we have

$$E\left[e^{-r(t'-t)}\frac{S_{t+\Delta t} + S_{t+2\Delta t} + \cdots + S_{t'}}{n_{t'}} \middle| S_t = y\right]$$
$$\leqslant \frac{1}{n_t+1}E[e^{-r\Delta t}S_{t+\Delta t} + e^{-2r\Delta t}S_{t+2\Delta t} + \ldots + e^{-r(t'-t)}S_{t'} | S_t = y]$$
$$= \frac{1}{n_t+1}(n_{t'} - n_t)y$$
$$\leqslant \frac{1}{n_t+1}(N - n_t)y, \tag{12}$$

where the first inequality follows from $n_{t'} \geqslant n_t + 1$ and second inequality follows from the martingale property of $e^{-rt}S_t$. Suppose t_i is the next exercisable date after time t. For any stopping time $\tau > t$, we have

$$E\left[e^{-r(\tau-t)}\frac{S_{t+\Delta t}+S_{t+2\Delta t}+\cdots+S_\tau}{n_\tau}\middle| S_t=y\right]$$
$$=E\left[\sum_{t'=t_i}^{T} e^{-r(t'-t)}\frac{S_{t+\Delta t}+S_{t+2\Delta t}+\cdots+S_{t'}}{n_{t'}}\mathbf{1}\{\tau=t'\}\middle| S_t=y\right]$$
$$\leqslant \sum_{t'=t_i}^{T} E\left[e^{-\tau(t'-t)}\frac{S_{t+\Delta t}+S_{t+2\Delta t}+\cdots+S_{t'}}{n_{t'}}\middle| S_t=y\right]$$
$$\leqslant \frac{\eta-i+2}{n_t+1}(N-n_t)y, \tag{13}$$

where the last inequality results from (12). We denote the bound as $C_t \triangleq \frac{\eta-i+2}{n_t+1}(N-n_t)y$, which is independent of stopping times $\tau > t$. Taking $x \geqslant \max(\frac{n_t+1}{n_t}K, (n_t+1)C_t)$, for any stopping time $\tau > t$, we have

$$E\left[e^{-r(\tau-t)}(\overline{S}_\tau - K)^+ \middle| \overline{S}_t = x, S_t = y\right]$$
$$= E\left[e^{-r(\tau-t)}\left(\left(\frac{n_t x}{n_\tau} - K\right) + \frac{S_{t+\Delta t} + \cdots + S_\tau}{n_\tau}\right)^+ \middle| S_t = y\right]$$
$$\leqslant E\left[e^{-r(\tau-t)}\left(\left(\frac{n_t x}{n_\tau} - K\right)^+ + \frac{S_{t+\Delta t} + \cdots + S_\tau}{n_\tau}\right) \middle| S_t = y\right]$$
$$\leqslant \left(\frac{n_t x}{n_t+1} - K\right)^+ + C_t$$
$$\leqslant \left(\frac{n_t x}{n_t+1} - K\right)^+ + \frac{x}{n_t+1} = x - K = \psi(x),$$

where the first inequality follows from $(a+b)^+ \leqslant a^+ + b$ for any $b > 0$, the second inequality follows from (13) and $n_\tau \geqslant n_t + 1$, and the last line follows from the choice of x. Taking the supremum over all stopping times leads to $c(x, y, t) \leqslant \psi(x)$, so that the set over which $F_t^*(y)$ is defined is nonempty:

$$\{x : c(x, y, t) \leqslant \psi(x)\} \neq \varnothing.$$

By definition of $F_t^*(y)$, there exists a decreasing sequence $x_{t,k}$ that approaches $F_t^*(S_t)$ such that $x_{t,k} \geqslant F_t^*(y)$ and $c(x_{t,k}, y, t) \leqslant \psi(x_{t,k})$ for every k. By the continuity of $c(\cdot, y, t)$ for the first variable (from Lemma 1) and $\psi(\cdot)$, we know that

$$c(F_t^*(y), y, t) \leqslant \psi(F_t^*(y)), \tag{14}$$

i.e., the infimum is attainable and well defined. Therefore, it suffices to show that

$$c(F_t^*(y) + \epsilon, y, t) - \psi(F_t^*(y) + \epsilon) \leqslant 0 \tag{15}$$

for any $\epsilon > 0$, which means that it is optimal to exercise the option at time t if the asset price is $S_t = y$ and average asset price is $\overline{S}_t = F_t^*(y) + \epsilon$. Using (14), we can write

$$c(F_t^*(y) + \epsilon, y, t) - \psi(F_t^*(y) + \epsilon)$$
$$\leqslant (c(F_t^*(y) + \epsilon, y, t) - \psi(F_t^*(y) + \epsilon)) - (c(F_t^*(y), y, t) - \psi(F_t^*(y)))$$
$$= (c(F_t^*(y) + \epsilon, y, t) - c(F_t^*(y), y, t)) - (\psi(F_t^*(y) + \epsilon) - \psi(F_t^*(y)))$$
$$\leqslant \epsilon - (\psi(F_t^*(y) + \epsilon) - \psi(F_t^*(y))),$$

where the last line follows from Lemma 1. Since $F_t^*(y) \geqslant K$,

$$(F_t^*(y) + \epsilon - K)^+ - (F_t^*(y) - K)^+ = \epsilon,$$

establishing (15) and concluding the proof. □

Remark on Theorem 1. If we define $\tau = \inf\{t : \overline{S}_t \geqslant F_t^*(S_t)\} \wedge T$ ($\wedge$ denotes the minimum operator), then from Theorem 1 we know that τ is an optimal stopping time. So the results from Theorem 1 imply the existence of an optimal stopping time.

Proof of Theorem 2. Suppose, on the contrary that $F_t^*(\cdot)$ is bounded, i.e., there exists a constant M_t, such that

$$F_t^*(y) \leqslant M_t$$

for all y. Then by (14),

$$c(F_t^*(y), y, t) \leqslant \psi(F_t^*(y)) = (F_t^*(y) - K)^+$$

should also be bounded, so it suffices to show the contradiction that $c(F_t^*(y), y, t) \to \infty$ as $y \to \infty$. Suppose t_i is the next exercisable date after time t and consider $\tau = t_i$, a fixed stopping time. Then we have

$$\begin{aligned} &c(F_t^*(y), y, t)\\ &\quad \geqslant E\big[e^{-r(t_i-t)}(\overline{S}_{t_i}-K)^+|\overline{S}_t=F_t^*(y), S_t=y\big]\\ &\quad = E\left[e^{-r(t_i-t)}\left(\frac{n_tF_t^*(y)+S_{t+\Delta t}+\ldots+S_{t_i}}{n_{t_i}}-K\right)^+\middle| S_t=y\right]\\ &\quad \geqslant e^{-r(t_i-t)}\left(E\left[\frac{n_tF_t^*(y)+S_{t+\Delta t}+\ldots+S_{t_i}}{n_{t_i}}\middle| S_t=y\right]-K\right)^+\\ &\quad = e^{-r(t_i-t)}\left(\frac{n_tF_t^*(y)}{n_{t_i}}+\frac{1}{n_{t_i}}(e^{r\Delta t}+\ldots+e^{r(t_i-t)})y-K\right)^+, \end{aligned} \tag{16}$$

where the second inequality follows from Jensen's inequality and the last equation results from the martingale property of $e^{-rt}S_t$. It is easy to see that as $y \to \infty$, the right-hand side of (16) goes to infinity, so $c(F_t^*(y), y, t) \to \infty$. □

Proof of Theorem 3. Suppose on the contrary that the opposite is true. Then there would exist a pair of prices y_1 and y_2 with $y_1 < y_2$, such that

$$F_t^*(y_1) > F_t^*(y_2).$$

It follows from Theorem 1 that the option will not be exercised at time t if the asset price is $S_t = y_1$ and the average asset price is $\overline{S}_t = F_t^*(y_2)$, i.e.,

$$c(F_t^*(y_2), y_1, t) > \psi(F_t^*(y_2)). \tag{17}$$

On the other hand, by the definition of $F_t^*(\cdot)$, we know that

$$\psi(F_t^*(y_2)) \geqslant c(F_t^*(y_2), y_2, t). \tag{18}$$

Since $y_2 > y_1$, by Assumption 1, we have

$$c(F_t^*(y_2), y_2, t) \geqslant c(F_t^*(y_2), y_1, t). \tag{19}$$

Combining (18) and (19) leads to a contradition of (17). □

Proof of Theorem 4. We provide the detailed proof only for the multiplicative case. The proof for the additive case is essentially identical, with the additive relationship substituted in the appropriate places. For $y_2 < y_1$, we will show that

$$F_t^*(\alpha y_1 + (1-\alpha)y_2) \leqslant \alpha F_t^*(y_1) + (1-\alpha)F_t^*(y_2) \tag{20}$$

for any $\alpha \in (0, 1)$. Write $y = \alpha y_1 + (1-\alpha)y_2$. By Theorem 1 it suffices to show that

$$\begin{aligned} &c(\alpha F_t^*(y_1) + (1-\alpha)F_t^*(y_2), y, t)\\ &\quad \leqslant \psi(\alpha F_t^*(y_1) + (1-\alpha)F_t^*(y_2)), \end{aligned} \tag{21}$$

because (21) is the exercise condition for an option at time t with asset price $S_t = y$ and average asset price $\overline{S}_t = \alpha F_t^*(y_1) + (1-\alpha)F_t^*(y_2)$, which by Theorem 1 is equivalent to the condition $\overline{S}_t = \alpha F_t^*(y_1) + (1-\alpha)F_t^*(y_2) \geqslant F_t^*(y)$.

Now let $\tau^*(> t)$ be the optimal stopping time for the state with average asset price $\overline{S}_t = \alpha F_t^*(y_1) + (1-\alpha)F_t^*(y_2)$ and asset price $S_t = y$ at time t. Then we have

$$\begin{aligned} &c(\alpha F_t^*(y_1)+(1-\alpha)F_t^*(y_2), y, t)\\ &\quad = E\Big[e^{-r(\tau^*-t)}\big(((n_t(\alpha F_t^*(y_1)+(1-\alpha)F_t^*(y_2))\\ &\qquad\quad +S_{t+\Delta t}+\cdots+S_{\tau^*})/n_{\tau^*})-K\big)^+|S_t=y\Big]\\ &\quad = E\Big[e^{-r(\tau^*-t)}\big(((n_t(\alpha F_t^*(y_1)+(1-\alpha)F_t^*(y_2))\\ &\qquad\quad +y(X_{t,t+\Delta t}+\cdots+X_{t,t+\Delta t}\cdots X_{\tau^*-\Delta t,\tau^*}))/n_{\tau^*})-K\big)^+\Big]\\ &\quad = E\Big[e^{-r(\tau^*-t)}\big[\alpha\big(((n_tF_t^*(y_1)+y_1(X_{t,t+\Delta t}+X_{t,t+\Delta t}X_{t+\Delta t,t+2\Delta t}\\ &\qquad\quad +\cdots+X_{t,t+\Delta t}\cdots X_{\tau^*-\Delta t,\tau^*}))/n_{\tau^*})-K\big)\\ &\qquad\quad +(1-\alpha)\big(((n_tF_t^*(y_2)+y_2(X_{t,t+\Delta t}+X_{t,t+\Delta t}X_{t+\Delta t,t+2\Delta t}\\ &\qquad\quad +\cdots+X_{t,t+\Delta t}\cdots X_{\tau^*-\Delta t,\tau^*}))/n_{\tau^*})-K\big)\big]^+\Big]\\ &\quad \leqslant \alpha E\left[e^{-r(\tau^*-t)}\left(\frac{n_tF_t^*(y_1)+S_{t+\Delta t}+\cdots+S_{\tau^*}}{n_{\tau^*}}-K\right)^+\middle| S_t=y_1\right]\\ &\qquad + (1-\alpha)E\left[e^{-r(\tau^*-t)}\left(\frac{n_tF_t^*(y_2)+S_{t+\Delta t}+\cdots+S_{\tau^*}}{n_{\tau^*}}-K\right)^+\middle|\right.\\ &\qquad\qquad \left. S_t=y_2\right]\\ &\quad \leqslant \alpha\sup_{\tau>t}E[e^{-r(\tau-t)}(\overline{S}_\tau-K)^+|\overline{S}_t=F_t^*(y_1), S_t=y_1]\\ &\qquad +(1-\alpha)\sup_{\tau>t}E[e^{-r(\tau-t)}(\overline{S}_\tau-K)^+|\overline{S}_t=F_t^*(y_2), S_t=y_2]\\ &\quad = \alpha c(F_t^*(y_1), y_1, t)+(1-\alpha)c(F_t^*(y_2), y_2, t)\\ &\quad \leqslant \alpha\psi(F_t^*(y_1))+(1-\alpha)\psi(F_t^*(y_2)) \ \text{ (by definition of } F_t^*(\cdot))\\ &\quad = \alpha(F_t^*(y_1)-K)^+ + (1-\alpha)(F_t^*(y_2)-K)^+\\ &\quad = \alpha F_t^*(y_1)+(1-\alpha)F_t^*(y_2)-K \quad (\text{since } F_t^*(\cdot)\geqslant K)\\ &\quad = \psi(\alpha F_t^*(y_1)+(1-\alpha)F_t^*(y_2)), \end{aligned}$$

where the first inequality follows from $(a+b)^+ \leqslant a^+ + b^+$. □

Proof of Lemma 2. Note that $F_i(\cdot)$ is nondecreasing, and

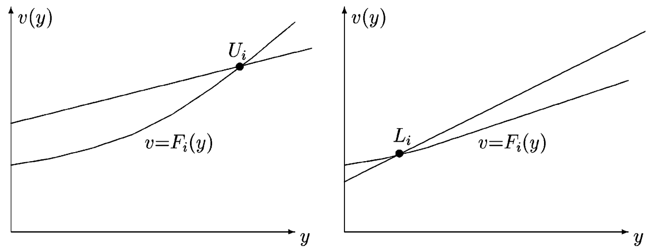
$$v = \frac{n_i - 1}{n_i}z + \frac{y}{n_i}$$

is a straight line as a function of y with slope $\frac{1}{n_i} > 0$. We consider four possible cases:

(i) If the entire straight line is above the curve $v = F_i(y)$, we can have

$$L_i(z) = 0_+ \quad \text{and} \quad U_i(z) = +\infty.$$

Figure 5. Determining L_i and U_i: Case (ii).



(ii) If the straight line intersects with $v = F_i(y)$ at only one point y_1, but it is not tangent to the curve, then either

$$L_i(z) = 0_+ \quad \text{and} \quad U_i(z) = y_1$$

or

$$L_i(z) = y_1 \quad \text{and} \quad U_i(z) = +\infty$$

(see Figure 5). If the line is tangent to the curve, we have

$$L_i(z) = U_i(z) = y_1.$$

(iii) If the equation

$$\frac{n_i - 1}{n_i} z + \frac{y}{n_i} = F_i(y) \tag{22}$$

has two solutions y_1 and y_2 with $y_1 < y_2$, by the convexity of $F_i(\cdot)$ (see Figure 6), we have

$$L_i(z) = y_1 \quad \text{and} \quad U_i(z) = y_2.$$

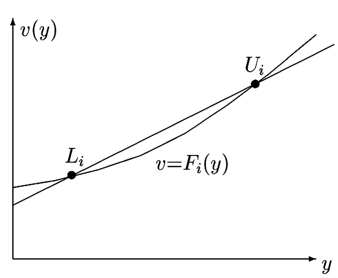
(iv) If Equation (22) has more than two solutions, let y_1 and y_2 denote the smallest and largest solutions, respectively, where y_2 could be $+\infty$. Then, by the convexity of F_i, it is easy to show that (22) is satisfied $\forall y \in (y_1, y_2)$. Thus, we may choose

$$L_i(z) = y_1 \quad \text{and} \quad U_i(z) = y_2.$$

Derivation of PA Estimator for the $\eta = 2$ Case. The sample performance for the $\eta = 2$ case is given by

$$\begin{aligned}
\mathscr{L} &= \mathbf{1}\{\overline{S}_{t_1} \geqslant F_1(S_{t_1})\}(\overline{S}_{t_1} - K)e^{-rt_1} \\
&\quad + \mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} \geqslant F_2(S_{t_2})\}(\overline{S}_{t_2} - K)e^{-rt_2} \\
&\quad + \mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} < F_2(S_{t_2})\}(\overline{S}_T - K)^+ e^{-rT}. \qquad (23)
\end{aligned}$$

Figure 6. Determining L_i and U_i: Case (iii).



Recall that $F_i, i = 1, 2$, are used to characterize the threshold levels and depend on some parameters of interest. Furthermore, they are nondecreasing functions. Taking the expectation of the first term of $\mathscr{L}$ given by (23), we have

$$\begin{aligned}
&E[\mathbf{1}\{\overline{S}_{t_1} \geqslant F_1(S_{t_1})\}(\overline{S}_{t_1} - K)^+ e^{-rt_1}] \\
&\quad = E\Bigg[\int_{h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}^{h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)} \\
&\qquad\qquad \left(\frac{(n_1 - 1)\overline{S}_{t_1-\Delta t} + h(z; S_{t_1-\Delta t}, \Delta t)}{n_1} - K\right) \\
&\qquad\qquad \times e^{-rt_1} f(z)\,dz\Bigg]. \qquad (24)
\end{aligned}$$

Note that here $L_1(\cdot)$ and $U_1(\cdot)$, which are defined in Lemma 2, are dependent on the parameters of interest and

$$\overline{S}_{t_1} \geqslant F_1(S_{t_1}) \Longleftrightarrow L_1(\overline{S}_{t_1-\Delta t}) \leqslant S_{t_1} \leqslant U_1(\overline{S}_{t_1-\Delta t}).$$

Intuitively, this implies that the option will be exercised at time t_1 if and only if the asset price at time t_1 does not pull downward or upward the average asset price too much. Specific derivation of $L_1(\cdot)$ and $U_1(\cdot)$ can be seen more clearly from the example given in §5. Differentiating (24) and assuming an interchange of differentiation and expectation, we have

$$\begin{aligned}
&\frac{\partial}{\partial\theta} E[\mathbf{1}\{\overline{S}_{t_1} \geqslant F_1(S_{t_1})\}(\overline{S}_{t_1} - K)e^{-rt_1}] \\
&\quad = E\Bigg[E\Bigg[\frac{\partial h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}{\partial\theta} \\
&\qquad\qquad \times f(h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t))(\overline{S}_{t_1} - K)e^{-rt_1} \\
&\qquad\qquad \Bigg| \overline{S}_{t_1-\Delta t}, S_{t_1} = U_1(\overline{S}_{t_1-\Delta t})\Bigg]\Bigg] \\
&\qquad - E\Bigg[E\Bigg[\frac{\partial h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}{\partial\theta} \\
&\qquad\qquad \times f(h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t))(\overline{S}_{t_1} - K)e^{-rt_1} \\
&\qquad\qquad \Bigg| \overline{S}_{t_1-\Delta t}, S_{t_1} = L_1(\overline{S}_{t_1-\Delta t})\Bigg]\Bigg] \\
&\qquad + E\left[\mathbf{1}\{\overline{S}_{t_1} \geqslant F_1(S_{t_1})\}\frac{\partial}{\partial\theta}((\overline{S}_{t_1} - K)e^{-rt_1})\right].
\end{aligned}$$

For the second term of $\mathscr{L}$ given by (23), we have

$$\begin{aligned}
&E[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} \geqslant F_2(S_{t_2})\}(\overline{S}_{t_2} - K)e^{-rt_2}] \\
&\quad = E\Bigg[\left(\int_{h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}^{\infty} + \int_{-\infty}^{h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}\right) \\
&\qquad \times E\Bigg(\int_{h^{-1}(L_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)}^{h^{-1}(U_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)} \Big((((n_1 - 1)\overline{S}_{t_1-\Delta t} \\
&\qquad\qquad + h(z_1; S_{t_1-\Delta t}, \Delta t) + \cdots + h(z_2; S_{t_2-\Delta t}, \Delta t)) \\
&\qquad\qquad /n_2) - K\Big)e^{-rt_2} f(z_2)\,dz_2 \Bigg| \overline{S}_{t_2-\Delta t}, S_{t_2-\Delta t}\Bigg) \\
&\qquad\qquad \times f(z_1)\,dz_1\Bigg],
\end{aligned}$$

where $L_2(\cdot)$ and $U_2(\cdot)$ are defined as in Lemma 2. Therefore, we have

$$\frac{\partial}{\partial\theta}E[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} \geqslant F_2(S_{t_2})\}(\overline{S}_{t_2} - K)e^{-rt_2}]$$
$$= E\bigg[\frac{\partial h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)) \times E[\mathbf{1}\{\overline{S}_{t_2} \geqslant F_2(S_{t_2})\}(\overline{S}_{t_2} - K)e^{-rt_2} \,\Big|\, \overline{S}_{t_1-\Delta t}, S_{t_1} = L_1(\overline{S}_{t_1-\Delta t})]\bigg]$$
$$- E\bigg[\frac{\partial h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)) \times E[\mathbf{1}\{\overline{S}_{t_2} \geqslant F_2(S_{t_2})\}(\overline{S}_{t_2} - K)e^{-rt_2} \,\Big|\, \overline{S}_{t_1-\Delta t}, S_{t_1} = U_1(\overline{S}_{t_1-\Delta t})]\bigg]$$
$$- E\bigg[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1})\}\frac{\partial h^{-1}(L_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(L_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)) \times (\overline{S}_{t_2} - K)e^{-rt_2} \,\Big|\, \overline{S}_{t_2-\Delta t}, S_{t_2} = L_2(\overline{S}_{t_2-\Delta t})\bigg]$$
$$+ E\bigg[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1})\}\frac{\partial h^{-1}(U_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(U_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)) \times (\overline{S}_{t_2} - K)e^{-rt_2} \,\Big|\, \overline{S}_{t_2-\Delta t}, S_{t_2} = U_2(\overline{S}_{t_2-\Delta t})\bigg]$$
$$+ E\bigg[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} \geqslant F_2(S_{t_2})\}\frac{\partial}{\partial\theta}((\overline{S}_{t_2} - K)e^{-rt_2})\bigg].$$

Similarly, for the third term of $\mathscr{L}$ given by (23), we have

$$\frac{\partial}{\partial\theta}E[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} < F_2(S_{t_2})\}(\overline{S}_T - K)^+ e^{-rT}]$$
$$= E\bigg[\frac{\partial h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)) \times E[\mathbf{1}\{\overline{S}_{t_2} < F_2(S_{t_2})\}(\overline{S}_T - K)^+ e^{-rT} \,|\, \overline{S}_{t_1-\Delta t}, S_{t_1} = L_1(\overline{S}_{t_1-\Delta t})]\bigg]$$
$$- E\bigg[\frac{\partial h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)) \times E[\mathbf{1}\{\overline{S}_{t_2} < F_2(S_{t_2})\}(\overline{S}_T - K)^+ e^{-rt_2} \,|\, \overline{S}_{t_1-\Delta t}, S_{t_1} = U_1(\overline{S}_{t_1-\Delta t})]\bigg]$$
$$- E\bigg[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1})\}\frac{\partial h^{-1}(U_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(U_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)) \times E[(\overline{S}_T - K)^+ e^{-rT} \,|\, \overline{S}_{t_2-\Delta t}, S_{t_2} = U_2(\overline{S}_{t_2-\Delta t})]\bigg]$$
$$+ E\bigg[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1})\}\frac{\partial h^{-1}(L_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(L_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t))E[(\overline{S}_T - K)^+ e^{-rT} \,|\, \overline{S}_{t_2-\Delta t}, S_{t_2} = L_2(\overline{S}_{t_2-\Delta t})]\bigg]$$
$$+ E\bigg[\mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} < F_2(S_{t_2})\}\frac{\partial}{\partial\theta}((\overline{S}_{t_2} - K)^+ e^{-rT})\bigg].$$

Combining all these results, we obtain the PA estimator for $\eta = 2$ case:

$$\frac{\partial h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}{\partial\theta} f(h^{-1}(L_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)) \times \{E[\mathscr{L} \,|\, \overline{S}_{t_1-\Delta t}, S_{t_1} = L_1(\overline{S}_{t_1-\Delta t})_-] - E[(\overline{S}_{t_1} - K)e^{-rt_1} \,|\, \overline{S}_{t_1-\Delta t}, S_{t_1} = L_1(\overline{S}_{t_1-\Delta t})]\}$$
$$- \frac{\partial h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(U_1(\overline{S}_{t_1-\Delta t}); S_{t_1-\Delta t}, \Delta t)) \times \{E[\mathscr{L} \,|\, \overline{S}_{t_1-\Delta t}, S_{t_1} = U_1(\overline{S}_{t_1-\Delta t})_+] - E[(\overline{S}_{t_1} - K)e^{-rt_1} \,|\, \overline{S}_{t_1-\Delta t}, S_{t_1} = U_1(\overline{S}_{t_1-\Delta t})]\}$$
$$+ \mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1})\}\frac{\partial h^{-1}(L_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(L_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)) \times [E[\mathscr{L} \,|\, \overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2-\Delta t}, S_{t_2} = L_2(\overline{S}_{t_2-\Delta t})_-] - E[(\overline{S}_{t_2} - K)e^{-rt_2} \,|\, \overline{S}_{t_2-\Delta t}, S_{t_2} = L_2(\overline{S}_{t_2-\Delta t})]]$$
$$+ \mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1})\}\frac{\partial h^{-1}(U_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)}{\partial\theta} \times f(h^{-1}(U_2(\overline{S}_{t_2-\Delta t}); S_{t_2-\Delta t}, \Delta t)) \times [E[\mathscr{L} \,|\, \overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2-\Delta t}, S_{t_2} = U_2(\overline{S}_{t_2-\Delta t})_+] - E[(\overline{S}_{t_2} - K)e^{-rt_2} \,|\, \overline{S}_{t_2-\Delta t}, S_{t_2} = U_2(\overline{S}_{t_2-\Delta t})]]$$
$$+ \mathbf{1}\{\overline{S}_{t_1} \geqslant F_1(S_{t_1})\}\frac{\partial}{\partial\theta}[(\overline{S}_{t_1} - K)e^{-rt_1}]$$
$$+ \mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} \geqslant F_2(S_{t_2})\}\frac{\partial}{\partial\theta}[(\overline{S}_{t_2} - K)e^{-rt_2}]$$
$$+ \mathbf{1}\{\overline{S}_{t_1} < F_1(S_{t_1}), \overline{S}_{t_2} < F_2(S_{t_2})\}\frac{\partial}{\partial\theta}[(\overline{S}_T - K)^+ e^{-rT}].$$

ENDNOTE

1. Here, as in much of the related literature we cite, we take American-Asian options to mean Asian options with

early exercise opportunities at *discrete* points in time (as opposed to continuously exercisable).

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