Optimal Joint Preventive Maintenance and Production Policies*

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Abstract: We study joint preventive maintenance (PM) and production policies for an unreliable production-inventory system in which maintenance/repair times are non-negligible and stochastic. A joint policy decides (a) whether or not to perform PM and (b) if PM is not performed, then how much to produce. We consider a discrete-time system, formulating the problem as a Markov decision process (MDP) model. The focus of the work is on the structural properties of optimal joint policies, given the system state comprised of the system's age and the inventory level. Although our analysis indicates that the structure of optimal joint policies is very complex in general, we are able to characterize several properties regarding PM and production, including optimal production/maintenance actions under backlogging and high inventory levels, and conditions under which the PM portion of the joint policy has a control-limit structure. In further special cases, such as when PM set-up costs are negligible compared to PM times, we are able to establish some additional structural properties. © 2005 Wiley Periodicals, Inc. Naval Research Logistics 52: 668–681, 2005.

Keywords: optimal preventive maintenance; joint maintenance and production policies; unreliable production systems; Markov decision processes; control-limit structure

1. INTRODUCTION

We study the problem of jointly optimizing preventive maintenance (PM) and production policies in the context of an unreliable production system. Consider a make-to-stock production system with the stock of completed goods consumed by external demand. The production system can produce at any rate from 0 (idle) to its maximal rate if it is in working state. However, the system may fail, in which case corrective maintenance (CM) must be commenced immediately to restore the system to the working state. To reduce the likelihood of failure, the system can also be preventively maintained (PM) in lieu of production. The failure process is assumed stochastic, and the times to complete CM or PM are nonzero and also stochastic. Thus, in addition to the direct cost for performing CM or PM, the nonnegligible maintenance completion time leads to an indirect "cost" of lost production capacity due to system unavailability. The goal is to find optimal joint PM and production policies that minimize the total PM/CM and inventory costs.

The literature on optimal production control and optimal PM has been predominantly separate. On the one hand, although the problem of production control in failure-prone production systems has been studied extensively (see, for example, [1, 7, 8, 10]), models taking into account the action of PM appear scarce. One reason for the neglect is possibly due to the modeling of the failure processes as two-state (on–off) continuous-time Markov chains, which essentially means that the lifetimes are exponentially distributed and so have constant failure rates, thus precluding PM from being included in these models.

On the other hand, optimal PM problems have not been addressed sufficiently in the context of production-inven-

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tory systems. A wide range of maintenance models has been developed, and their structural properties have been analyzed extensively in the literature (see the survey papers [3, 11, 16, 17]), but most maintenance models have concentrated on utilizing data solely on the reliability of machines, and have not taken into account other system state information, such as buffer/inventory levels. However, in many settings, for example, in semiconductor manufacturing, this information plays a critical role in PM decisions [20].

Some recent work on optimal PM policies for production systems do consider the impact of inventory levels, but with the assumption of predetermined (fixed) production policies. For example, Van der Duyn Schouten and Vanneste [6] investigate an integrated maintenance-production problem, in which the preventive maintenance policy is based not only on the age of the device, but also on the level of the subsequent buffer. A predetermined production policy, specifically, a hedging point policy, is assumed. Das and Sarkar [4] evaluate the performance of an *a priori* preventive maintenance policy, when the system's inventory is managed according to an (s, S) policy.

Overall, the problem of joint optimization of PM and production control has not been considered until very recently. Boukas and Liu [2] consider the continuous flow model in which the machine has three working states: good, average and bad, and one failure state, whose transitions are governed by a continuous-time Markov chain. The objective is to optimize the production rate and maintenance rates (the transition rates to the good state) in order to minimize discounted total costs including inventory holding, backlog, and maintenance costs. Finding an optimal policy involves solving the corresponding Hamilton-Jacobi-Bellman equations, which often lack closed-form solution. Iravani and Duenyas [9] also consider an integrated maintenance and production control policy using a semi-Markov decision process model. Because the joint optimal policy is extremely complex, a heuristic policy with simple structure is proposed and analyzed. Other related work includes the papers by Sloan and Shanthikumar [14, 15], in which they address the problem of joint equipment maintenance scheduling and production dispatching. The recent work of Sloan [13] studies a joint production-maintenance problem, and explores some structural properties of optimal policies, but the model assumes instantaneous repairs.

The problem setting in this paper is similar to those in [6, 9], but the research focus differs, in that our emphasis is on characterizing the structural properties of optimal joint policies rather than on the study of heuristic policies. The analysis of structural properties provides important insights for the development of heuristic policies that have simple structural forms and hence are easily implemented in practice.

It is well known that for maintenance only problems (i.e., without production/inventory control), control-limit type PM policies are often optimal. In such a policy, there exists a threshold parameter called the control limit such that if the system age exceeds the control limit, then it is optimal to perform PM; otherwise it is not optimal to perform PM. Similarly, for failure-prone production control problems (without PM), hedging-point (essentially base-stock) policies are often optimal, where such a policy specifies that production should be carried out until the inventory level reaches the hedging point. Both types of policies are very appealing, because they have a simple structure and thus can be easily implemented in practice. However, when PM and production are considered simultaneously, we find that the optimal joint policies are more complex than a simple combination of those two structures, and only under certain conditions can some of the desirable structure be recovered. The purpose of our work is to try to characterize the settings in which such structural properties exist, as well as those in which they do not.

The main results of this paper are the following. In a general setting, we establish two structural properties intuitively consistent with hedging point policies: (a) When inventory is below a certain (negative) point, the optimal action is either to do PM or to produce at the maximal production rate; and (b) when inventory is above a certain (positive) point, the optimal action is either to do PM or not to produce at all. In between these two inventory levels, however, the optimal policy may not have a simple structure, as a numerical example clearly shows. In the context of classical maintenance models [5], it is well known that under the condition of increasing failure rate (IFR), the optimal maintenance policy has a control-limit structure. We are able to recover similar results for the production-maintenance problem for a limited class of problems; i.e., we prove, under various scenarios, that there exists an optimal joint policy that exhibits a control-limit structure with respect to PM actions. Specifically, we show that the joint optimization problem reduces to classical maintenance models when the inventory level approaches infinity, either positive or negative, leading to the existence of a control-limit optimal policy. We also study the model with a special hedging-point policy such that the hedging point is zero, and show the optimal PM policy has the control-limit structure if the machine failure distribution has IFR, and the result is not dependent on either PM/CM cost or PM/CM time duration. Finally, we study the model under the assumption of zero PM setup cost, which is also assumed in [2, 6, 9], and could be justified in situations where the main effect of PM is system unavailability. Again we establish, under some appropriate conditions, the control-limit structure with respect to PM actions for optimal joint policies.

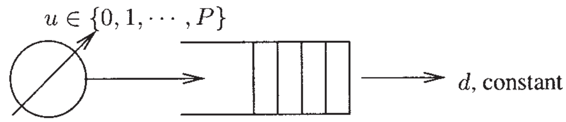

Figure 1. An unreliable production system with time-dependent failures.

In the remainder of the paper, we start with the problem definition and the development of a discrete-time Markov decision process (MDP) model in Section 2. Section 3 contains the results characterizing the structural properties of the optimal cost function and the associated optimal policies. In Section 4 we consider three special cases in which optimal PM policies have control-limit structure. Finally, we conclude the paper with some remarks.

2. PROBLEM DEFINITION

Consider an unreliable make-to-stock discrete-time production system as in Figure 1. The stock of completed goods is consumed by a constant per period demand d. The system experiences *time-dependent* failures (see, e.g., [7]), which means it deteriorates as long as it is up, even if it is not producing, and can fail when idle. If the system fails, a CM must be performed. In order to avoid failures, PM can be scheduled on the system. Both CM and PM will take some random time to complete. The system can produce any amount up to the maximal rate P ($P > d$) when it is up; however when it is in PM or CM, it produces nothing. For the sake of simplicity, we assume d and P are integer. (For noninteger cases, we can apply the same analysis throughout, and the same results follow.) The problem is to find an optimal joint policy to decide whether or not to perform PM, and if PM is not performed, then how much to produce, under an appropriate objective function.

The following cost structure will be imposed. Completed goods in stock will incur an inventory holding cost, while backlogged demands are allowed but with a penalty cost. If the system fails, then a CM must be conducted with a setup cost c_{CM}. If a PM is to be performed, then a setup cost c_{PM} will be incurred. Both c_{CM} and c_{PM} are one-time costs, and are independent of the time for CM and PM, respectively. The objective is to minimize the expected discounted cost over the infinite horizon. The problem is formulated as a discounted-cost MDP model.

2.1. MDP Formulation

We first summarize in the following the notation used throughout the paper.

- S_t inventory level at the beginning of period t.
- X_t system state at the beginning of period t, $X_t \in \{0, 1, 2\}$, where $X_t = 1$ if the system is up; $X_t = 0$ if the system is down for CM; $X_t = 2$ if system is down for PM
- N_t number of time periods that the system has been in the current state X_t at the beginning of period t. If $X_t = 1$, N_t is the age of the system since the last PM or CM. If $X_t = 0$ or 2, N_t is the time that the system has been in CM or PM, respectively
- f_n conditional failure probability at age $n + 1$, $n \geq 0$, given the system is up at age n; i.e., $f_n = \Pr(X_{t+1} = 0 | X_t = 1, N_t = n)$. In other words, f_n is the failure rate at age $n + 1$. Note it also implies that the system will not fail at age 0 (the newest state)
- p_n conditional PM completion probability at the beginning of the next period after the PM has lasted for n periods; i.e., $p_n = \Pr(X_{t+1} = 1 | X_t = 2, N_t = n)$, $n \geq 0$. Note it implies that a PM takes at least one period to complete
- r_n conditional CM completion probability at the beginning of the next period after the CM has lasted for n periods; i.e., $r_n = \Pr(X_{t+1} = 1 | X_t = 0, N_t = n)$. Note it implies that a CM takes at least one period to complete
- d incoming constant demand each period
- P the maximal production rate, $P > d$
- c_{CM} one-time setup cost for performing CM, incurred when the CM initiated
- c_{PM} one-time setup cost for performing PM, incurred when the PM initiated
- c^+ unit inventory holding cost per period, $c^+ > 0$
- c^- unit backlog penalty cost per period, $c^- > 0$
- τ_r time for CM
- F_r c.m.f. of τ_r
- τ_p time for PM
- F_p c.m.f. of τ_p
- β discount factor, $0 < \beta < 1$.

The system state at the beginning of period t is denoted by (S_t, X_t, N_t). When the system is in state $X_t = 1$, the available control u_t is either to do PM ($u_t = PM$) or to produce amount $u_t \in \{0, 1, \ldots, P\}$. When $u_t = PM$ is applied to state $(s, 1, n)$, the state jumps immediately to state $(s, 2, 0)$. When the system is in state $X_t = 0$ or 2, there is no admissible control, other than to continue the CM or PM until it finishes.

Denote by $g(S_t)$ the one-period inventory cost function, which is piecewise linear given by

$$g(S_t) = c^+ \cdot |S_t| \cdot I(S_t \geq 0) + c^- \cdot |S_t| \cdot I(S_t < 0), \quad (1)$$

where $I(A)$ is the indicator function of the event A.

Denote by $J(s, x, n)$ the optimal total discounted cost over the infinite horizon when the system starts at the initial state (s, x, n). An optimal policy μ^* can be obtained by solving the following dynamic programming optimality equations [12], based on the system dynamics:

$$J(s, 0, n) = g(s) + \beta \cdot r_n \cdot J(s - d, 1, 0) + \beta \cdot (1 - r_n) \cdot J(s - d, 0, n + 1), \quad (2)$$

$$J(s, 2, n) = g(s) + \beta \cdot p_n \cdot J(s - d, 1, 0) + \beta \cdot (1 - p_n) \cdot J(s - d, 2, n + 1), \quad (3)$$

$$J(s, 1, n) = \min\{Q^{PM}(s, 1, n), \min_{u \in \{0,1,\ldots,P\}} Q^u(s, 1, n)\}, \quad (4)$$

where

$$Q^{PM}(s, 1, n) := c_{\mathrm{PM}} + g(s) + \beta \cdot p_0 \cdot J(s - d, 1, 0) + \beta \cdot (1 - p_0) \cdot J(s - d, 2, 1) = c_{\mathrm{PM}} + J(s, 2, 0), \quad (5a)$$

$$Q^u(s, 1, n) := g(s) + \beta \cdot f_n \cdot (c_{\mathrm{CM}} + J(s + u - d, 0, 0)) + \beta \cdot (1 - f_n) \cdot J(s + u - d, 1, n + 1). \quad (5b)$$

Equations (2) and (3) state that when the system has been under CM and PM, respectively, for n periods, the corresponding maintenance will finish by the next period with probability r_n and p_n, respectively, and the inventory level will decrease by d in each period. In Eq. (4), $Q^{\mathrm{PM}}(s, 1, n)$ is the cost function corresponding to the policy that chooses to do PM in state $(s, 1, n)$, and then follows an optimal policy. $Q^u(s, 1, n)$ is the cost function corresponding to the policy that chooses not to do PM but to produce at the rate of u in state $(s, 1, n)$, and then follows an optimal policy thereafter.

For the sake of simplicity, denote by state $(s, 1, F)$ when the system just fails. Note the system will jump to state $(s, 0, 0)$ immediately with a cost c_{CM} incurred, and so $J(s, 1, F) := c_{\mathrm{CM}} + J(s, 0, 0)$. Equation (5b) now can be written as

$$Q^u(s, 1, n) := g(s) + \beta \cdot f_n \cdot J(s + u - d, 1, F) + \beta \cdot (1 - f_n) \cdot J(s + u - d, 1, n + 1). \quad (6)$$

The form of optimal joint policies may be very complex. This is evidenced by the following numerical example. Consider the case in which the failure distribution is a discretized Weibull distribution with p.m.f. $\{w_k, k = 1, 2, \cdots\}$, obtained by discretizing the continuous c.d.f.

$$F(t) = 1 - e^{-(t/\eta)^\gamma}, \qquad t \ge 0, \quad \gamma > 0, \quad \eta > 0, \quad (7)$$

with the discretizing interval δ, such that

$$w_k = F(k\delta) - F((k - 1)\delta), \qquad k = 1, 2, \ldots. \quad (8)$$

Its conditional failure probability $\{f_k, k = 0, 1, \cdots\}$ is given by

$$f_k = \frac{w_{k+1}}{\sum_{j=k+1}^{\infty} w_j} = 1 - e^{-(\delta/\eta)^\gamma[(k+1)^\gamma - k^\gamma]}, \qquad k = 0, 1, \ldots. \quad (9)$$

Definition (Increasing Failure Rate (IFR) for discrete distributions): A discrete distribution with p.m.f. $\{p_n, n = 1, 2, \cdots\}$ has IFR if its failure rate $r_n = p_{n+1}/\sum_{k=n+1}^{\infty} p_k$, $n = 0, 1, \ldots$, is *nondecreasing* in n.

NOTE: Throughout the paper, we follow the convention of using the terms increasing (decreasing) and nondecreasing (nonincreasing) interchangeably, adding the qualifier "strictly" to indicate otherwise.

It is easy to see that, similar to the continuous Weibull distribution, a discretized Weibull distribution has IFR if $\gamma \ge 1$, and has decreasing failure rate if $\gamma \in (0, 1)$.

We truncate the discretized Weibull distribution at the maximal age 100, i.e., $w_{100} = F(\infty) - F(99 \cdot \delta) = 1 - F(99 \cdot \delta)$, and so $f_{99} = 1$. Let $\delta = 0.2$, $\eta = 5$, and $\gamma = 4$; then $\{f_k, k = 0, 1, \ldots, 98\}$ can be obtained by (9). Furthermore, let the time for PM be uniformly distributed on $\{1, 2, 3\}$, and the time for CM also be uniformly distributed, on $\{1, 2, 3, 4, 5, 6\}$. Let $\beta = 0.95$, $d = 1$, $P = 3$, $c_{\mathrm{PM}} = 50$, $c_{\mathrm{CM}} = 2 \cdot c_{\mathrm{PM}}$, $c^+ = 1$, and $c^- = 10$. The inventory level is truncated to the range $[-40, 80]$.

An optimal policy can be obtained numerically by solving the corresponding dynamic programming equations (2)–(5b). Figure 2 shows the optimal policy. The boundary area between the three larger regions is enlarged in the bottom part of Figure 2, where "1" indicates states in which $u = 1$ is optimal and "2" indicates states in which $u = 2$ is optimal. The solution clearly indicates that the structure of the optimal joint policy is not simple. However, for any fixed inventory level, it also reveals a control-limit structure for performing PM with respect to system age.

To see the benefit of the optimal joint policy, we compare it with a sequentially optimized maintenance and production policy. The sequentially optimized policy is obtained by first optimizing the PM policy without consideration of production and inventory, and then under the determined PM policy, optimizing the production policy. Using the

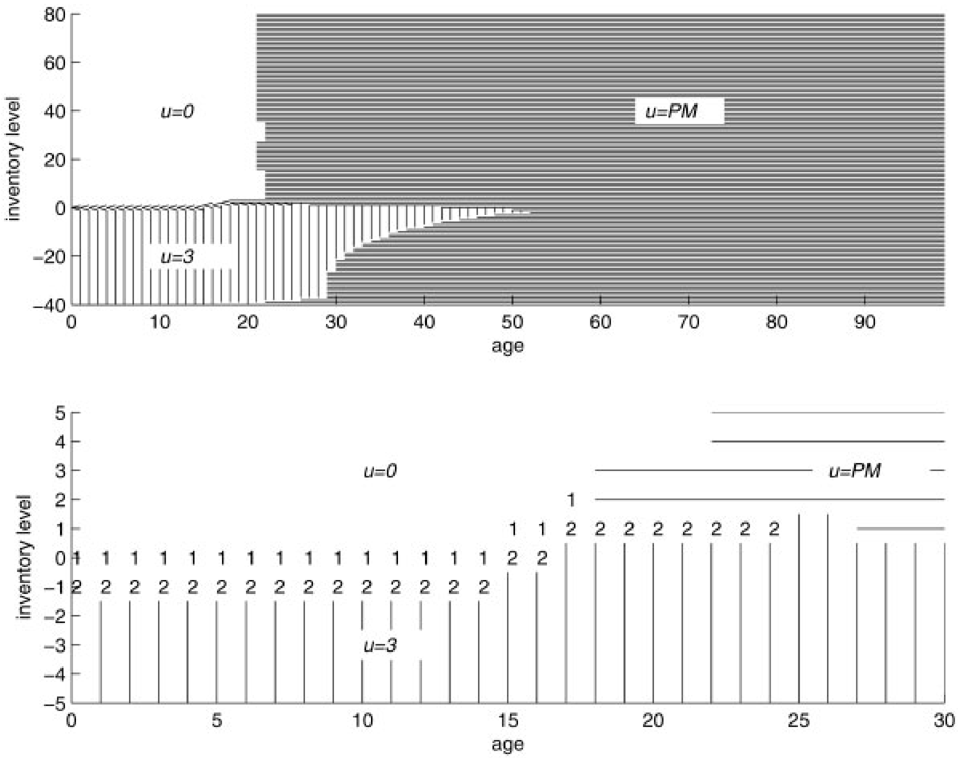

Figure 2. Optimal joint policy. (a) The optimal actions on the entire state space. (b) The optimal actions on the space $[-5, 5] \times [0, 100]$.

same parameters as in the joint optimization, the optimal PM policy can be easily calculated [5], and it has a control-limit structure with a control-limit $CL = 21$. After the PM policy is determined, the production policy can be obtained by numerically solving Eqs. (2)–(5b), with (4) replaced by the following equation:

$$J(s, 1, n) = \begin{cases} Q^{PM}(s, 1, n), & \text{if } n \geq CL, \\ \min_{u \in \{0,1,\ldots P\}} Q^{u}(s, 1, n), & \text{if } 0 \leq n < CL. \end{cases} \quad (10)$$

Figure 3 shows the sequentially optimized policy over the entire state space, using the same legend as in Figure 2. Let J_{opt} be the cost function corresponding to the optimal joint policy, and J_{seq} the cost function corresponding to the sequentially optimized policy. Figure 4, which shows the relative difference, $(J_{seq} - J_{opt})/J_{opt} \times 100\%$, over the entire state space, indicates the sequentially optimized policy can lead to poor performance for some states, especially in ranges where one is likely to be operating, with the maximal difference up to 60%. Although these conclusions are specific to this particular example, the substantial performance improvement of optimal joint policies over sequentially optimized policies motivates us to explore the structural properties of optimal joint policies.

3. STRUCTURAL PROPERTIES OF OPTIMAL POLICIES

In this section, we show some structural properties related to the optimal cost function and optimal policies. The main technique employed is value iteration (see, for example, [12]). Let $J_k(\cdot, \cdot, \cdot)$ be the approximated cost function at iterative step k, and $J_0(\cdot, \cdot, \cdot) = 0$. Then

$$J_{k+1}(s, 0, n) = g(s) + \beta \cdot r_n \cdot J_k(s - d, 1, 0) + \beta \cdot (1 - r_n) \cdot J_k(s - d, 0, n + 1), \quad (11)$$

$$J_{k+1}(s, 2, n) = g(s) + \beta \cdot p_n \cdot J_k(s - d, 1, 0) + \beta \cdot (1 - p_n) \cdot J_k(s - d, 2, n + 1), \quad (12)$$

$$J_{k+1}(s, 1, n) = \min\{Q^{PM}_{k+1}(s, 1, n), \min_{u \in \{0,1,\ldots P\}} Q^{u}_{k+1}(s, 1, n)\}, \quad (13)$$

$$Q^{PM}_{k+1}(s, 1, n) = c_{PM} + J_{k+1}(s, 2, 0), \quad (14)$$

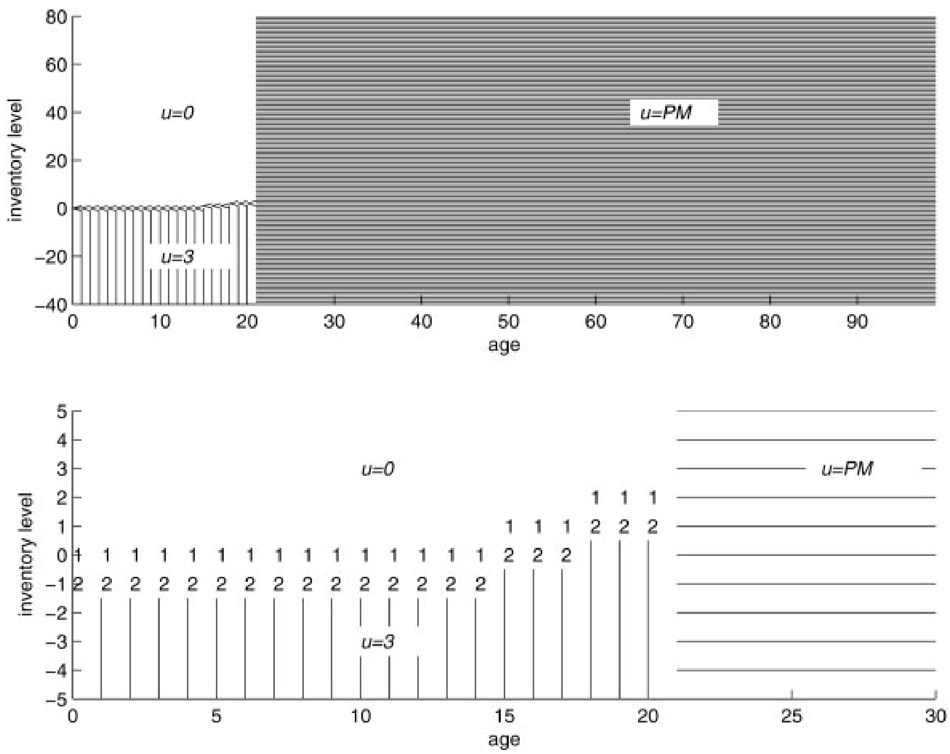

Figure 3. Sequential optimized policy. (a) The optimal actions on the entire state space. (b) The optimal actions on the space $[-5, 5] \times [0, 100]$.

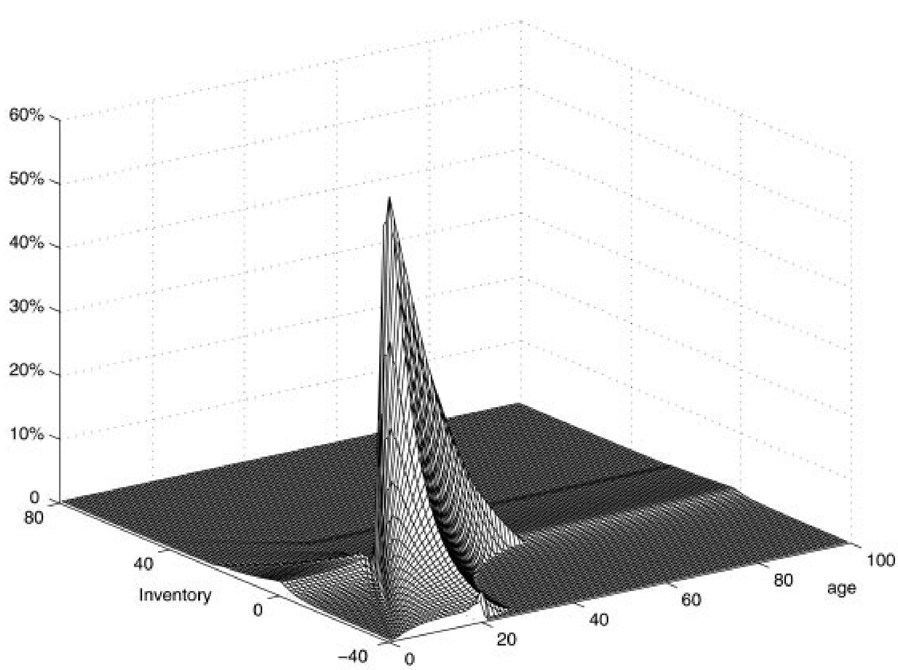

Figure 4. Performance difference of two policies.

$$Q^u_{k+1}(s, 1, n) = g(s) + \beta \cdot f_n \cdot J_k(s + u - d, 1, F) + \beta \cdot (1 - f_n) \cdot J_k(s + u - d, 1, n + 1). \quad (15)$$

Note that the system state is countable, and the number of actions is finite for every state. The following lemma states some results regarding optimal policies and the optimal cost function.

LEMMA 1: For the dynamic programming equations (2)–(5b), the following hold:

(a) Among all history dependent and randomized policies, there exists an optimal policy that is Markov deterministic or stationary deterministic.
(b) Let w be an arbitrary positive real-valued function on the state space; the optimality equations have a unique solution in the space of real-valued functions v satisfying $\|v\|_w < \infty$ (for the definition of $\|\cdot\|_w$, refer to [12], p. 231).
(c) Value iteration or policy iteration converges to the optimal cost function J.

PROOF: Let $w(s, x, n) := |s|\max\{c^+, c^-\} + \max\{c_{\text{PM}}, c_{\text{CM}}, 1\}$, a positive real-valued function on the state space. We first want to prove the following two conditions hold:

C1: (Assumption 6.10.1 on p. 232 of [12]). There exists a constant $\mu < \infty$ such that the one stage reward-(cost) function $r(s, x, n; u)$ at state (s, x, n) with action u satisfies

$$\sup_u |r(s, x, n; u)| \le \mu w(s, x, n).$$

C2: (Assumption 6.10.2 on pp. 232–233 of [12]). There exists a constant κ, $0 \le \kappa < \infty$, such that

$$E\{w(S_{t+1}, X_{t+1}, N_{t+1})|S_t = s, X_t = x, N_t = n; u\} \le \kappa w(s, x, n),$$

for all (s, x, n) and admissible u; and for each β, $0 \le \beta < 1$, there exists an α, $0 \le \alpha < 1$ and an integer J such that

$$E_\pi\{w(S_{t+J}, X_{t+J}, N_{t+J})|S_t = s, X_t = x, N_t = n\} \le \alpha\beta^{-J} w(s, x, n),$$

for all Markov deterministic policies π and for all (s, x, n).

It can easily be seen that C1 holds with $\mu = 1$. We can show C2 also holds by applying Proposition 6.10.5.a on page 237 of [12] with $L = P\max\{c^+, c^-\} > 0$, as in one stage the change of inventory level cannot exceed P. The results then follow immediately from Theorem 6.10.4 on page 236 of [12]. □

Therefore, by value iteration,

$$J(s, \cdot, \cdot) = \lim_{k\to\infty} J_k(s, \cdot, \cdot).$$

LEMMA 2: $J(s, \cdot, \cdot)$ is decreasing (nonincreasing) in s, for $s \le 0$.

PROOF: We proceed by value iteration.

(i) $J_0(s, \cdot, \cdot) = 0$.
(ii) Assume $J_k(s, \cdot, \cdot)$ is decreasing in s, for $s \le 0$. It is obvious that $J_{k+1}(s, 0, n)$ and $J_{k+1}(s, 2, n)$ are decreasing in s, for $s \le 0$, since $g(s)$ is decreasing in s, for $s \le 0$.

Now we show $J_{k+1}(s, 1, n)$ is also decreasing in s, for $s \le 0$. It is easily seen that $Q^{\text{PM}}_{k+1}(s, 1, n)$ is decreasing in s, for $s \le 0$. Let

$$Q^{\overline{PM}}_{k+1}(s, 1, n) := \min_{u\in\{0,1,\ldots,P\}} Q^u_{k+1}(s, 1, n) = g(s) + \beta \cdot f_n \cdot c_{\text{CM}} + \min_u\{\beta f_n J_k(s + u - d, 0, 0) + \beta(1 - f_n)J_k(s + u - d, 1, n + 1)\},$$

and denote $W_k(s) := \beta \cdot f_n \cdot J_k(s, 0, 0) + \beta \cdot (1 - f_n) \cdot J_k(s, 1, n + 1)$, so $W_k(s)$ is decreasing in s, for $s \le 0$. Thus,

$$Q^{\overline{PM}}_{k+1}(s, 1, n) = g(s) + \beta \cdot f_n \cdot c_{\text{CM}} + \min_{u=0,\ldots,P} W_k(s + u - d) = g(s) + \beta \cdot f_n \cdot c_{\text{CM}} + \begin{cases} W_k(s + P - d) & \text{if } s \le d - P, \\ \min_{x=0,\ldots,s+P-d} W_k(x) & \text{if } d - P < s \le 0. \end{cases}$$

Similarly, for $(s - 1, 1, n)$, we have

$$Q^{\overline{PM}}_{k+1}(s - 1, 1, n) = g(s - 1) + \beta \cdot f_n \cdot c_{\text{CM}} + \begin{cases} W_k(s + P - d - 1) & \text{if } s \le d - P, \\ \min_{x=0,\ldots,s+P-d-1} W_k(x) & \text{if } d - P < s \le 0. \end{cases}$$

Therefore, $Q^{\overline{PM}}_{k+1}(s - 1, 1, n) > Q^{\overline{PM}}_{k+1}(s, 1, n)$, for $s \le 0$. It follows that $J_{k+1}(s, 1, n)$ is decreasing in s, for $s \le 0$, $\forall n$.

Let $k \to \infty$, so $J(s, \cdot, \cdot) = \lim_{k\to\infty} J_k(s, \cdot, \cdot)$ is decreasing in s, for $s \le 0$. $\square$

The lemma yields the following theorem immediately.

THEOREM 3: If the inventory level is negative, then the optimal action is either to do PM or to produce at least enough items such that the inventory becomes non-negative. Furthermore, if the inventory level is less than or equal to $d - P$, then the optimal action is either to do PM or produce at the maximal rate P.

PROOF: It is easily seen, from Eq. (5b) and by Lemma 2,

$$\min_{u\in\{0,1,\ldots,P\}} Q^u(s, 1, n) = \begin{cases} Q^P(s, 1, n) & \text{if } s \le d - P, \\ \min\limits_{d-s\le u\le P} Q^u(s, 1, n) & \text{if } d - P < s < 0. \end{cases}$$

Therefore, by optimality equation (4), if $s \le d - P$, then $J(s, 1, n) = \min\{Q^{PM}(s, 1, n), Q^P(s, 1, n)\}$, i.e., the optimal action is either to do PM or to produce P. If $d - P \le s < 0$, $J(s, 1, n) = \min\{Q^{PM}(s, 1, n), Q^{d+|s|}(0, 1, n), \ldots, Q^P(P - d - |s|, 1, n)\}$. $\square$

The theorem reflects intuition that negative inventory is not preferable. More importantly, it characterizes one feature of the optimal production policy such that when there are too many backorders, it is optimal to produce at the maximal rate, unless the likelihood of failure becomes too great, in which case PM should be commenced.

Next, we show that for fixed inventory level s, there is an optimal control-limit policy with respect to machine's age to do PM. Observe $Q^{PM}(s, 1, n)$ is a constant with respect to the machine's age n. If we can show $\min_{u\in\{0,1,\ldots,P\}} Q^u(s, 1, n)$ is increasing (nondecreasing) in n, then it follows that $J(s, 1, n)$ is increasing in n, and the optimal policy has the control-limit form, as stated in the following lemma.

LEMMA 4: For fixed s, if $\min_{u\in\{0,1,\ldots,P\}} Q^u(s, 1, n)$ is increasing in n, then optimal policies have "control-limit" structure, such that if in state $(s, 1, n)$, it is optimal to do PM, then it is also optimal to do PM in state $(s, 1, n + 1)$.

PROOF: Straightforward, so omitted here. $\square$

Note that $Q^u(s, 1, n)$ increasing in n is a sufficient condition of Lemma 4. From Eqs. (2) and (3), the cost $J(s, 2, n)$ and $J(s, 0, n)$ can be expressed in terms of $J(\cdot, 1, 0)$. Specifically,

$$J(s, 0, 0) = \sum_{k=0}^{\infty} \beta^k \prod_{i=0}^{k-1} (1 - r_i) g(s - kd)$$

$$+ \sum_{k=1}^{\infty} \beta^k r_{k-1} \prod_{i=0}^{k-2} (1 - r_i) J(s - kd, 1, 0), \qquad (\text{let } r_{-1} = 0)$$

$$= \sum_{k=0}^{\infty} \beta^k \overline{F_r}(k) g(s - kd) + E[\beta^{\tau_r} J(s - \tau_r d, 1, 0)]. \quad (16)$$

Similarly,

$$J(s, 2, 0) = \sum_{k=0}^{\infty} \beta^k \overline{F_p}(k) g(s - kd) + E[\beta^{\tau_p} J(s - \tau_p d, 1, 0)], \quad (17)$$

where $\overline{F_r}(\cdot)$ and $\overline{F_p}(\cdot)$ are the tail distributions of CM time τ_r and PM time τ_p, respectively.

We are now ready to prove the following theorem.

THEOREM 5: The optimal policy is of control-limit type with respect to the machine's age, if the following conditions are satisfied:

(1) f_n is increasing (nondecreasing) in n; i.e., the machine failure distribution has IFR.
(2) $c_{\text{PM}} \le c_{\text{CM}}$.
(3) $\tau_p = \tau_r$, in the sense of stochastic equivalence; i.e., $F_p(\cdot) = F_r(\cdot)$.

PROOF: By Eqs. (16), (17), and condition (3), $J(s, 0, 0) = J(s, 2, 0)$ for all s. Moreover,

$$J(s, 1, n) = \min\{Q^{PM}(s, 1, n), \min_u Q^u(s, 1, n)\}$$
$$\le Q^{PM}(s, 1, n) = c_{\text{PM}} + J(s, 2, 0) \le c_{\text{CM}} + J(s, 0, 0), \qquad \text{by condition (2)} = J(s, 1, F).$$

It is clear that using value iteration, at each step k we have also $J_k(s, 1, n) \le J_k(s, 1, F)$. Now, we claim $Q^u(s, 1, n)$ is increasing in n. To prove the monotonicity, we proceed by value iteration:

(1) Obviously, it holds at step 0.
(2) Assume at step k, $Q_k^u(s, 1, n)$ is increasing in n. So is $J_k(s, 1, n)$. At step $k + 1$, we have

$$Q^u_{k+1}(s, 1, n+1) - Q^u_{k+1}(s, 1, n) = \beta[(f_{n+1} - f_n)J_k(s+u-d, 1, F) + (1-f_{n+1})J_k(s+u-d, 1, n+2) - (1-f_n)J_k(s+u-d, 1, n+1)] \geq \beta[(f_{n+1}-f_n)J_k(s+u-d, 1, F) + (1-f_{n+1})J_k(s+u-d, 1, n+1) - (1-f_n)J_k(s+u-d, 1, n+1)] = \beta[(f_{n+1}-f_n)(J_k(s+u-d, 1, F) - J_k(s+u-d, 1, n+1))] \geq 0;$$

i.e.,

$$Q^u_{k+1}(s, 1, n+1) \geq Q^u_{k+1}(s, 1, n). \tag{18}$$

Let $k \to \infty$, we have $Q^u(s, 1, n+1) \geq Q^u(s, 1, n)$. It follows from Lemma 4, the optimal policy has control-limit structure. □

REMARK: Condition (3) is somewhat strict. In practice, one may find it is rarely satisfied. We will show under some special cases, the optimal policy has the control-limit structure without the requirement of condition (3).

The following theorem states another intuitive property about optimal joint policies, such that when the system has a high inventory level and if no PM is performed, then the optimal production rate is 0.

THEOREM 6: There exists s^* such that for any inventory level is greater than s^*, the optimal action is either to do PM or not to produce. Furthermore, s^* is independent of machine age.

PROOF: We decompose the optimal cost $J(s, 1, n)$ into two parts

$$J(s, 1, n) = G(s, 1, n) + V(s, 1, n), \tag{19}$$

where G is the total discounted inventory holding/backlogging cost and V is the total discounted PM/CM cost. Now consider a starting inventory level $s = k \cdot d + \delta$ with $k = 0, 1, 2, \ldots,$ and $0 \leq \delta < d$. Obviously, the total discounted inventory holding cost satisfies

$$G(s, 1, n) \geq \sum_{t=0}^{k} c^+(s - t \cdot d)\beta^t \equiv A(s). \tag{20}$$

Let μ_1 be any policy such that $\mu_1(s, 1, n) \geq 1$. Thus the total discounted inventory holding cost starting at $(s, 1, n)$ under μ_1 satisfies

$$G^{\mu_1}(s, 1, n) \geq c^+ s + \sum_{t=1}^{k} c^+(s + 1 - t \cdot d)\beta^t \tag{21}$$

$$= A(s) + c^+ \frac{\beta - \beta^{k+1}}{1-\beta}. \tag{22}$$

Now, consider the cost related to PM/CM only. Let $h(s, 1, n)$ be the minimal total discounted PM/CM cost. This will be achieved by an optimal PM policy. (Actually, because the machine deterioration process is not affected by the inventory process, the problem can be reduced to the classic problem of finding the optimal PM policy for a Markov chain; see, for example [5].) Denote the corresponding optimal PM policy by a function $v(s, 1, n)$. Obviously, for all joint policies μ, the total discounted PM/CM cost V^μ satisfies

$$V^\mu(s, 1, n) \geq h(s, 1, n). \tag{23}$$

Thus, from (20) and (23),

$$J(s, 1, n) \geq A(s) + h(s, 1, n). \tag{24}$$

From (22) and (23)

$$J^{\mu_1}(s, 1, n) \geq A(s) + c^+ \frac{\beta - \beta^{k+1}}{1-\beta} + h(s, 1, n). \tag{25}$$

We now construct a simple joint PM/production policy μ_0 from the optimal PM policy v, such that

$$\mu_0(s, 1, n) = \begin{cases} 0, & \text{if } v(s, 1, n) \neq PM, \\ PM, & \text{if } v(s, 1, n) = PM. \end{cases}$$

It is obvious that $V^{\mu_0}(s, 1, n) = h(s, 1, n)$, and

$$G^{\mu_0}(s, 1, n) = \sum_{t=0}^{\infty} g(s - td)\beta^t = \sum_{t=0}^{k} c^+(s - td)\beta^t + \sum_{t=k+1}^{\infty} c^-(td - s)\beta^t = A(s) + \beta^k \sum_{t=1}^{\infty} c^-(td - \delta)\beta^t \leq A(s) + \beta^k \sum_{t=1}^{\infty} c^- \cdot td \cdot \beta^t = A(s) + \frac{\beta^{k+1} c^- d}{(1-\beta)^2}. \tag{26}$$

Therefore,

$$J^{\mu_0}(s,1,n) \le A(s) + h(s,1,n) + \frac{\beta^{k+1}c^- d}{(1-\beta)^2}. \quad (27)$$

From (24) and (27),

$$A(s) + h(s,1,n) \le J(s,1,n) \le A(s) + h(s,1,n) + \frac{\beta^{k+1}c^- d}{(1-\beta)^2}. \quad (28)$$

It then follows from (25) that $\mu(s, 1, n) \in \{0, PM\}$ if

$$c^+ \frac{\beta - \beta^{k+1}}{1-\beta} > c^- \frac{\beta^{k+1} d}{(1-\beta)^2}, \quad (29)$$

or equivalently if

$$k > \left\lceil \frac{\ln \dfrac{c^+}{c^+ + c^- d/(1-\beta)}}{\ln \beta} \right\rceil = k^*, \quad (30)$$

where $\lceil a \rceil$ is the minimal integer that is greater or equal to a. So, $s^* = k^* d$, and it is not dependent on n. □

4. SPECIAL CASES

The complex structure of optimal policies under general conditions turns us to study the problem in the presence of special conditions. One structure particularly interesting is control-limit structure, which is often seen with optimal PM policies in classical maintenance models. Theorem 5 establishes the control-limit structure, yet under some strict conditions, especially the condition of stochastic equivalence between duration of PM and duration of CM. In this section, we show that when the inventory level goes to either positive or negative infinity, the optimal PM policy is of control-limit form. This is accomplished by making a connection between classical one-dimensional PM problems, i.e., without consideration of inventory level, and our joint policy problem. We also consider a hedging point production policy with the hedging point at zero, and present some structural properties regarding PM action. Finally, we consider a special case with the PM setup cost $c_{\mathrm{PM}} = 0$.

4.1. Limiting Cases When $s \to -\infty$ or $s \to +\infty$

Theorems 3 and 6 show that when the inventory level is either sufficiently low or sufficiently high, the optimal production amount is fixed. Specifically, it is optimal to produce at the maximal rate when the inventory is low, or not to produce when the inventory is high. In both cases, since the production portions of joint policies are determined, the two-dimensional (PM and production) problem reduces to the one-dimensional PM problem. Furthermore, we show in the following when $s \to -\infty$, the optimal PM policy for the original problem (2)–(5b) is the same as the optimal PM policy for the following problem (32)–(34).

Although we restrict ourselves to the case of $s \to -\infty$, the analysis for the case of $s \to +\infty$ is similar, and hence will be omitted.

Let I_t be the indicator function of not performing PM in period t when the machine is up, i.e.,

$$I_t = \begin{cases} 1 & u_t \ne PM \text{ and } X_t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3 states that when inventory is below $(d - P)$, it is optimal to produce at the maximal rate P if in period t the machine is up and not performing PM; and thus the inventory will decrease by $P - d$ for each such period. Let $\tau = \min\{t | S_t > d - P, S_0 = s\}$, i.e., the first time when inventory is larger than $(d - P)$. Under an optimal policy, the decomposed total inventory cost $G(s, 1, n)$ as in (19) is given by

$$\begin{aligned} G(s,1,n) &= E\left[\sum_{t=0}^{\infty} \beta^t g(S_t) | S_0 = s\right] \\ &= E\left[\sum_{t=0}^{\tau-1} \beta^t g(S_t) + \sum_{t=\tau}^{\infty} \beta^t g(S_t) | S_0 = s\right] \\ &= E\left[\sum_{t=0}^{\tau-1} \beta^t c^- \left(-S_0 + t \cdot d - \sum_{t'=0}^{t} I_{t'} \cdot P\right) \right. \\ &\quad \left. + \sum_{t=\tau}^{\infty} \beta^t g(S_t) | S_0 = s\right] = E\left[\sum_{t=0}^{\tau-1} \beta^t c^- (-s + t \cdot d) \right. \\ &\quad - \frac{c^- P}{1-\beta} \cdot \sum_{t=0}^{\tau-1} \beta^t \cdot I_t + c^- P \cdot \frac{\beta^{\tau-1}}{1-\beta} \sum_{t=0}^{\tau-1} I_t \\ &\quad \left. + \sum_{t=\tau}^{\infty} \beta^t g(S_t) | S_0 = s\right]. \end{aligned} \quad (31)$$

Note when $s \to -\infty$, $\tau \to \infty$ almost surely, and the third and fourth terms of the RHS of Eq. (31) vanish. The first term is a constant, and the second term can be regarded as the reward for the machine in production. As a result, when $s \to -\infty$, the optimal policy for minimizing total PM/CM cost plus total inventory cost $G(s, 1, n)$ is equivalent to that for minimizing the total PM/CM cost plus only the second term of Eq. (31), i.e., the running reward q for each period the machine is up, where $q = c^- P/(1 - \beta)$. Therefore, the optimal PM policy for the original problem (2)–(5b) when $s \to -\infty$ is equivalent to the optimal PM policy for the following PM problem (32)–(34).

Let n be the age of the machine. The machine can be preventively maintained at any time with a one-time setup cost c_{PM}, and the time for PM is a random variable τ_p. If the machine is not maintained at age n, then it gains a revenue g, but it fails at the next time period with probability f_n, or its age is updated to $n + 1$ with probability $1 - f_n$. When it fails, a one-time CM setup cost c_{CM} is incurred, and the time for CM τ_r is also random. The objective is to find an optimal PM policy to minimize the total PM/CM cost minus the total revenue gains.

The system state (x, n) is defined as before, and we can write the optimality equations as follows:

$$J(1, n) = \min\{c_{PM} + J(2, 0), -q + \beta \cdot f_n[c_{CM} + J(0, 0)] + \beta(1 - f_n)J(1, n + 1)\}, \quad (32)$$

$$J(2, n) = \beta \cdot p_n J(1, 0) + \beta(1 - p_n)J(2, n + 1), \quad (33)$$

$$J(0, n) = \beta \cdot r_n J(1, 0) + \beta(1 - r_n)J(0, n + 1). \quad (34)$$

Note that

$$J(2, 0) = E[\beta^{\tau_p}]J(1, 0), \quad (35)$$

$$J(0, 0) = E(\beta^{\tau_r})J(1, 0). \quad (36)$$

Plugging them into Eq. (32), it turns out the state variable x can be discarded. Equivalently, Eq. (32) can be rewritten as

$$J(n) = \min\{c_{PM} + E[\beta^{\tau_p}]J(0), -q + \beta \cdot f_n[c_{CM} + E(\beta^{\tau_r})J(0)] + \beta(1 - f_n)J(n + 1)\}. \quad (37)$$

LEMMA 7: If f_n is increasing in n, then $J(n)$ is an increasing function of n.

PROOF: Consider a history dependent policy π consisting of not doing PM until the first failure of the machine and then switching to an optimal policy. Denote by $T_r(n)$ the time of the first failure. Under such a policy,

$$J^{\pi}(n) = E\left[\sum_{t=0}^{T_r(n)} \beta^t \cdot (-q)\right] + E(\beta^{T_r(n)})(c_{CM} + E(\beta^{T_r})J(0)) < c_{CM} + E(\beta^{\tau_r})J(0).$$

Let $J(F) := c_{CM} + E(\beta^{\tau_r})J(0)$. Thus

$$J(n) \le J^{\pi}(n) < J(F), \qquad \forall\, n.$$

Using value iteration and the arguments similar to those in the proof of Theorem 5, and, by the condition of f_n increasing in n, it follows that $J(n)$ is increasing in n. □

THEOREM 8: If the machine failure distribution has IFR, then the optimal PM policy is a control-limit policy.

PROOF: It follows immediately from the monotonicity of the optimal cost function $J(n)$. □

REMARK: The control-limit for doing PM can be obtained by solving Eq. (37). Following the same steps, for the case $s \to +\infty$, we can construct a similar optimal PM problem, except that we set q to be zero for each period the machine is up, and it is easy to see that under the assumption of IFR, the optimal PM policy has the control-limit structure.

4.2. A Hedging Point Policy

We now consider a hedging point production policy with the hedging point at zero, i.e., the production policy is prescribed such that if the optimal action is not to do PM, then the amount to be produced is determined by $u = \min\{P, -s + d\}$, for all $s \le 0$, where s is the system inventory level at decision epochs. The optimality equations are the same as (2)–(5b), except that the minimization over $u \in \{0, 1, \ldots, P\}$ in (7) is replaced by u that is determined by the hedging point policy.

Because the hedging point is assumed at zero, we are only interested in the case of $s \le 0$ in the following analysis.

LEMMA 9: $J(s, 1, n)$ is decreasing in s, for all $s \le 0$.

PROOF: By value iteration. The proof is similar to that of Lemma 2 and thus omitted here. □

LEMMA 10: $J(s, 1, F) \ge J(s, 1, n)$, for all n, and for all $s \le 0$.

PROOF: Denote by $T_r(n)$ the (random) time epoch when the machine fails if it is kept producing when the system starts from state $(\cdot, 1, n)$ at time 0. It is readily seen that for system state $(0, 1, n)$, if we apply a policy such as keep the machine producing until it fails at $T_r(n)$, then $J(0, 1, n) \le E[\beta^{T_r(n)}J(0, 1, F)] \le J(0, 1, F)$.

In the following we consider system states $(s, 1, n)$, $s < 0$. We decompose the total cost $J(s, 1, \cdot)$ into inventory cost $G(s, 1, \cdot)$ and PM/CM cost $V(s, 1, \cdot)$.

Suppose that $\pi^F = \{\mu_0^F, \mu_1^F, \cdots\}$ is an optimal PM policy for the system starting at $(s, 1, F)$. Now we construct a specific policy $\pi = \{\mu_0, \mu_1, \cdots\}$ for the system starting at $(s, 1, n)$ by (a) letting the machine produce until it fails at time $T_r(n)$ and then (b) following the optimal policy π^F with common random variables (i.e., failures and PM/CM times) for both policies such that $\mu_{t+T_r(n)}(S_{t+T_r(n)}, 1, \cdot) = \mu_t^F(S_t^F, 1, \cdot)$, where S_t and S_t^F are the inventory levels at the time epoch t when the system starts from state $(s, 1, n)$ and $(s, 1, F)$, respectively. This implies that the sequence of PM/CM related events starting at $(s, 1, F)$ at time 0 is the same as that under policy π starting at time $T_r(n)$.

CASE 1: $T_r(n) = 1$. By construction,

$$V^\pi(s, 1, n) = \beta \cdot V(s, 1, F) < V(s, 1, F).$$

Furthermore, under the policy π, $S_{t+1} \geq S_t^F$, $\forall t \geq 0$. We then consider the following two subcases:

(a) $s \geq -P + d$. Then $S_1 = 0$, and

$$G^\pi(s, 1, n) = E\left[g(s) + \sum_{t=2}^{\infty} \beta^t \cdot g(S_t)\right] \leq E\left[g(s) + \sum_{t=2}^{\infty} \beta^t \cdot g(S_{t-1}^F)\right] \leq E\left[g(s) + \sum_{t=1}^{\infty} \beta^t \cdot g(S_t^F)\right] = G(s, 1, F).$$

Therefore, $J(s, 1, n) \leq J^\pi(s, 1, n) = G^\pi(s, 1, n) + V^\pi(s, 1, n) \leq J(s, 1, F)$.

(b) $s < -P + d$. Let $T := \min\{t : S_t = 0\}$. It can be easily checked that $S_t \geq S_t^F$, $\forall t < T$. Hence

$$G^\pi(s, 1, n) = E\left[\sum_{t=0}^{T-1} \beta^t \cdot g(S_t) + \sum_{t=T+1}^{\infty} \beta^t \cdot g(S_t)\right] \leq E\left[\sum_{t=0}^{T-1} \beta^t \cdot g(S_t^F) + \sum_{t=T+1}^{\infty} \beta^t \cdot g(S_{t-1}^F)\right] \leq E\left[\sum_{t=0}^{T-1} \beta^t \cdot g(S_t^F) + \sum_{t=T}^{\infty} \beta^t \cdot g(S_t^F)\right] = G(s, 1, F).$$

It follows that $J(s, 1, n) \leq J(s, 1, F)$.

CASE 2: $T_r(n) = 2$. Note that, at time $t = 1$, the state of the system is $(s', 1, n + 1)$ with $T_r(n + 1) = 1$ for the case $T_r(n) = 2$ while the state of the system is $(s', 1, F)$ for the case $T_r(n) = 1$. As a result, the sample path of the case $T_r(n) = 2$ can be constructed from that of the case $T_r(n) = 1$ in the same way as the sample path of the case $T_r(n) = 1$ was constructed from that starting from $(s, 1, F)$. It follows that

$$E\left[\sum_{t=0}^{\infty} \beta^t \cdot g(S_t^n) | T_r = 2\right] \leq E\left[\sum_{t=0}^{\infty} \beta^t \cdot g(S_t^n) | T_r = 1\right]. \tag{38}$$

Repeating the same argument, we have

$$E\left[\sum_{t=0}^{\infty} \beta^t \cdot g(S_t^n) | T_r = k + 1\right] \leq E\left[\sum_{t=0}^{\infty} \beta^t \cdot g(S_t^n) | T_r = k\right] \leq \cdots \leq G(s, 1, F). \tag{39}$$

Hence,

$$G^\pi(s, 1, n) = E\left[E\left[\sum_{t=0}^{\infty} \beta^t \cdot g(S_t^n) | T_r\right]\right] \leq G(s, 1, F).$$

It follows immediately that

$$J(s, 1, n) \leq G^\pi(s, 1, n) + V^\pi(s, 1, n) \leq G(s, 1, F) + V(s, 1, F) = J(s, 1, F),$$

for all n and for all $s \leq 0$. □

LEMMA 11: If f_n is increasing in n, then $J(s, 1, n)$ is increasing in n.

PROOF: By Lemma 10, $J(s, 1, n) \leq J(s, 1, F)$. Using value iteration and the arguments similar to those in the proof of Theorem 5, and by the condition of f_n increasing in n, it follows that $J(s, 1, n)$ is increasing in n. □

THEOREM 12: Under the hedging point policy with hedging point at zero, if the machine failure distribution has IFR, then the optimal PM policy is a control-limit policy.

PROOF: The proof is immediate from Lemma 11. □

REMARK: The result of control-limit structure is not dependent on the PM/CM costs, nor the PM/CM time durations.

4.3. No PM Setup Cost

We consider a special case in which the PM setup cost $c_{PM} = 0$. Without the cost c_{PM}, PM cost is factored in, though implicitly, through the inventory cost in the following sense: Since production is impossible while PM is being performed, large backorder penalty costs may be incurred if PM is undertaken when the inventory level is low. In literature on maintenance and production, it is not uncommon to assume PM setup cost being zero (for example, see [2, 6, 9]).

The following assumptions will be used in this subsection:

(A1) $c_{PM} = 0$.
(A2) p_n is decreasing in n.
(A3) $p_n \geq r_n$, $\forall n$.
(A4) f_n is increasing in n.

By assumption (A1), and the optimality equations (4) and (5a), it follows immediately that $J(s, 1, n) \leq J(s, 2, 0)$, for all n, s. We show the optimal cost function has the following structural properties.

LEMMA 13: If assumptions (A1) and (A2) are satisfied, then $J(s, 2, n)$ is increasing in n.

PROOF: We proceed by value iteration. Assume $J_k(s, 2, n) \leq J_k(s, 2, n + 1)$. Then

$$J_{k+1}(s, 2, n) = g(s) + \beta \cdot p_n \cdot J_k(s - d, 1, 0) + \beta \cdot (1 - p_n) \cdot J_k(s - d, 2, n + 1) \leq g(s) + \beta \cdot p_n \cdot J_k(s - d, 1, 0) + \beta \cdot (1 - p_n) \cdot J_k(s - d, 2, n + 2) = g(s) + \beta \cdot J_k(s - d, 2, n + 2) - \beta \cdot p_n \cdot (J_k(s - d, 2, n + 2) - J_k(s - d, 1, 0)).$$

The above inequality is by the induction hypothesis. Since $p_n \geq p_{n+1}$, and $J_k(s, 1, 0) \leq Q_k^{PM}(s, 1, 0) = J_k(s, 2, 0) \leq J_k(s, 2, n)$, it follows that

$$J_{k+1}(s, 2, n) \leq g(s) + \beta \cdot J_k(s - d, 2, n + 2) - \beta \cdot p_{n+1} \cdot (J_k(s - d, 2, n + 2) - J_k(s - d, 1, 0)) = J_{k+1}(s, 2, n + 1).$$

Let $k \to \infty$, and it follows $J(s, 2, n) \leq J(s, 2, n + 1)$. □

REMARK: Geometrically distributed PM time, i.e., $p_n = p$ for all n, is a special case of assumption (A2). If PM time is deterministic, and takes only one period, i.e., $p_0 = 1$, $p_n = 0$, $\forall n \geq 1$, then assumption (A2) is also satisfied.

LEMMA 14: If assumptions (A1)–(A3) are satisfied, then $J(s, 2, n) \leq J(s, 0, n)$, for all n, s.

PROOF: Again, by value iteration,

$$J_{k+1}(s, 2, n) = g(s) + \beta \cdot p_n \cdot J_k(s - d, 1, 0) + \beta \cdot (1 - p_n) J_k(s - d, 2, n + 1) \leq g(s) + \beta \cdot p_n \cdot J_k(s - d, 1, 0) + \beta \cdot (1 - p_n) J_k(s - d, 0, n + 1) = g(s) + \beta \cdot J_k(s - d, 0, n + 1) - \beta \cdot p_n \cdot (J_k(s - d, 0, n + 1) - J_k(s - d, 1, 0)).$$

Because $p_n \geq r_n$, and $J_k(s, 1, 0) \leq J_k(s, 2, 0) \leq J_k(s, 2, n + 1) \leq J_k(s, 0, n + 1)$, it follows that

$$J_{k+1}(s, 2, n) \leq g(s) + \beta \cdot J_k(s - d, 0, n + 1) - \beta \cdot r_n \cdot (J_k(s - d, 0, n + 1) - J_k(s - d, 1, 0)) = J_{k+1}(s, 0, n).$$

The result follows when $k \to \infty$. □

From Lemma 13 and 14,

$$J(s, 1, \cdot) \leq J(s, 2, n) \leq J(s, 0, n), \ \forall n. \tag{40}$$

LEMMA 15: If assumptions (A1)–(A4) are satisfied, then $J(s, 1, n)$ is increasing in n.

PROOF: By value iteration,

$$Q_{k+1}^u(s, 1, n) = g(s) + \beta \cdot f_n(J_k(s + u - d, 0, 0) + c_{CM}) + \beta(1 - f_n) J_k(s + u - d, 1, n + 1) \leq g(s) + \beta \cdot f_n(J_k(s + u - d, 0, 0) + c_{CM}) + \beta(1 - f_n) J_k(s + u - d, 1, n + 2) = g(s) + \beta \cdot J_k(s + u - d, 1, n + 2) + \beta \cdot f_n \cdot (J(s + u - d, 0, 0) + c_{CM} - J_k(s + u - d, 1, n + 2)) \leq g(s) + \beta \cdot J_k(s + u - d, 1, n + 2) + \beta \cdot f_{n+1} \cdot (J(s + u - d, 0, 0) + c_{CM} - J_k(s + u - d, 1, n + 2)) = Q_{k+1}^u(s, 1, n + 1).$$

It follows from Eqs. (13) and (14) that

$$J_{k+1}(s, 1, n) \leq J_{k+1}(s, 1, n + 1).$$

Let $k \to \infty$, and the result follows. □

THEOREM 16: If assumptions (A1)–(A4) hold, then the optimal joint policies have the control-limit structure for doing PM with respect to the machine's age.

PROOF: It is immediate from Lemma 15. □

REMARK: If $c_{PM} = 0$, $p_0 = 1$, and f_n increasing in n, then leaving the machine idle is not optimal. This can be seen from Eq. (40) and Lemma 15 that $Q^{PM}(s, 1, n) \leq Q^0(s, 1, n)$, and hence the action $u = 0$ is not optimal. Furthermore, it follows from Theorem 6 that there exists s^* such that $\mu(s, 1, n) = PM$, for all $s \geq s^*$, and for all n. This is equivalent to saying that the machine is prevented from failure if idle.

It is worth noting that we don't make any assumptions on the CM setup cost c_{CM}, except that it is nonnegative. It can be shown that the above results still hold under the condition such that there is an additional running cost for each period of PM and CM.

5. CONCLUSIONS

We have studied optimal joint PM and production policies for an unreliable production–inventory system. Our analysis and numerical example indicate that the form of jointly optimal policies can be complex in general. We characterize several structural properties related to the optimal policy. We show the intuitively consistent result that when there is a high level of backorders, it is optimal either to utilize the maximal rate or to perform PM, and when there is a high level of inventory, idling the production system or doing PM is optimal. We also prove a control-limit structure with respect to doing PM under various conditions. Specifically, we investigate policy structure for situations such as when there is very high stockups or backorders, when a hedging point production policy is predetermined, or when PM setup cost is not taken in account. Since some of these conditions are somewhat strict, a challenging avenue for further research is to be able to relax some of these conditions.

Also note that an important assumption in the model is that failures are time dependent. While this type of failure is commonly assumed, another type—*operation-dependent* failures [7], where a machine deteriorates only when it is working—might be more appropriate for some systems. In this case, the system deterioration process is now strongly affected by the production actions. Some preliminary results have been obtained [19].

Another interesting direction for future research is to consider the problem under the framework of bandit processes. There seems to be a strong connection between the joint PM and production problem with the so-called "restless bandit problems" [18]. Indeed, our problem can be regarded as a two-armed restless bandit process, with one arm corresponding to production and the other to preventive maintenance. At each decision epoch, we choose to operate one of the two arms, in order to minimize the overall operating costs. Although the restless bandit problem is usually extremely difficult to solve explicitly, it would shed some light on the problem from a different perspective.

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